3 Like A Function But Not A Function

3-1 Start From the Dice

组合数学 Combinatorics

Tsinghua University
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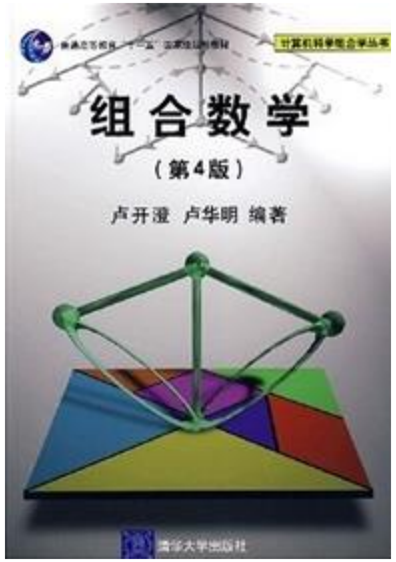
The Definition of Generating Function

Function

$$G(x)= x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9+3x^{10}+2x^{11}+x^{12}$$



Definition 2-1 For sequence $c_0, c_1, c_2\ldots$, forms a function

$$G(x)= c_0+c_1x+c_2x^2+\ldots,$$

Name $G(x)$ as the generating function for $c_0, c_1, c_2\ldots$。

Is generating function a function?

The main content of combinatorics is counting, the mostly used counting tool is generating function.

Generating Function

Generating Function， is an important theory and tool in combinatorics, especially in counting. By using this mathematic method, it will usually improve the application performance and speed tremendously.

Generating Function Method not only holds an important status in probability and counting, it is also a very important method in combinatorics.

Generating Function & Counting Rule

AND: Multiplication Rule
OR : Addition Rule

- E.g.: For 2 dice to roll out 6 points, how many possibilities are there?

$$\square + \square = 6$$

- **Solution 1:** 1st digit **1-5, and** the 2nd digit is determined by the 1st digit **$5 \times 1 = 5$**
- **Solution 2:** 1+5 = 5+1; **or** 2+4 = 4+2; **or** 3+3 **$2 + 2 + 1 = 5$**

Jakob Bernoulli
Swiss Mathematician Year 1654—1705

- **For *m* number of dice, how many different ways to get the sum of the points on dice as *n*?**

Generating Function & Counting Rules

AND： Multiplication Rule
OR ： Addition Rule

- E.g.： How many ways for 2 dice to roll n points?

$$\square + \square = n$$

○ x ○ $(○)^2$ x^2

$x^6 \leftarrow$ x^2 × Exponent is corresponding to points x^4

x^4

$$(x + x^2 + x^3 + x^4 + x^5 + x^6) \times (x + x^2 + x^3 + x^4 + x^5 + x^6)$$

$$x^6: \quad x^1 x^5 + x^2 x^4 + x^3 x^3 + x^4 x^2 + x^5 x^1 = 5x^6$$

The coefficient of x^6 is 5， means there are 5 ways for 2 dice to roll 6 points

$$G(x)=(x+x^2+\ldots+x^6)^2= x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9+3x^{10}+2x^{11}+x^{12}$$

The number of possibilities for 2 dice to roll ***n*** points is corresponding to the coefficient of x^n in $G(x)=(x+x^2+\ldots+x^6)^2$。

The coefficient in function is corresponding to counting sequence.

Generating Function & Counting Rule

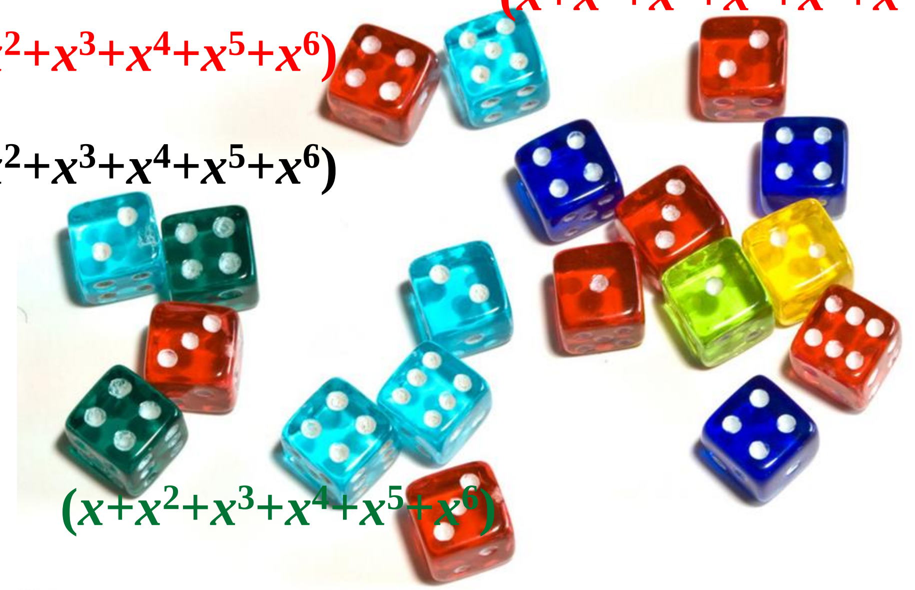
Generating Function is mother， counting sequence is a child。

雅各布 伯努利
Jakob I. Bernoulli
Swiss Mathematician
Year 1654—1705

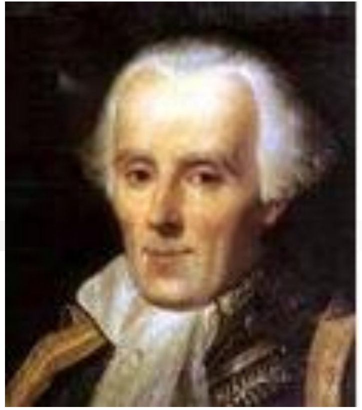
- **For *m* number of dice, what is the number of possibilities for the summation of points equals to *n*?**

$$G(x)=(x+x^2+x^3+x^4+x^5+x^6)^m$$

the coefficient of x^n in the expansion equation



§ 1.Generating Function & Counting Rule



Laplace

- **Definition 2-1** for sequence $c_0, c_1, c_2\ldots,$

$$G(x) = c_0 + c_1 x + c_2 x^2 + \ldots\ldots$$

Function $G(x)$ is the generating function for $c_0, c_1, c_2 \ldots$。

- In 1812， French mathematician Laplace was studying on generating function method and its theories while writing the 1st volume of "The Analysis Theory of Probability"
  - Counting Tool Like a function but not a function **Yes？ ？**
  - Do not consider the convergence
  - Do not consider the actual value
  - Formal power series

§ 1. Generating Function & Counting Rule

Corresponding Relation

count $p = 23^{\frac{1}{3}} 41^{\frac{1}{8}}$ (original)

Logarithmic mapping →

$\lg p = \frac{1}{3}\lg(23) + \frac{1}{8}\lg(41)$ (mapping)

Difficult

Simplify exponential calculation to addition and/or multiplication

Get the value of $\lg p$ (mapping)

Inverse logarithm ←

Solve the value for original p

Generating Function & Counting Rule Mapping Relationship

Primary Image

$\square + \square = n$ **Mapping of Generating Function** → Mapping $(x+x^2+x^3+x^4+x^5+x^6)(x+x^2+x^3+x^4+x^5+x^6)$

