组合数学 Combinatorics

2 Journey of a Pingpang ball

2-1 Fundamental Counting Principals

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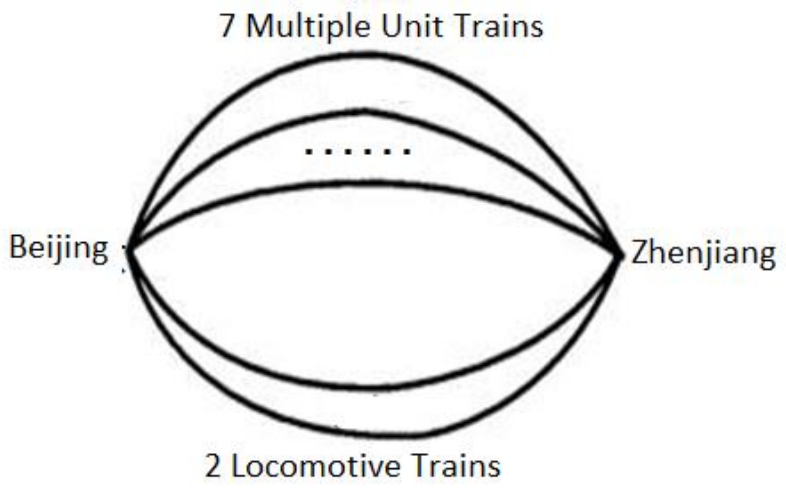
Table Tennis

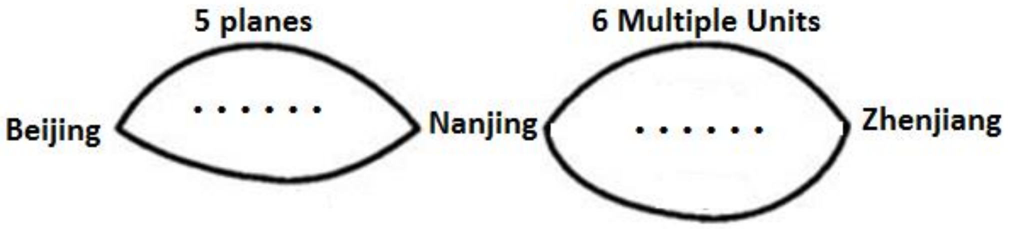
- Table Tennis
  - The 52th table tennis world championship was held in Tokyo during April 28th, 2014 and May 5th, 2014.
  - In 2014, the Chinese National Team Trial started in Jan 25th , 2014 in Zhenjiang.

Journey Planning

- Beijing-Zhenjiang
  - High Speed Trains: 7； Other trains: 2
  - 7+2=9 different journeys.
  - **Classification: Addition**



- Beijing-Zhenjiang
  - 5 planes from Beijing to Nanjing in the morning, 6 high speed trains from Nanjing to Zhenjiang in the afternoon
  - 5×6=30 different journeys
  - **Segmentation: Multiplication**



Sum Rule & Multiplication Rule

[The Addition Rule] Assume that event A can happen in m ways, event B can happen in n ways. Event A **or** B can happen in $m + n$ ways.

Language of Set Theory：

If $|A| = m$, $|B| = n$, $A \cap B = \varnothing$, Then $|A \cup B| = m + n$ 。

25Male Students, 5 Female Students, Total: 25+5 = 30 。

Non-art class

1. 1 Addition Rule and Multiplication Rule

[The Multiplication Rule]Assume that event A can happen in *m* ways, event B can happen in *n* ways, so event A **and** B can happen in *m**n* ways.

Language of Set Theory：

If $|A| = m$, $|B| = n$, $A\times B = \{(a,b) \mid a \in A, b \in B\}$,

Then $|A \times B| = m \times n$ 。

In the non-art class, if we pick one male monitor and one female class leader, how many arrangements are there?

Step 1，Pick the monitor，25 choices $25 \times 5 = 125$

Step 2，Pick the class leader, 5 choices

1.1 Addition Rule and Multiplication Rule

The Addition Rule: Classification

The Multiplication Rule: Segmentation

Be cautious that when using the Addition Rule and the Multiplication Rule, the events need to be **independent**.

In the class there's a male student "GFS", a female student "BFM".

They are siblings

If we pick one male monitor and one female class leader, how many arrangements are there?

1. 1 Addition Rule and Multiplication Rule

Classification

1. GFS to be the monitor: 4 females to choose $1 \times 4=4$
2. GFS not the monitor: 24 males to choose and 5 females to choose. $4+24 \times 5=124$

1. 1 Addition Rule and Multiplication Rule

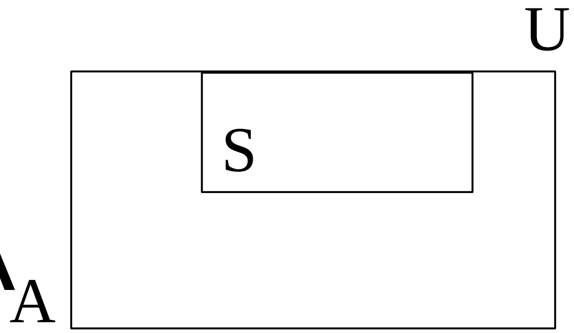
Simpler Solutions？

Legal arrangements requires classification, but there’s only 1 illegal arrangement

So, total number of arrangements – illegal ones

$$= 125 - 1 = 124$$

- Subtraction Rule：A is the set of solutions，U is the universal set
  - Define the complement of A in U : $\bar{A}$ = $U \setminus A = \{x \in U : x \notin A\}$
  - Calculate the size of A as:
  - $|A| = |U| - |\bar{A}|$



1. 1 Addition Rule and Multiplication Rule

Division Rule：

Divide 25 male students to 5 groups, in each group:

$25/5 = 5$ students

We've introduced the 4 basic counting principles

组合数学 Combinatorics

2 Journey of a Pingpang ball

2-2 Definition of Permutation and combination

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Would you like to solve the problems on the blackboard?