? ↓ **Difficult** ⟷ ? ↓ Polynomial Multiplication

$c_0, c_1, \ldots, c_n$ ← Corresponding Coefficient $= x^2+2x^3+3x^4+4x^5+5x^6+\ldots\ldots$

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)\times\left(\sum_{k=0}^{\infty} b_k x^k\right) = (a_0 + a_1x + a_2x^2 + \ldots)\times(b_0 + b_1x + b_2x^2 + \ldots)$$

$a_ib_{n-i}x^n \longleftarrow a_ix^i \qquad b_{n-i}x^{n-i}$

Multiplication Rule:

n **number of counting objects can be partition as** ***i*** **number of objects and** ***n-i*** **number of objects** $a_ix^i \times b_{n-i}x^{n-i} = a_ib_{n-i}x^n$

Addition Rule:

The accumulation of the partitioning strategy for ***n*** **number of counting objects**

$$c_n = \sum_{i=0}^{n} a_ib_{n-i},\ n = 0,1,2,\ldots$$

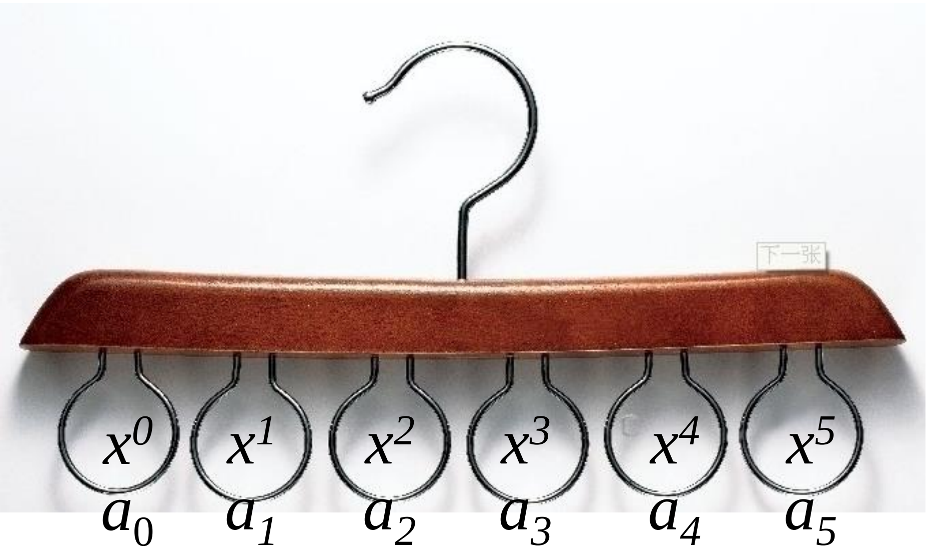
Polynomial multiplication enabled generating function to be equipped with counting ability

Generating function is a line of hangers which used to display a series of number sequences 。

— Herbert · Vere

$$G(x)=x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9+3x^{10}+2x^{11}+x^{12}$$

Function: $f(x)=\sum_{n=0}^{\infty} a_n x^n$ $G(x)=\sum_{n=0}^{\infty} a_n x^n$

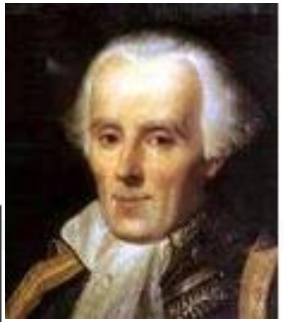


§3 Lesson Summary

Definition 2-1 For sequence $a_0, a_1, a_2\ldots$, form a function

$$G(x)= a_0+a_1x+a_2x^2+\ldots,$$

Name $G(x)$ as the generating function for sequence $a_0, a_1, a_2\ldots$。



Laplace
Year 1812

Bernoulli
Year 1705

Like A Function But Not A Function, It's Mapping

Euler
Year 1764

Found the mapping relationship is a "<u>Mathematic Discovery</u>".
Finding mapping is an important mathematic thinking.

3 似函数，非函数

3-2 母函数的计数问题

组合数学 Combinatorics

清华大学 马昱春

Generating Function & Counting Rule

Example 1： If there is 1g、2g、3g、4g of weights each, how different weight can be weighed? How many possible solutions?

$$(1+x)(1+x^2)(1+x^3)(1+x^4)$$
$$=1+x+x^2+2x^3+2x^4+2x^5+2x^6+2x^7+x^8+x^9+x^{10}$$

From the right generating function knows that it can generate from 1g to 10g , the coefficient is the solution number. For example, the right side there is $2x^5$, which mean there is 2 types of solutions for the 5g $5=2+3=1+4$

Similarly， $6=1+2+3=4+2$

$$10=1+2+3+4$$

The number of solutions for 6g weighed is 2, the solution for 10g weighed is 1

Example Question

Example 1： If there is 1、2、4、8、16、32g of weights each, how different weight can be weighed? How many possible solutions?

$$G(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})$$

$$(1+x)(1-x) = (1-x^2)$$

$$= \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^8}{1-x^4}\frac{1-x^{16}}{1-x^8}\frac{1-x^{32}}{1-x^{16}}\frac{1-x^{64}}{1-x^{32}}$$

$$= \frac{1-x^{64}}{1-x} = (1+x+x^2+\cdots+x^{63}) = \sum_{k=0}^{63} x^k$$

Example Question

From the generating function it can be known that these weights can be used to weigh from 1g to 63g of weightage. And, each solution is unique

This problem can be generalize to prove any decimal number n, can be represented as

$$n = \sum_{k \geq 0} a_k 2^k, \quad 0 \leq a_k \leq 1, \; k \geq 0$$

and it is unique.

Example Question

Example: Integer *n* is split into the summation of 1，2，3，…，*m*, and repetition is allowed, get its generating function.

If integer *n* is split into the summation of 1，2，3，…，*m*, and repetition is allowed, its generating function is $(1-x)^{-1}=1+x+x^2+.....$

$$G_1(x)=(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots$$

$$\cdots(1+x^m+x^{2m}+\cdots)$$

Example Question

If *m* appeared at least once，how is the generating function?

$$G_2(x) = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots(x^m+x^{2m}+\cdots)$$

$$= \frac{x^m}{(1-x)(1-x^2)\cdots(1-x^m)}$$

$$G_2(x) = \frac{1}{(1-x)(1-x^2)\cdots(1-x^m)} - \frac{1}{(1-x)(1-x^2)\cdots(1-x^{m-1})}$$

The above combination meaning: The partition number of integer *n* which is split into the summation of 1 to *m*, minus the partition number of the split 1 to *m*-1, is the partition number of *m* at least appeared once.

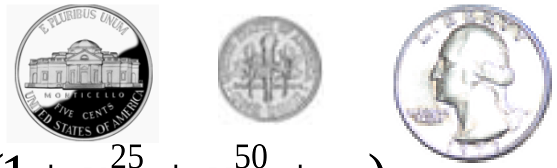
Combinations of Coins

- China Yuan (RMB) common coins: 10 cents, 50 cents, 1 dollar

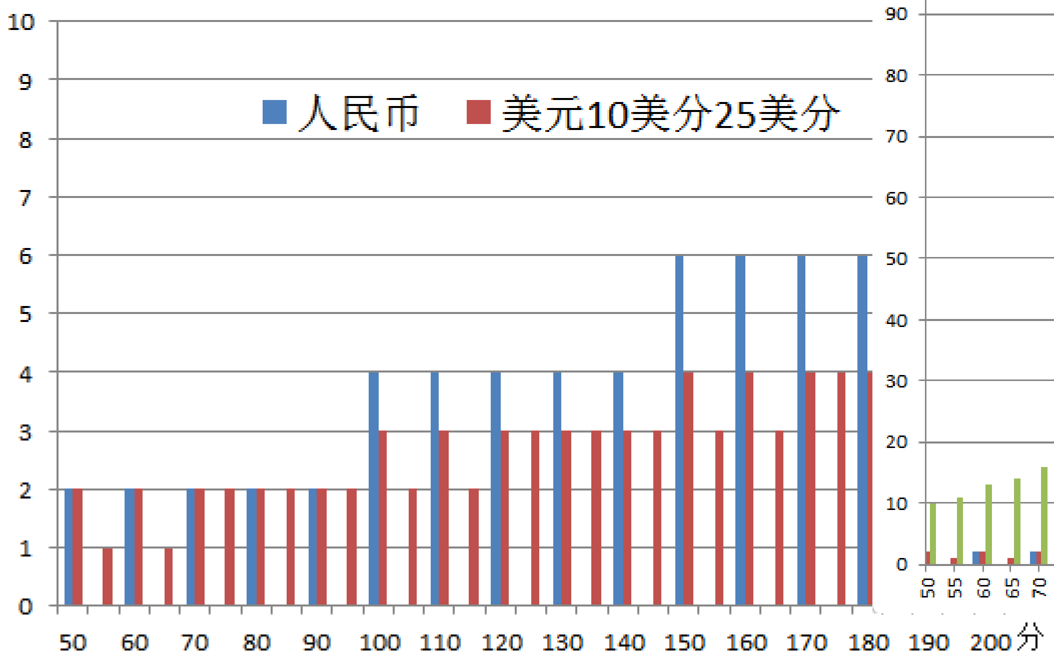
- The generating function for China Yuan coins

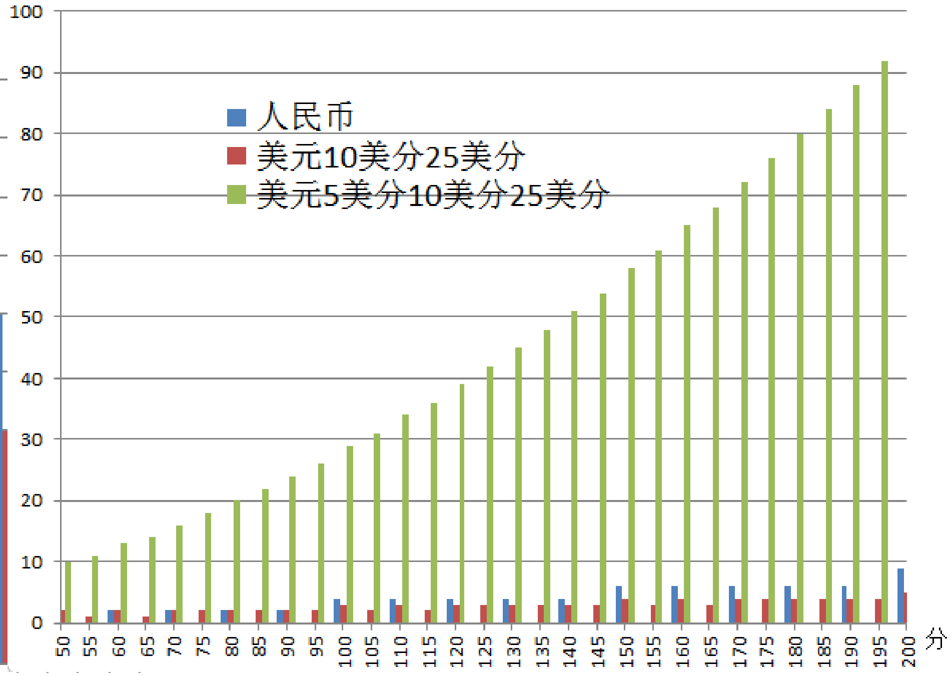
$$G(x) = (1 + x^{10} + x^{20} + \dots)(1 + x^{50} + x^{100} + \dots)(1 + x^{100} + x^{200} + \dots)$$

- USD common coins: 10 cents, 25 cents, 50 cents



$$G(x) = (1 + x^{5} + x^{10} + \dots)(1 + x^{10} + x^{20} + \dots)(1 + x^{25} + x^{50} + \dots)$$





3 Like A Function But Not A Function

3-3 Integer Partition

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The Partition of Integer

Natural number (positive number) partition is to express a positive number as the summation of several positive number:

Order is considered within various parts is named as orderly partition (Composition) ;

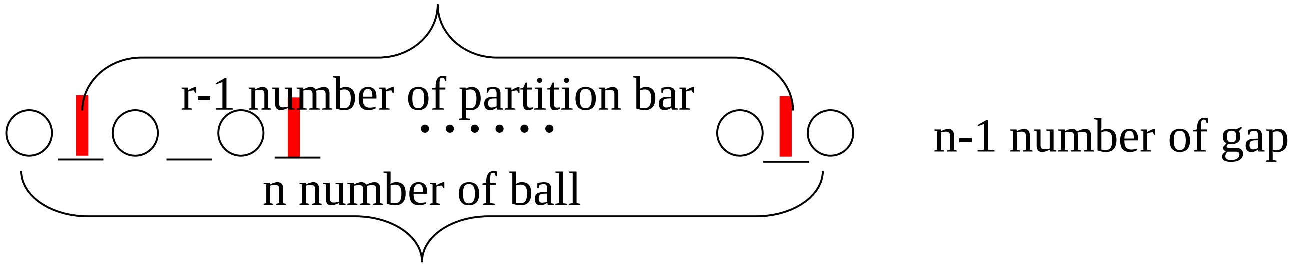
Otherwise, it is known as unordered partition (Partition).

3’s orderly 2-splitting: 3=2+1=1+2

n’s orderly r− splitting number is C(*n*-1,*r*-1)

n number of ball, split into r part，

Use r−1 of partition wall to put within n−1 gap between balls， solution number is C(n−1, r−1)



Ball Placing Model: *n*’s single *r*-splitting is equaled to put *n* identical ball into *r* labelled box. Box cannot be left empty.

Orderly partition of ball placing model: n’s single r-splitting is same as putting n identical ball into r **labelled** box, box cannot be left empty

- Unordered Partition
- 3’s unordered 2-splitting： 3=2+1
- 3’s all unordered splitting 3=3+0+0=2+1+0=1+1+1
- $x_1+x_2+\ldots+x_r=n$ number of solution of non-negative number? C(n+r-1,n)

n number of 1

1 ..1 | 1.. 1 | 1.. 1 | 1..1

$x_1$number of 1

r-1 number of door frame

Equals to put n identical ball into r **labelled** box, box can be left empty

0+3+0

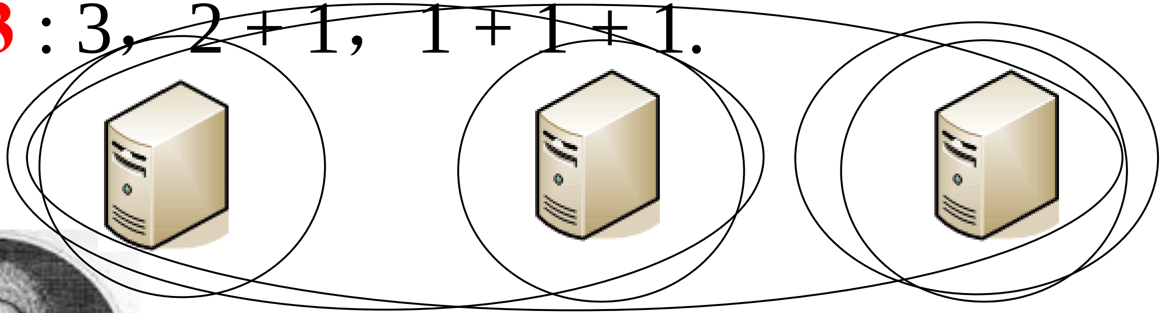
3+0+0

Unlike unordered partition

Integer Partition (**partition** of a positive integer n) is to partition integer into the summation of several integer , same as putting n identical ball into n unlabeled box, box can be left empty, also allows to place more than 1 ball. Integer is partition into the summation of several integers with different ways, the total number of different splitting methods is known as partition number.