Put pingpang balls to boxes

- Permutation and Combination
- Indexed pingpang balls：
  - 4 pingpang balls：#1, #2, #3, #4
  - Pick 3 of them
  - If consider the order ⇒ Permutation P(4,3)
    - $P(4,3)=4\times 3\times 2 = 24$

    Permutation without repetitions
  - If not **consider** the order ⇒ Combination C(4,3)
    - $C(4,3)=24/3!\ =4$

    Combination without repetitions

Permutation and Combination

Def [Permutation]Pick r non-repeated elements from n and order them. This is called the r-permutation without repetition of n. It's denoted by $P(n,r)$ or P_n^r. When r=n, it's the permutation of n. We can omit "without repetition"($n \geq r$)

Def[Combination]Pick r non-repeated elements from n and ignore their orders. It's called the r-combination without repetition of n.

We use $C(n,r)$ or C_n^r to denote r-combination of n.

Models of Permutation

[Permutation] A classic model for r-permutation of n is to pick *r* balls from *n* and put them into *r* different boxes, 1 per each.($n \geq r$)

•There are *n* choices for the 1st box, *n-1* choices for the 2nd box, ⋯⋯, *n-r+1* choices for the *r*th box.

So we have $P(n,r)=n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$

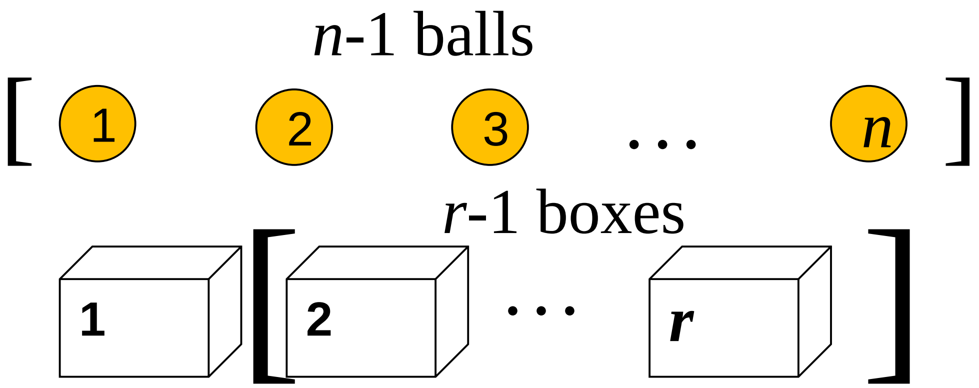
Sometimes we use $[n]_r$ to denote $n(n-1)\cdots(n-r+1)$

Permutation of n： $P(n,n) = n!$

Recurrence Relation of $P(n, r)$

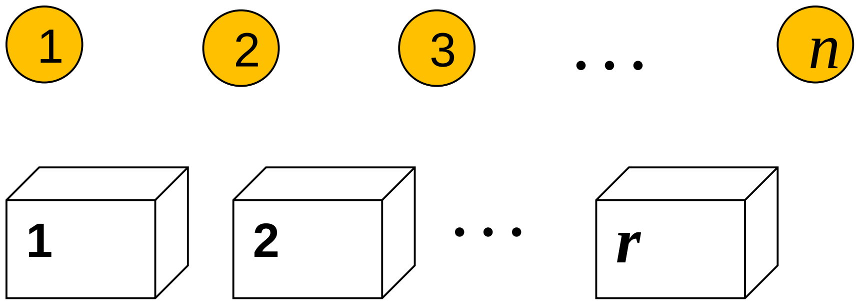
$\boxed{P(n,r) = nP(n-1,r-1)}$

- Recurrence by Segmentation
  - Choose the ball for box #1.
    - ***N choices***
  - Pick *r-1* balls from *n-1* balls into *r-1* boxes
    - $\boldsymbol{P(n\text{-}1,r\text{-}1)}$



$\boxed{P(n,r) = P(n-1,r) + rP(n-1,r-1)}$

- Recurrence by Classification
  - Not choose the first ball？
    - $\boldsymbol{P(n\text{-}1,r)}$
  - Choose the first ball？
    - $\boldsymbol{rP(n\text{-}1,r\text{-}1)}$



Model for Combination

If the balls are different and boxes are the same, so it's r-combination of n.

If we differentiate the boxes after doing combination, it's converted to a permutation problem. Each combination has *r!* indexing ways.

So we have $C(n,r)\, r! = P(n,r) = \dfrac{n!}{(n-r)!}$ $\quad \boldsymbol{C(n,r) = \dfrac{n!}{r!(n-r)!}}$

$C(n,r)=C(n,n-r)$

The number of ways to choose *r* balls from *n* is equal to the number of ways to choose the remaining *n-r* balls.