§ 2.The Application of Generating Function: Integer Partition Number

- **Unordered Partition of Positive Integer**: Split a positive integer n into the summation of several integer, the order between numbers is ignored and allow repetition, its different partition number is $p(n)$。
  - Cryptography, Statistics, Biology……
- ***p*(3)=3** : 3, 2 + 1, 1 + 1 + 1.



Philippe Naudé

Year 1740, Naude asked Euler in the letter:

Solution number of partition of integer?

Exponent correspondence value

Euler

$G(x) = (1+x+x^2+\ldots)(1+x^2+x^4+\ldots)\ldots(1+x^m+x^{2m}+\ldots)\ldots$ coefficient of x^n

Generating Function of "1" — Generating Function of "2" — Generating Function of "m"

Weil A. Number Theory: An Approach Thought History [M]. Boston: Birkhouser, 1984. 284

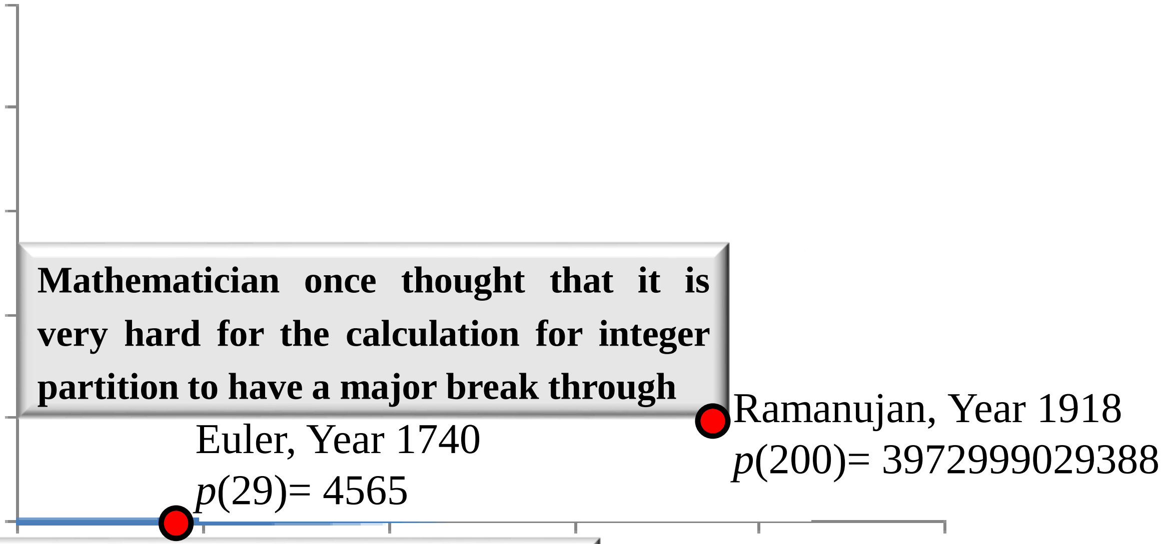
§ 2.The Application of Generating Function: Integer Partition Number

- OEIS: On-line Encyclopedia of Integer Sequences
  - Number Theory Related Authoritative Database and Algorithm Library
  - $p(n)$: A000041 sequence
- Generating function of integer partition $p(n)$

§ 2.The Application of Generating Function: Integer Partition Number

- OEIS: On-line Encyclopedia of Integer Sequences
  - Number Theory Related Authoritative Database and Algorithm Library
  - $p(n)$: A000041 sequence
- Generating function of integer partition p(n)

$$G(x) = (1+x+x^2+\ldots)(1+x^2+x^4+\ldots)\,(1+x^3+x^6+\ldots)\,\ldots\,(1+x^m+x^{2m})\ldots$$



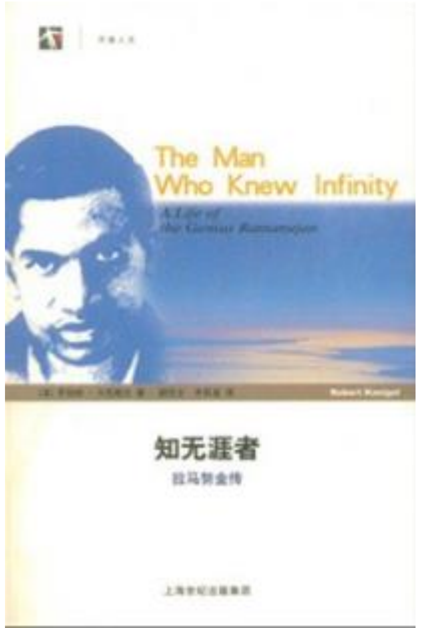
Mathematician once thought that it is very hard for the calculation for integer partition to have a major break through

Euler's generating function method is ingenious; but it is not suitable for calculating $p(n)$ 。

Manual Calculation for Polynomial Calculation

Son of India, Ramanujan （1887-1920）

“The weirdest person in the history of mathematic and also science”
He had never exposed to any proper mathematic training but he owned very amazing sixth sense towards mathematic, he discovered almost 3900 mathematics formulas and propositions independently.
In year 2012, America mathematician Ken Ono and his colleagues had proved that as Ramanujan was laid dying, he left a miraculous function which can be used directly to explain the partial secret of our black holes.
He wrote down all his foreseen mathematic propositions into 3 notebooks; and many of them got proven later. For example, mathematician V. Deligne had proved in year 1973 on Ramanujan’s guess which was placed in the year 1916. And, he was awarded with Fields Metal in year 1973.
America University of Florida had founded 《 Ramanujan’s Periodic Magazine》 in year 1997, specifically to publish on the research papers which are related to “His Influenced Mathematic Field ” ；