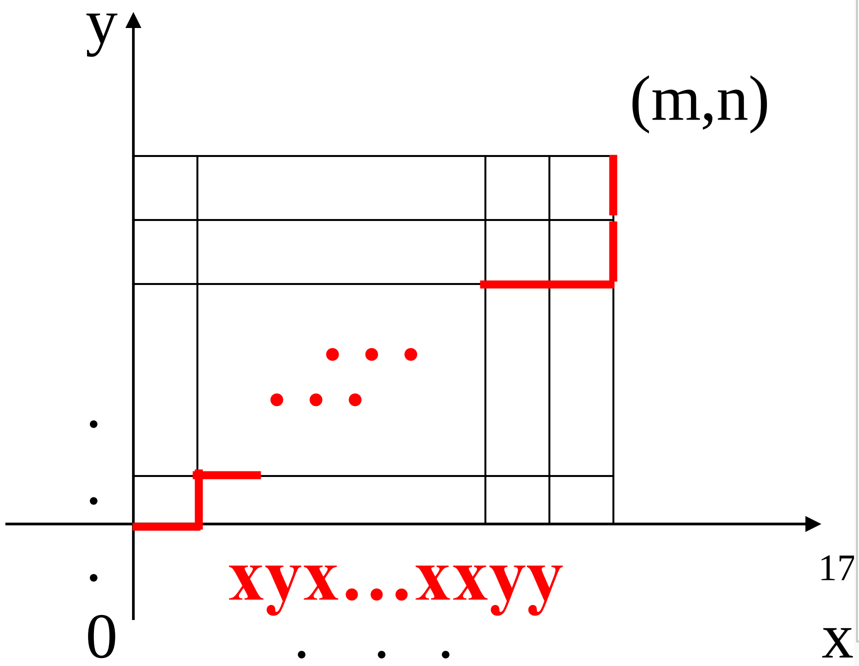
$\mathbf{C}(n,l)C(l,r)=C(n,r)C(n-r,l-r)$

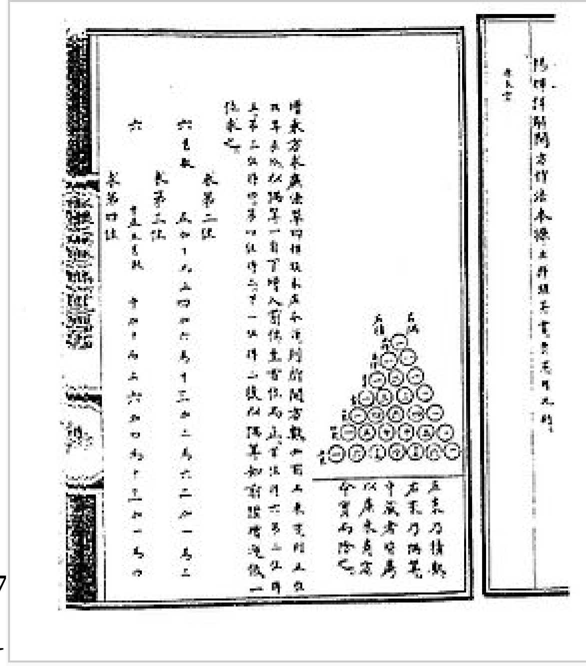
The non-art class has *n* students, elect *l* committee members and then elect *r* core members from them.

Elect r core members at first and then elect the other *l-r* members.

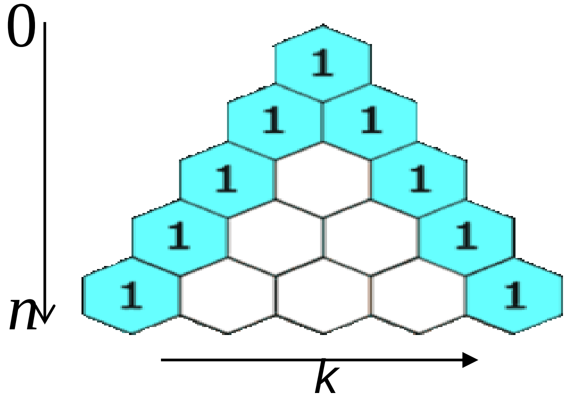
Models of Combination

- Lattice - path：
  - Walk along the positive directions of x-axis or y-axis from (0,0) to (m,n), 1 unit per step, there are $C(m+n,n)$ routes.





永乐大典一页
杨辉引用贾宪《释锁算书》中的贾宪三角形



The n^{th} row, k^{th} column : $C(n,k)$

Coefficients of $(a+b)^n$

Combinations and binomial coefficients?

Binomial Theorem

- $(a+b)^n = \mathrm{C}(n,0)a^n + \mathrm{C}(n,1)a^{n-1}b + \ldots + \mathrm{C}(n,n)b^n$
- (a+b)(a+b)…(a+b) n $(a+b)$’s

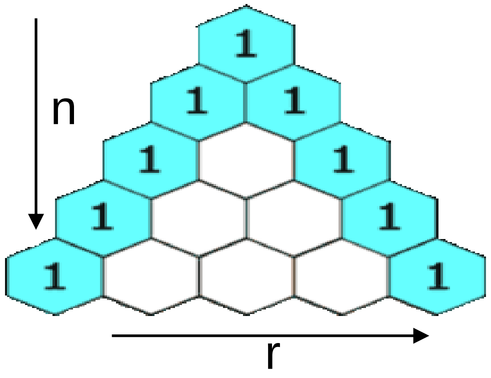
  a b … a

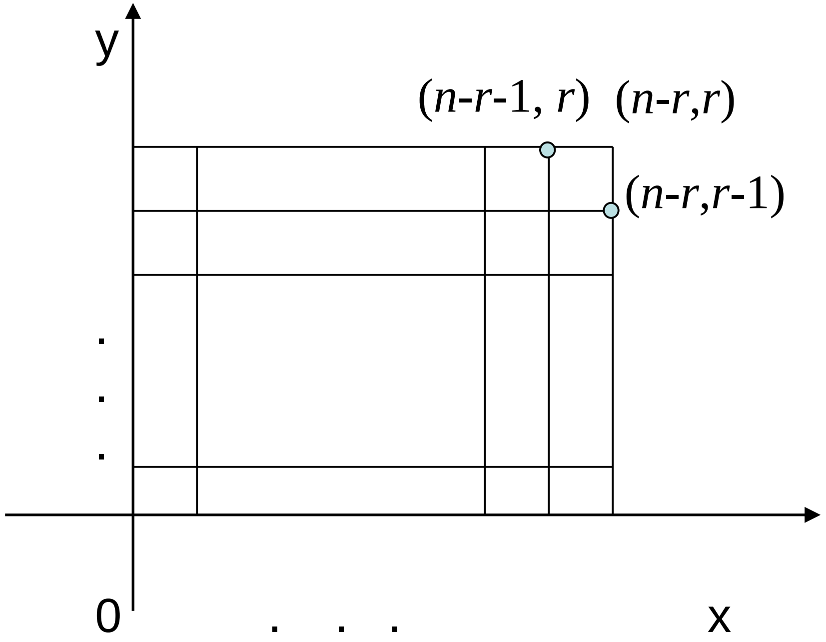
  Pick r b’s from n positions to form $a^{n-r}b^r$