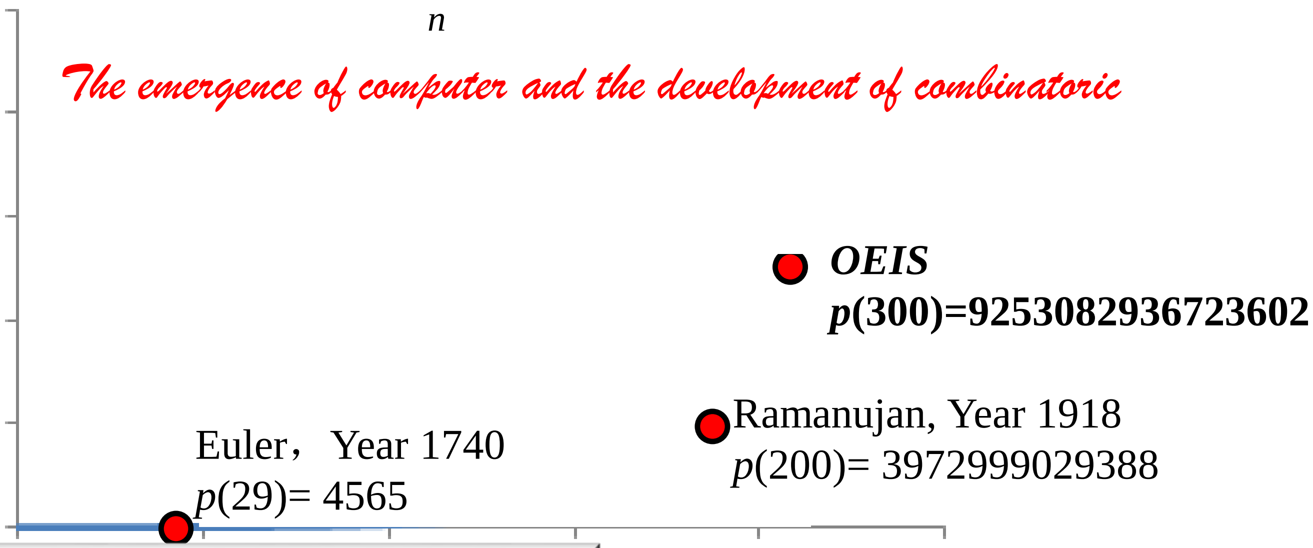


§ 2.The Application of Generating Function: Integer Partition Number

- OEIS: On-line Encyclopedia of Integer Sequences
  - Number Theory Related Authoritative Database and Algorithm Library
  - p(n): A000041 sequence
- Generating function of integer partition p(n)

$$G(x) = (1+x+2x^2+2x^3+2x^4+2x^4 \ldots)(1+x^3+x^6+\ldots)\ldots(1+x^m+x^{2m})\ldots$$

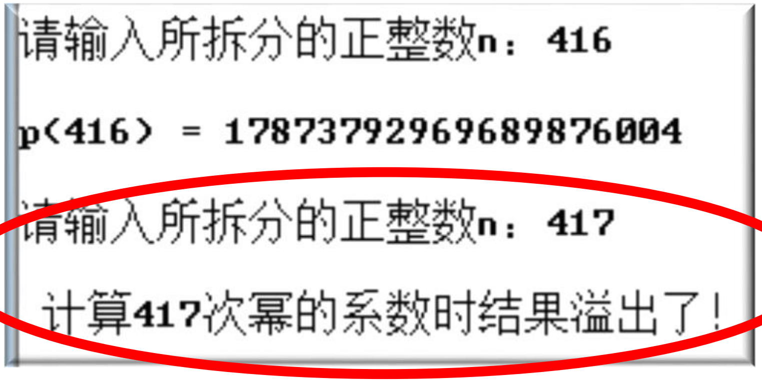
$$(f*g)[k] = \sum_{n} f[i]g[k-i]$$



Euler's generating function method is ingenious; but it is not suitable for calculating $p(n)$ 。

Manual Calculation for Polynomial Calculation

§ 2.The Application of Generating Function: Integer Partition Number



64-bit of computer unsigned integer

unsigned_int64 – largest representation is 2^{64}-1

18,446,744,073,709,551,615

p(416)= **17,8**73,792,969,689,876,004

p(417)= **18,9**87,964,267,331,664,557

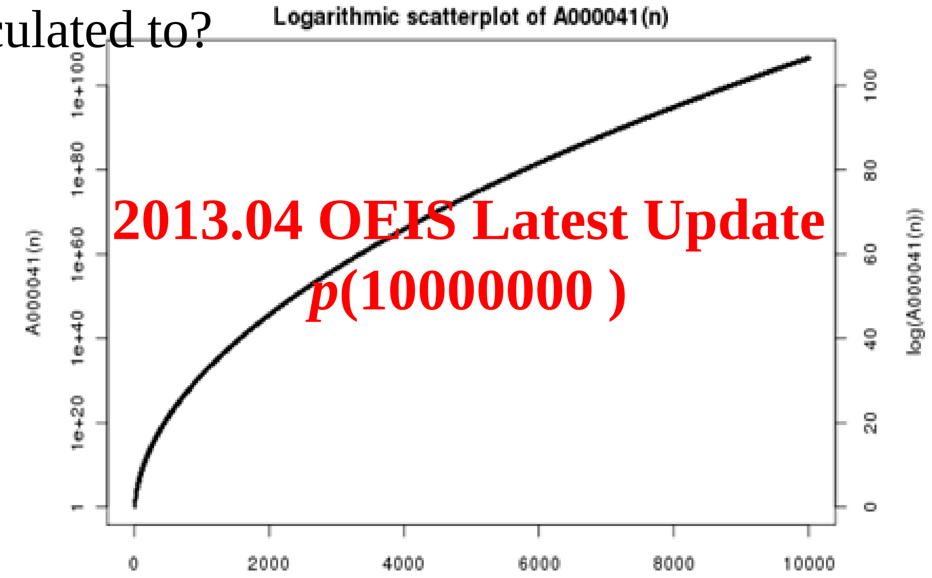
The polynomial calculation which based on integer representation can only be calculated until *p*(416)

- How large the integer partition number can be calculated to?

Algorithm for Big Number Calculation?

$$(f*g)[m]=\sum_{n} f[n]g[m-n]$$

- Related Guess： BSD Guess
  - Birch and Swinnerton-Dyer's Guess
  - 7 Big Problems of Mathematics
  - 1 million USD Awards



2013.04 OEIS Latest Update
***p*(10000000)**

Can you accurately calculate the largest integer partition number?

$$G(x)= x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9+3x^{10}+2x^{11}+x^{12}$$

In the18th Century, Euler came out with function symbol

1718 year Johann Bernoulli "x's function"

1694 year Wilhelm Leibniz

Tedious Generating Function Counting

In year 1812, Laplace raised the concept of generating function

In about year 1740, Euler was studying on integer partition

1705 years before Jakob I. Bernoulli was studying on dice

Function $$f(x)=\sum_{n=0}^{\infty} a_n x^n$$

Generating Function $$G(x)=\sum_{n=0}^{\infty} a_n x^n$$

3 Like A Function But Not A Function

3-4 Ferrers Diagram

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Ferrers Diagram

From top to bottom n level of grids, m_i is the number of grids for level i, when $m_i \geq m_{i+1}, (i = 1,2,\cdots n-1)$, where the total grid of level above is not less than the level below（weakly decreasing）， known as Ferrers diagram, as image 2-6-2。

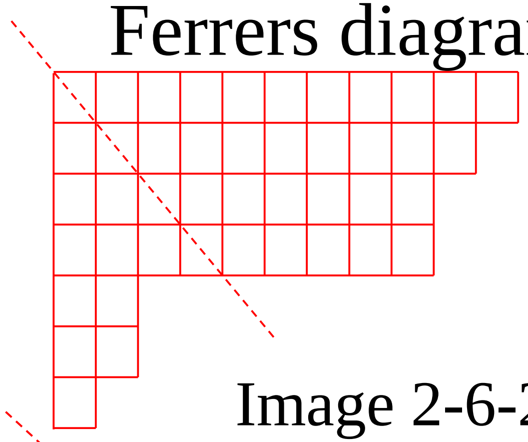
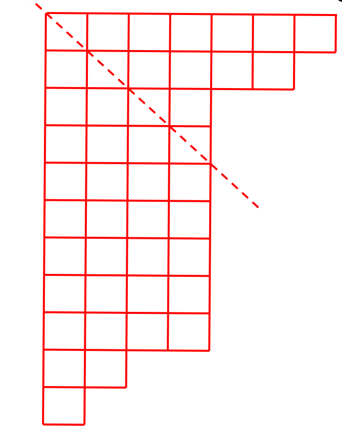


Image 2-6-2



Ferrers Diagram owns the following characteristics:

1. Each level contains at least 1 grid.
2. 1st row exchanged with 1st column, 2nd row exchanged with 2nd column, …, as image 2-6-2 is rotated by following the dotted line as axis; is still Ferrers diagram. 2 Ferrers diagram is known as a pair of conjugated Ferrers diagram.