  The number of $a^{n-r}b^r$’s is $C(n,r)$
- If $a=b=1$, then
  - $2^n = C(n,0) + C(n,1) + \ldots + C(n,n)$
- If $a=1$, $b=-1$, then
  - $0 = C(n,0) - C(n,1) + \ldots \pm C(n,n)$

Combinatorial Identities

- $C(n, r)=C(n\text{-}1,r)+C(n\text{-}1,r\text{-}1)$





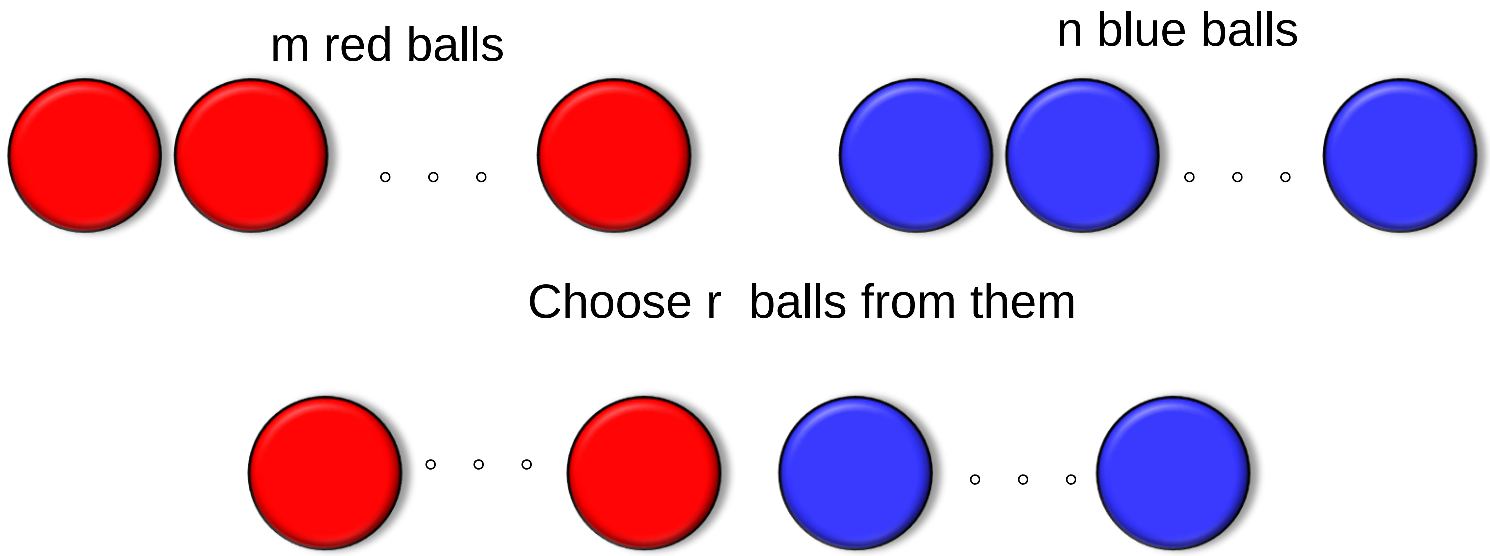
- Left hand side: All lattice paths from (0,0) to $(n\text{-}r,r)$
- Right hand side:
  - (0,0) to $(n\text{-}r\text{-}1,r)$
  - (0,0) to $(n\text{-}r,r\text{-}1)$

Identities

- **C**(m+n, r)=C(m,0)C(n, r)+C(m, 1)C(n,r-1)+ …+C(m, r)C(n, 0)
- **Vandermonde's identity**
- Alexandre-Théophile Vandermonde (28 February 1735 – 1 January 1796, **French musician and mathematician**
- Donald E. Knuth mentioned that a form of this identity has been introduced in Zhu Shijie’s *Jade Mirror of the Four Unknowns* (1303) in *The Art of Computer Programming Vol.ii* (1998).
- **Chu-** Vandermonde Identity

Prove identities by pingpang balls

- $\mathbf{C}(m+n,r)=C(m,0)C(n,r)+C(m,1)C(n,r-1)+\ldots+C(m,r)C(n,0)$
- $C(m,0)C(n,r)+C(m,1)C(n,r-1)+\ldots+C(m,r)C(n,0)$



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2 Journal of a Pingpang ball