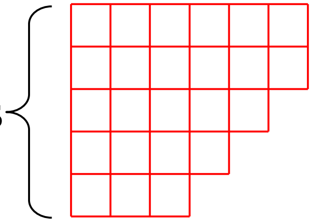
§ 2.6.3 Ferrers Diagram

Through Ferrers diagram, it managed to get a very interesting result for integer partition.

(a) the number of ways to partition n into k numbers would be the same to the number of ways to partition n with the largest number of k.

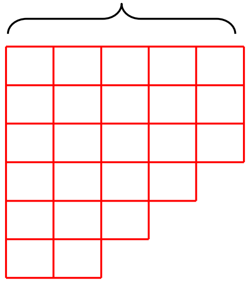
Because integer n is split into the summation of k numbers and its partition can use one k row of diagram to represent. The conjugated Ferrers diagram contains k grids on its top level.For example:

5 numbers



24=6+6+5+4+3 5 numbers， largest is 6

Largest number as 5



24=5+5+5+4+3+2 6 numbers, largest is 5

§ 2.6.3 Ferrers Diagram

(b) The partition number of integer n is split into the summation of not more than m numbers, is equaled to n is split with the partition number that is not more than m.

Reason is similar to (a).

The generating function for the partition number of partition where the summation of not more than m numbers is

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^m)}$$

The generating function of the partition number of partitioning into the summation of not more than m−1 numbers is

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^{m-1})}$$

The generating function of the partition number of the summation of exact partitioning into m numbers is

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^m)}-\frac{1}{(1-x)(1-x^2)\cdots(1-x^{m-1})}=\frac{x^m}{(1-x)(1-x^2)\cdots(1-x^m)}$$

§ 2.6.3 Ferrers Diagram

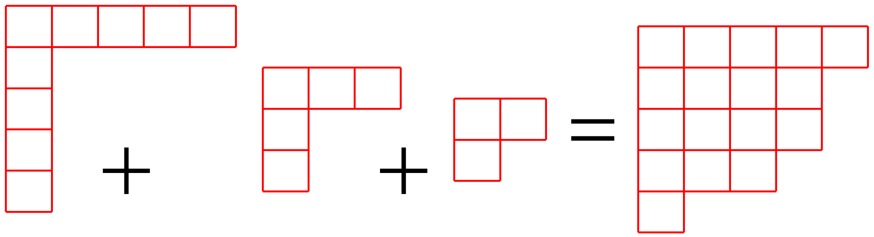
(c) The partition number of the summation of the partitioning of integer n into different odd numbers, is equaled to the partition number of n is partitioned into the self-conjugated Ferrers Diagram.

Set $n = (2n_1 + 1) + (2n_2 + 1) + \cdots + (2n_k + 1)$

where $n_1 > n_2 > \cdots > n_k$

To form a Ferrers Diagram, it's 1st row, it's 1st column is $n_1 + 1$ number of grid, corresponding to $2n_1 + 1$, the 2nd row, the 2nd column have $2n_2 + 1$ number of grids, corresponding to $n_2 + 1$, and so on. Through this, the Ferrers Diagrams are conjugated. It will look the same if it was reversed.

E.g. $17 = 9 + 5 + 3$ Corresponding Ferrers Diagram is



9+5+3=17

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3 Like A Function But Not A Function

3-5 Generating Function And Recurrence Relation

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Binomial Theorem

$$(1+x)^{-1} = 1 - x + x^2 + \cdots + (-1)^k x^k + \cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + ...$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2} x^2 + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!} x^k + \cdots$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k \qquad \alpha \in R$$

$$(a)^n = a^n$$

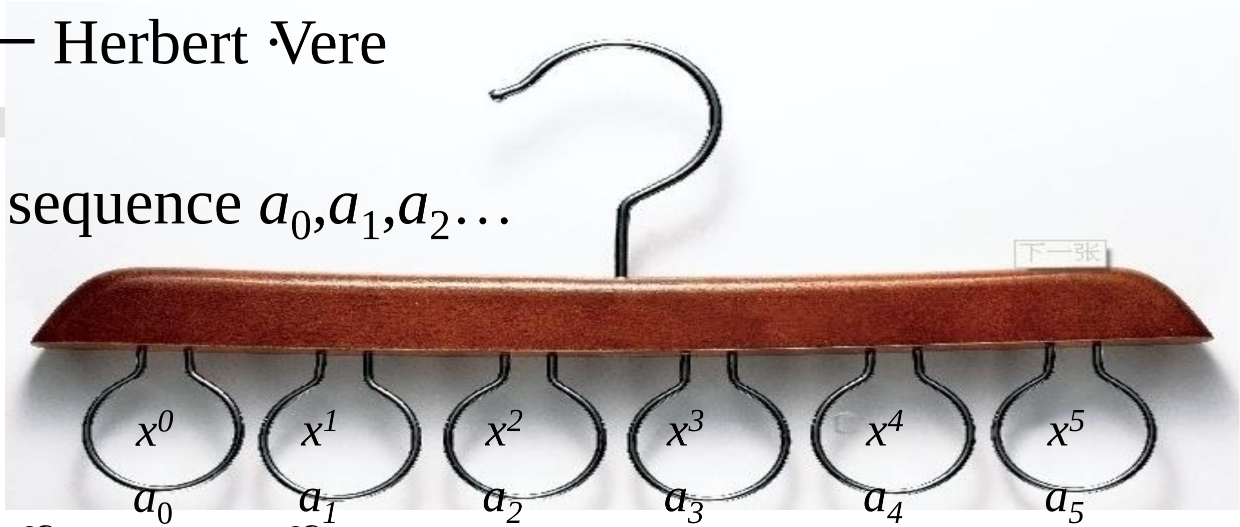
$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^{n-k} b^k$$

$$(a+b+c)^n = \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{n!}{l!(k-l)!(n-k)!} a^{n-k} b^{k-l} c^l$$

$$(a+b+c+d)^n = \sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{m=0}^{l} \frac{n!}{m!(l-m)!(k-l)!(n-k)!} a^{n-k} b^{k-l} c^{l-m} d^m$$

Generating function is a line of hangers which used to display a series of number sequences.