2-3 More on pingpang balls

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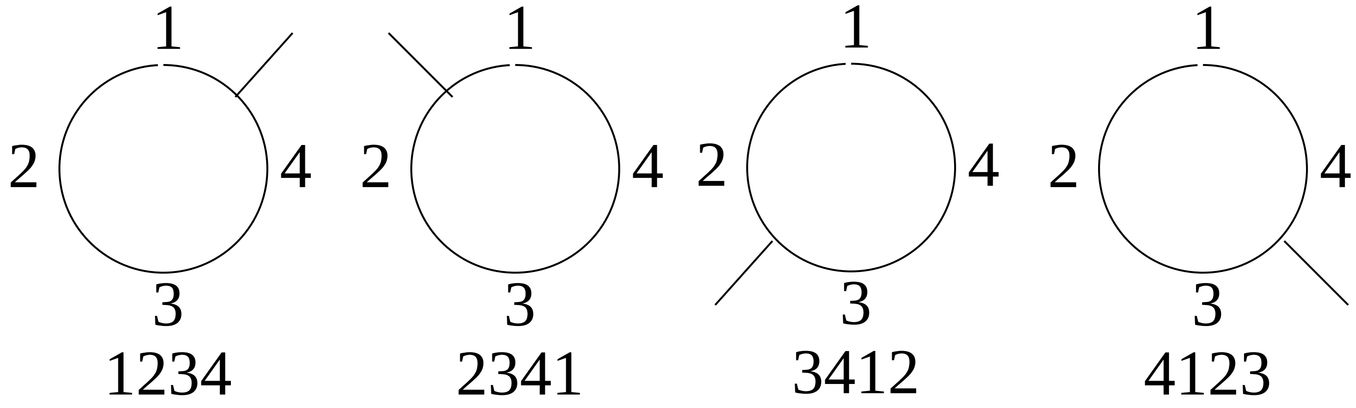
The origin of permutation and combination

- The problems of permutation and combinations firstly appeared in *I Ching*.
  - “Sixiang” is the permutation of two yao’s. “Bagua” is the permutation of 3 yao’s.
- Han Dynasty mathematician Xu Yue’s *Shu Shu Ji Yi* (2nd century) mentioned a Bagua used for divination.
  - It’s similar to the classic problem “8 people are sitting around a table, how many arrangements are there?”

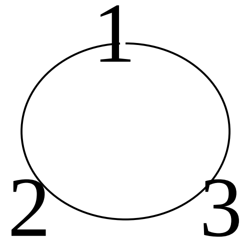
Circular Permutation

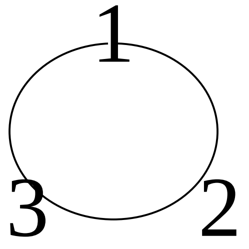
- Circular Permutation：The number of *r*-circular permutations of *n* is $P(n,r)/r$, $2 \le r \le n$
- Example: 4 elements



Necklace

- Necklace: Similar to real necklaces. Based on circular permutation. Here two sides of the necklace are considered a same permutation.
- **Eg.** The next two arrangements are actually the same permutation of 3 elements.
- The number of necklace permutation of picking r elements from n is :$P(n, r)/2r$, $3 \leq r \leq n$





Permutation with repetitions

Eg How many 4-strings could be formed with the 26 English letters?

26^4

Eg How many non-repeated 4-strings could be formed with 26 English letters？

P(26,4)

Eg How many non-repeating 4-strings such that *b* and *d* are not adjacent can be formed with 26 English letters?

P(26,4) －C(24,2)*3!*2

Permutations of multisets

- Permutations of multisets：
- How many permutations of "*pingpang*" are there？
  - 2 p's, 2 n's， 2 g's， 1 I's， 1 a's，
  - The permutations are denoted by $\binom{8}{2\quad 2\quad 2\quad 1\quad 1}$
  - Differentiate it with subscripts $p_1p_2n_1n_2g_1g_2ia$
    - There are 2! arrangements for labes of p,n,g

$$\binom{8}{2\quad 2\quad 2\quad 1\quad 1}2!2!2! = 8! \qquad \binom{8}{2\quad 2\quad 2\quad 1\quad 1} = \frac{8!}{2!2!2!}$$

Polynomial Expansion

- Calculate the permutation of r_1 1's， r_2 2's, … ,r_t t's,设 $r_1+r_2+\ldots+r_t=n$, denote the permutation as $P(n;r_1,r_2,\ldots,r_t)$
- Index 1,2,… ,t separately, get $\begin{pmatrix} & n & & \\ r_1 & r_2 & \ldots & r_t \end{pmatrix}$

$$P(n;r_1,r_2,\ldots,r_t)\cdot r_1!\cdot r_2!\cdot\ldots\cdot r_t! = n!$$

- $\therefore P(n;r_1,r_2,\ldots,r_t) = \dfrac{n!}{r_1!r_2!\ldots.r_t!}$
- Polynomial expansion

$$(a+b)^n = \sum_{0\le k\le n} \frac{n!}{k!(n-k)!} a^k b^{n-k} = \sum_{0\le k\le n} C(n,k) a^k b^{n-k}$$

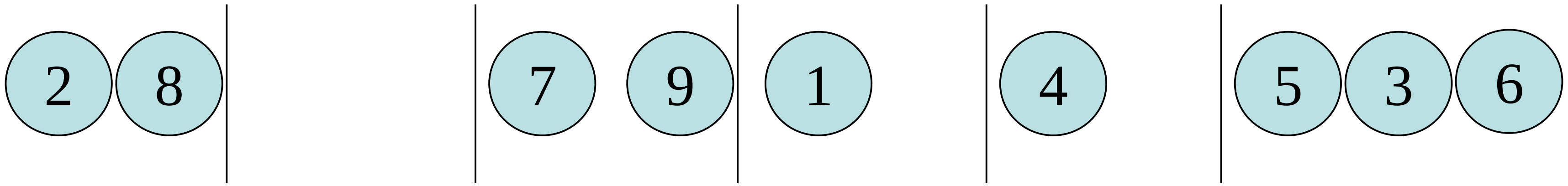
$$(a_1+a_2+\ldots+a_t)^n = \sum \frac{n!}{r_1!\ldots r_t!} a_1^{r1}\ldots a_t^{rt} \qquad \sum r_i = n$$

Permutations of multisets

Eg There are 6 holes and each time only one pingpang ball could be sent. How many arrangements are there to sent 9 balls into the 6 holes?

[Solution]Denote an arrangement as: XX11 XX 1X1X1XXX

In which a “X” denote a ball and a “1” is a bar. “X”s are different and “1”s are the same. Any arrangements to send the balls could be denoted as such a permutation.



Permutations of Multisets

XX11 XX 1X1X1XXX

[Solution 1]Indexing the bars could generate 14-permutations for each arrangement. Assume that x is the number of arrangements, we have:

$$x \cdot 5! = 14!$$

$$\therefore x = 14!/5! = 726485760$$

[Solution 2]We decide the positions for "1"s among the 14 elements at first. It's C(14,5). Then we decide the positions of the balls, there are 9! Choices.

So C(14,5) · 9! is the answer

1.2Permutation and Combination

[Solution 3]Dividing the process to several steps for the convenience of calculating.

•6 choices for #1；

•Except all choices of #1, #2 also need to choose whether to be in the front of #1 or not. So there are 7 choices for #2;

•Similar to #2, there are 8 choices for #3;

• ……

•So forth, there are 14 choices for #9.

So there are 6*7*8*….*14 =14!/5!=726485760 arrangements

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2-4 Combinations

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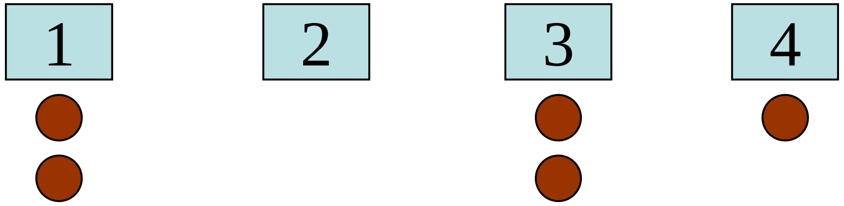
Preparing a fruit compote

Apples	Pears
1	3
2	2
3	1
4	0

- Choose 4 fruits from apples and pears to make a compote.
- Multisets: elements are repeatable. t_i=0, 1, …∞ denotes the maximum number of element a_i. A multiset of n elements could be denoted by $\{t_1 a_1, t_2 a_2 \ldots, t_n a_n\}$.
- $t_i = \infty$ Multisets
- **Combinations of multisets**： Pick r elements from $\overline{A} = \{1,2,3 \cdots n\}$:$\{a_1, a_2, \ldots a_r\}$, $a_i \in A$， i=1,2,..r, and $a_i = a_j$, $i \neq j$ are allowed. 记为C(n,r)。

Combinations of Multisets

- Take 5 elements from $A=\{1,2,3,4\}$, elements are repeatable.
- **Combinations of Multisets**：


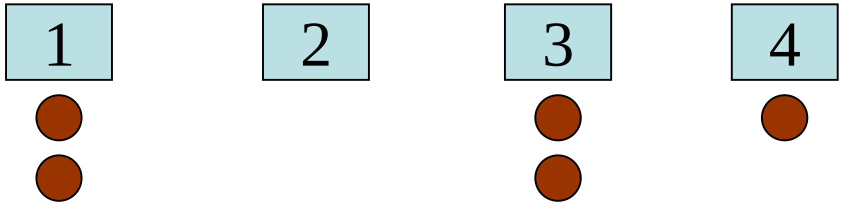
- {1 1 3 3 4}
- Model for combinations of multisets： take *r* **non-labeled** balls, *n* **different** boxes. In each box there could be **0 or more then 1 balls.**
- Model for combinations without repetitions： *n* balls are **different**, *r* boxes are **the same**, put *r* balls into boxes, **1 ball** per box.


Combinations of Multisets

Construct a related combination without repetition

$$C(8,5)\quad \begin{array}{ccccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \{1, & 1, & 3, & 3, & 4\} \\ \{\mathbf{1}, & \mathbf{2}, & \mathbf{5}, & \mathbf{6}, & \mathbf{8}\} \end{array}$$

(1) …… (4+5-1)



Are the r numbers non-repeated?

$\mathbf{0}$ $\mathbf{1}$ $\boldsymbol{r}\mathbf{-1}$

$\{a_1, a_2 \ldots \ldots a_r\}$ $1 \le a_1 \le a_2 \le \ldots \le a_r \le n$

r numbers $a_1, a_2+1, a_3+2 \ldots\ldots a_k+k-1, \ldots a_r+r-1$

(1) …… (n+r-1)

Proof by contradiction:

Assume $\exists i<j$, $a_i+i-1=a_j+j-1$,

Then $a_i>a_j$,

But when $i<j$, $a_i \le a_j$,

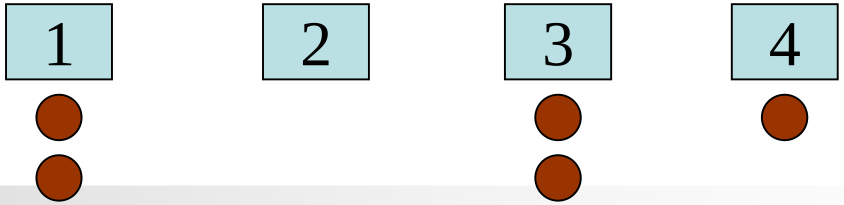
Contradiction ⇒ non-repeated

When elements are repeatable, there are C(n+r-1,r) combinations to pick r elements from n.