— Herbert ·Vere



- G(x) is the generating function for counting sequence $a_0, a_1, a_2 \ldots$
  - **G(x)=$a_0+a_1x+a_2x^2+\ldots\ldots$**

$$(1-ax)^{-1}=1+ax+a^2x^2+\ldots..$$

$$\frac{2-3x}{(1-x)(1-2x)}=\frac{1}{1-x}+\frac{1}{1-2x}=\sum_{k=0}^{\infty}x^k+\sum_{k=0}^{\infty}2^kx^k \quad =\sum_{k=0}^{\infty}(1+2^k)x^k$$

$\dfrac{2-3x}{(1-x)(1-2x)}$是数列$f(k)=2^k+1$的母函数

Generating Function		Number Sequence
$\dfrac{2-3x}{(1-x)(1-2x)}$	Partial Fraction Decomposition ⟺ $(1-ax)^{-1}=1+ax+a^2x^2+\ldots..$	$f(k)=2^k+1$
G(x)	⟺ ?	$h(k)=2h(k-1)+1$ Recurrence Relation

Recurrence Relation

Recurrence Relation： Is difference equation, which is a recursively defined the formulae for a **sequence**： Each item of the sequence is defined as the function of “**Several Former Items**”。

- E.g.Hanoi Problem： Year 1883 France Mathematician Edouard Lucas
  - When the Great Brahma created the world, he made 3 diamond pillars, there are 64 golden discs from smallest to largest, from top to the bottom in each pillar.
  - The Great Brahma ordered Brahma to move all these discs to another pillar by following smallest to largest arrangement order, starting from the bottom.
  - No disc may be placed on top of its smaller disc, among the 3 pillars, only one disc may be moved at a time.
  - When the movement is completed, it will be the time when the world is destroyed
    - Algorithm design;
    - Estimation of complexity.

摘自《从一到无穷大》

Recurrence Relation

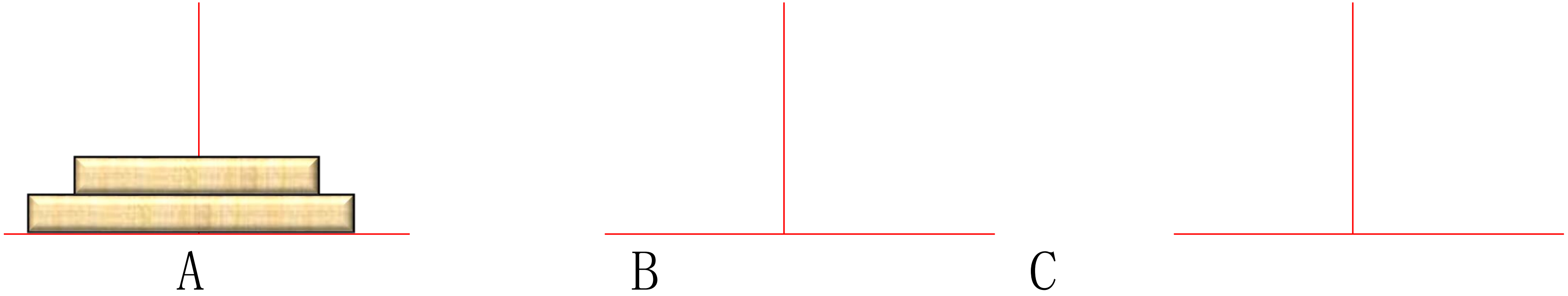
Algorithm： When N=2

1st Step: Move the top most disc to B

2nd Step: Move the bottom disc to C

Lastly, move the disc from B to C

The transmission is completed



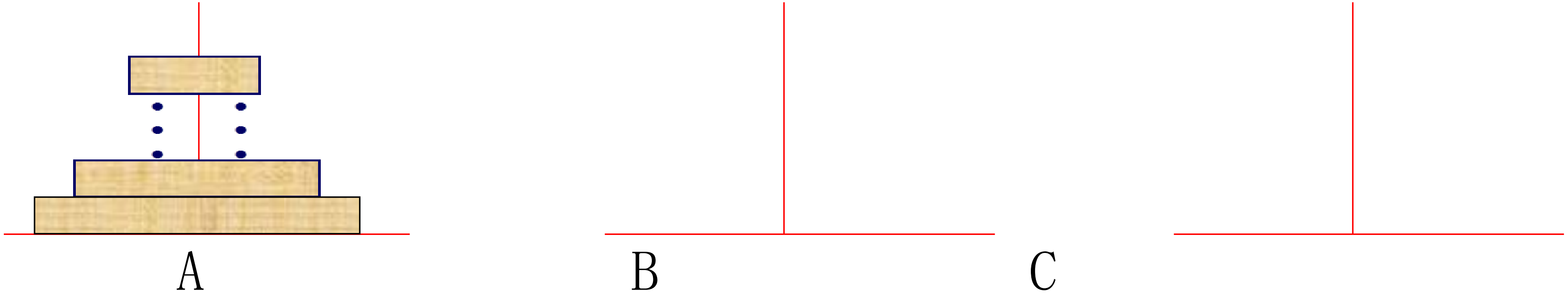
Recurrence Relation

Let $h(n)$ to represent the number of moves for n number of discs

- If the number of movements for n-1 discs is known to be with the complexity of $h(n-1)$
  - For typical problem like n number of discs, firstly, move the top n-1 of discs from C to B: $h(n-1)$
  - 2nd Step: Move the last disc from A to C: $h(1)$
  - Lastly, move n-1 number of disc from B to C through A:$h(n-1)$

Complexity of Algorithm： $h(n) = 2h(n-1)+1,\ h(1)=1$

Structure of Generating Function: $H(x) = h(1)x + h(2)x^2 + h(3)x^3 + \cdots,$



$$(1-x)^{-1}=1+x+x^2+.....$$

$$h(n)=2h(n-1)+1,\ \ h(1)=1$$

$h(0)=0$

Recurrence Relation

If these exponent value is performing 4 arithmetic operations, it is same like the finite algebra expression.

$$H(x)=h(1)x+h(2)x^2+h(3)x^3+\cdots,$$

$$+)\ \ -2xH(x)=\qquad -2h(1)x^2-2h(2)x^3+\cdots,$$

$$(1-2x)H(x)=h(1)x+[h(2)-2h(1)]x^2$$

$$+[h(3)-2h(2)]x^3+\cdots$$

$$\because\ \ h(1)=1, h(2)-2h(1)=1, h(3)-2h(2)=1,\cdots$$

$$\therefore\ \ (1-2x)H(x)=x+x^2+x^3+\cdots=x/(1-x)$$

$$\therefore\ \ H(x)=\frac{x}{(1-2x)(1-x)}$$

$$h(n) = 2h(n-1) + 1, \quad h(1) = 1$$

$h(0)=0$

Recurrence Relation

$$H(x) = h(1)x + h(2)x^2 + h(3)x^3 + \cdots,$$

Apply Recurrence Relation $x^2 : h(2) = 2h(1) + 1$

$$x^3 : h(3) = 2h(2) + 1$$

+) ....