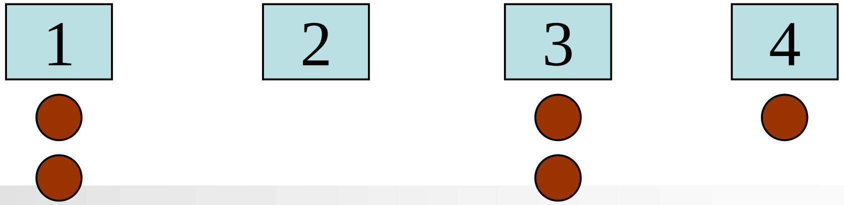
0 1 2 3 4

{1, 1, 3, 3, 4}

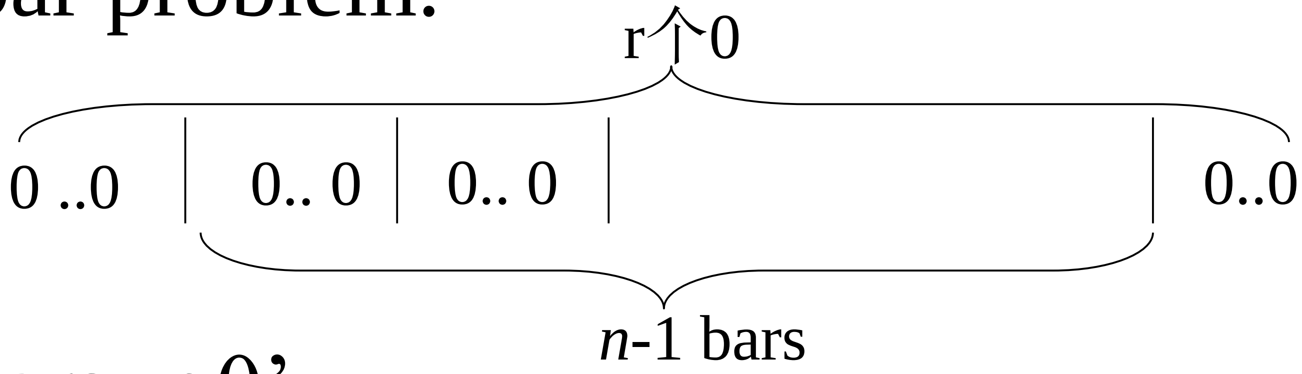
{1, 2, 5, 6, 8}



- **Pick *r* from *n* different elements, if it's repeatable, the number of combinations is C(n+r-1,r)**
- **Proof1:** Construct a bi-jection to C(n+r-1,r)
- Assume the *n* elements are{1,2…*n*}, we get a repeatable combination $\{a_1,a_2..a_r\}$, $a_1 \le a_2 \le \ldots \le a_r$
- Construct $a_1,a_2+1,a_3+2\ldots..a_k+k-1,\ldots a_r+r-1$,
- Prove that the numbers in this sequence are different:
  - Assume ∃ $i<j$, $a_i+i-1=a_j+j-1$, then $a_i>a_j$, but when $i<j$, $a_i<a_j$,, contradiction, construct a sequence with r numbers ∈ $[i,n+r-1]$
- Pick *r* numbers among the *n*+*r*-1 numbers in the constructed sequence **without repetition**.
- Conversely, we could use each combination of picking *r* numbers from *n*+*r*-*1* ones to form a combination of multisets. So there's a bi-jection relation between the constructed sequence and the original sequence.
- The constructed sequence is from 1 to *n*+*r*-*1*. So it's equal to picking *r* numbers from *n*+*r*-*1* ones. So combinations of multisets are C(*n*+*r*-1,*r*)



- **Picking *r* elements from *n* different ones, if repeatable, the number of combinations is C(*n*+*r*-1,*r*)**
- **Proof 2**： Convert it to the bar problem.



- Total: $n+r-1$ elements, $n-1$ bars, r 0's
- If both bars and 0's are indexed, it's a (n+r-1)-permutation.
- So the result is $\dfrac{(n+r-1)!}{r!(n-1)!} = C(n+r-1, r)$

Integer roots of linear equations

- $x_1+x_2\ldots+x_n=b$ has C($n+b$-1,b) non-negative integer roots.

$$\overbrace{\underset{x1\text{ 1's}}{1\ ..1}\ \underbrace{\ |\ 1..\ 1\ |\ 1..\ 1\ |\qquad\qquad |\ }_{\text{n-1 bars}}\ 1..1}^{\text{b 1's}}$$

- Number of roots: C($n+b$-1,b)

1.8 Non-adjacent combination

- Non-adjacent combination is to pick r non-adjacent numbers from $A=\{1,2,\ldots n\}$(**Not repeatable**), such that adjacent numbers j, j+1 don't appear together.
- Eg. The non-adjacent combinations of n=6,r=3 include:
- {1 3 5}{2 4 6}
- Pick r non-adjacent numbers from A={1,2,…n}中, the number of combination is C(n-r+1,r)

$$\begin{array}{ll} & \mathbf{0} \quad \mathbf{1} \quad \mathbf{2} \quad \mathbf{3} \quad \mathbf{4} \\ \text{B:} & \{1,\ 3,\ 5,\ 7,\ 9\} \\ \text{C:} & \{1,\ 2,\ 3,\ 4,\ 5\} \end{array}$$

1.8 Non-adjacent combination

- **Take *r* non-adjacent numbers from $A=\{1,2,\ldots n\}$, the number of combinations is C(n-r+1,r)**
- **Proof:** Assume $B=\{b_1,b_2\ldots b_r\}$ is a set of non-adjacent combinations
- Suppose $b_1<b_2\ldots.<b_r$, let $c_1=b_1$, $c_2 = b_2-1,\ldots c_r=b_r-r+1\le n-r+1$, then $c_1<c_2\ldots<c_r$,$\{c_1,c_2\ldots c_r\}$ is the combination without repetition of picking r from $\{1,2,\ldots n-r+1\}$

$$\begin{array}{ll} & \color{red}{\mathbf{0 \quad 1 \ldots \quad r\text{-}1}} \\ & \{b_1, b_2 \ldots \ldots b_r\} \quad 1\le b_1<b_2<\ldots < b_r \le n \\ r \text{ numbers} & b_1, b_2-1, b_3-2\ldots..b_k-k+1,\ldots b_r-r+1 \\ & (1) \qquad\qquad \ldots\ldots \qquad\qquad (n-r+1) \end{array}$$