Left side:

$$h(2)x^2 + h(3)x^3 + \cdots = H(x) - h(1)x = H(x) - x$$

1st term on the right side:

$$2h(1)x^2 + 2h(2)x^3 + \cdots = 2x[h(1)x + h(2)x^2 + \cdots]$$

$$= 2xH(x)$$

2nd term on the right side:

$$x^2 + x^3 + \cdots = x^2/(1-x)$$

$$\therefore \quad H(x) - x = 2xH(x) + x^2/(1-x)$$

$$H(x) = \frac{x}{(1-x)(1-2x)}$$

$$H(x)=\sum_{k=1}^{\infty}h(k)x^k=\frac{x}{(1-x)(1-2x)}$$

$$h(64)=18446744073709551615$$

How to find the number sequences based on generating function? $h(1),h(2),\cdots$

Transformed into partial fractional algorithm.

$$H(x)=\frac{A}{1-x}+\frac{B}{1-2x}=\frac{A(1-2x)+B(1-x)}{(1-x)(1-2x)}$$

$$=\frac{(A+B)-(2A+B)x}{(1-x)(1-2x)}$$

$$\therefore\quad (A+B)-(2A+B)x=x$$

From the above equation:

$$\begin{cases} A+B=0 \\ -2A-B=1 \end{cases}\Longrightarrow A=-1,\quad B=1.$$

Be: $H(x)=\frac{1}{1-2x}-\frac{1}{1-x}$

$$=(1+2x+2^2x^2+2^3x^3+\cdots)-(1+x+x^2+\cdots)$$

$$=(2-1)x+(2^2-1)x^2+(2^3-1)x^3+\cdots$$

$$=\sum_{k=1}^{\infty}(2^k-1)x^k$$

$$\therefore\quad h(k)=2^k-1$$

E.g. 2. Find the total occurrence of even number of 5 in n length of decimal number

Start with analysing the structure for occurrence of even number of 5 in n length of decimal numbers

Set $p_1 p_2 \cdots p_{n-1}$ as n−1 length of decimal number，

If $p_1p_2\ldots p_{n-1}$ contains even number of 5, then p_n get anything other than 5 like 0，1，2，3，4，6，7，8，9, one of the nine numbers，

If $p_1p_2\ldots p_{n-1}$ contains odd number of 5 , then p_n get 5, make $p_1p_2\ldots p_{n-1}p_n$ as the decimal number for the occurred even number of 5.

Solution 1：Let a_n as the total occurrence of even number of 5 in n length of decimal number, b_n as the total occurrence of odd number of 5 in n length of decimal number.

$$\begin{cases} a_n = 9a_{n-1} + b_{n-1} \\ b_n = 9b_{n-1} + a_{n-1} \end{cases}$$

Set the generating function of sequence $\{a_n\}$ as A(x)，generating function of $\{b_n\}$as B(x)。

$$\begin{cases} a_n = 9a_{n-1} + b_{n-1} \\ b_n = 9b_{n-1} + a_{n-1} \end{cases} \qquad a_1=8,\ b_1=1$$

$$\begin{aligned} A(x) &= a_1 + a_2 x + a_3 x^2 + \cdots \\ -9xA(x) &= \qquad -9a_1 x - 9a_2 x^2 - \cdots \\ +)\quad -xB(x) &= \qquad -b_1 x - b_2 x^2 - \cdots \end{aligned}$$

$$(1-9x)A(x) - xB(x) = a_1 = 8$$

$$\begin{aligned} x &: \quad a_2 = 9a_1 + b_1 \\ x^2 &: \quad a_3 = 9a_2 + b_2 \\ x^3 &: \quad a_4 = 9a_3 + b_3 \\ +) & \quad \cdots\cdots \end{aligned}$$

$$A(x) - 8 = 9xA(x) + xB(x)$$

$$\therefore \quad (1-9x)A(x) - xB(x) = 8$$

$$\begin{aligned} B(x) &= b_1 + b_2 x + b_3 x^2 + \cdots \\ -9xB(x) &= \qquad -9b_1 x - 9b_2 x^2 - \cdots \\ +)\quad -xA(x) &= \qquad -a_1 x - a_2 x^2 - \cdots \end{aligned}$$

$$(1-9x)B(x) - xA(x) = 1$$

Recurrence Relation

Hence, it will get generating function of A(x) and B(x) formula sets：

$$\begin{cases}(1-9x)A(x)-xB(x)=8\\ -xA(x)+(1-9x)B(x)=1\end{cases}$$

$$\therefore \quad D=\begin{vmatrix}1-9x & -x\\ -x & 1-9x\end{vmatrix}=(1-9x)^2-x^2=1-18x+80x^2$$

$$=(1-8x)(1-10x)$$

$$A(x)=\frac{1}{1-18x+80x^2}\begin{vmatrix}8 & -x\\ 1 & 1-9x\end{vmatrix}=\frac{-71x+8}{(1-8x)(1-10x)}$$

$$B(x)=\frac{1}{(1-8x)(1-10x)}\begin{vmatrix}1-9x & 8\\ -x & 1\end{vmatrix}=\frac{1-x}{(1-8x)(1-10x)}$$

$$\therefore \quad A(x)=\frac{1}{2}\left(\frac{7}{1-8x}+\frac{9}{1-10x}\right)=\frac{1}{2}\sum_{k=0}^{\infty}(7\cdot 8^k+9\cdot 10^k)x^k$$

$$A(x)=a_1+a_2x+a_3x^2+\cdots \qquad \therefore \quad a_k=\frac{7}{2}8^{k-1}+\frac{9}{2}10^{k-1}$$

Recurrence Relation

Solution 2： n-1 length of decimal number contains $9\times 10^{n-1}$ (the 1st position cannot be 0)， set the desired value as a_n, set $A(x)= a_1x+a_2x^2+\ldots$, categorized by the last digit if it was a 5：

Last digit is not 5： $9a_{n-1}$

Last digit is 5, previous n-1 positions contains odd number of 5:

$$b_{n-1}=9\times 10^{n-2}-a_{n-1}$$

$$a_n=9a_{n-1}+9\times 10^{n-2}-a_{n-1}$$

$$\therefore \quad a_n=8a_{n-1}+9\times 10^{n-2}, \qquad a_1=8$$

$$\begin{aligned} x^2 &: \ a_2=8a_1+9 \\ x^3 &: \ a_3=8a_2+90 \\ x^4 &: \ a_4=8a_3+900 \\ +) & \quad \cdots\cdots\cdots\cdots \end{aligned}$$

$$A(x)-a_1x=8xA(x)+9x^2(1+10x+10^2x^2\ldots\ldots)$$

Recurrence Relation

$$\therefore \quad (1-8x)A(x) = 8x + \frac{9x^2}{1-10x}$$

$$\therefore \quad A(x) = \frac{x(8-71x)}{(1-8x)(1-10x)} = \frac{1}{2}\left(\frac{7x}{1-8x} + \frac{9x}{1-10x}\right)$$

$$= \frac{1}{2}\sum_{k=1}^{\infty}(7\cdot 8^{k-1} + 9\cdot 10^{k-1})x^k$$

$$\therefore \quad a_k = \frac{7}{2}\cdot 8^{k-1} + \frac{9}{2}\cdot 10^{k-1}$$

Verification： a_1=8, a_2=73

Conclusion

- $\mathbf{G}(x)=a_0+a_1x+a_2x^2+\ldots\ldots$
- From **G(*x*)** obtains sequence **$\{a_n\}$.** The key is over the bridge between sequence to generating function, and between generating function to sequence.

$$x^2 : h(2) = 2h(1)+1$$
$$x^3 : h(3) = 2h(2)+1$$
$$+)\qquad \cdots\cdots\cdots\cdots$$

$$H(x)=\sum_{k=1}^{\infty} h(k)x^k = \frac{x}{(1-x)(1-2x)}$$
$$= \frac{1}{1-2x} - \frac{1}{1-x} = \sum_{k=1}^{\infty}(2^k-1)x^k$$

$$(1-ax)^{-1} = 1+ax+a^2x^2+\ldots..$$

Itemize representation of rational fraction
The denominator coefficients contain any special meaning?

The suitability of generating function method towards recurrence relation?