In converse, conduct a r-combination without repetition of $\{1,2,\ldots n\text{-}r+1\}$ and gets $\{d_1,d_2\ldots d_r\}$, assume $d_1<d_2\ldots<d_r$

$c_1=d_1$, $c_2=d_2+1,\ldots c_r=d_r+r-1\leq n-r+1+r-1=n$

$c_1<c_2\ldots<c_r$, $c_{i+1}-c_i = (d_{i+1}+i)-(d_i+i-1)=d_{i+1}-d_i+1>1$,

So c_{i+1} and c_i are not adjacent. $\{c_1,c_2\ldots c_r\}$ are picking r non-adjacent numbers among $\{1,2,\ldots,n\}$.

So there's a bijection relation between picking r non-adjacent numbers among $A=\{1,2,\ldots n\}$ and r-combination of $(n-r+1)$. The number of combinations is $C(n-r+1,r)$.

Application Example

- **Eg** We need to use multiple keys at the same time to open a secure lock. Now we have 7 persons and each one has multiple locks. Only if 4 are together, they could open the lock.
- ①How many different keys are needed?
- ②How many keys do one need to hold?
- **Solution** ①Every 3 persons lack at least 1 keys but every 4 persons are not in lack of keys. So every 3 persons lack different keys.
  - If *abc* and *abd* are in lack of the same key, *abcd* couldn't open the lock.
- So there's at least C(7,3)=35 different keys.

Application Example

- Every 4 persons are not in lack of keys. So everyone should have a key corresponding to every other 3 persons. So everyone must hold at least C(6,3)=20 keys.
- A simple example: 3 persons among 4 need to be together to open a lock. So there are a total of C(4,2)=6 different keys and everyone has C(3,2) = 3 keys.

		keys					
		1	2	3	4	5	6
persons	A	√	√	√			
	B			√	√	√	
	C	√				√	√
	D		√		√		√

Type	Sample	Order counts?	Repetition allowed?	Number of ways
Combination w/o repetitions	Pick r balls from n balls.	No	No	$C(n, r)$
Permutation w/o repetitions	Line up r people from n	Yes	No	$P(n, r)$
Combination of multisets	Make compote with r fruits out of n kinds	No	Yes	$C(n+r-1, r)$
Permutation w/ repetitions	r-string formed by n letters	Yes	Yes	n^r
Permutation of multisets	n-string formed by r_1 a's, r_2 b's	Yes	Yes	$n!/(r_1!\, r_2!)$

组合数学 Combinatorics

2 Journey of a Pingpang ball

2-5 Generating all permutations

Yuchun Ma,
Tinsghua University

Generating algorithms of permutations

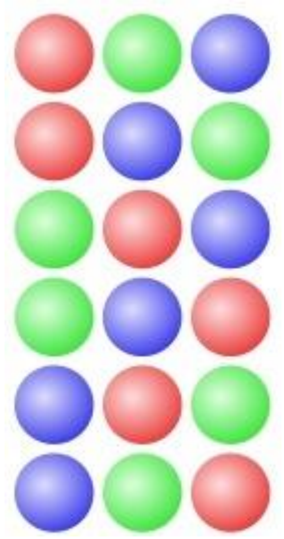
In the reality we usually need to list all the permutations

Permutations of words

Permutations of magic squares

… …

Generating algorithms of permutations: use the given alphabets to list the permutations without repetitions and omissions.



- **Donald Ervin Knuth**
  - One of the founders of modern computer science
- *The Art of Computer Programming*
  - One of the most respected text in the field of computer science
  - Donald Knuth. *The Art of Computer Programming*, Volume 4: ***Generating All Tuples and Permutations***, Fascicle 2 first printing. Addison-Wesley, 2005.
- http://en.wikipedia.org/wiki/Permutation
- http://www.jjj.de/fxt/demo/perm/

Change ringing

Rose in British in the 16th/17th century.

Hit a series of chimes in the churches (5-12 chimes). One person is responsible to a chime

The sequence is defined as number sequences

For example 12345, 21345, 23145 etc.

Bell ringing practice in Stoke Gabriel parish church, Devon, England

When Obama took office, New York Trinity Church change ringing

Report: The famous New York Trinity Church ringed 12 chimes to celebrate the taking office of the Obama government. This special change ringing lasted for 3 hours and a half.

HINT quiz, Calculate how many different tunes could be played in the Trinity Church?

组合数学 Combinatorics

2 Journal of a Pingpang ball

2-6 Generating algorithms of permutations

Yuchun Ma,
Tinsghua University

The generation of permutations

- Simple ⇒ Complex
  - Permutations of{1}

    1
  - Permutations of {1 2}

    1 2

    2 1
  - Permutations of {1 2 3}

    1 2 3

    1 3 2

    3 1 2

    3 2 1

    2 3 1

    2 1 3

High complexity!
Waste of memory!

2 1 2

3 1 3 2 3

3 2 3 1 3

Idea of recursion：Generate permutations of {1,2…n} with permutations of {1,2,…n-1}

Generating algorithms of permutations

- Any permutation of a *n*-string is corresponding to a permutation of *n*-digit numbers. There's a linear order relation among permutations of *n*-digit numbers.
- The next permutation of "abc" is "acb"
- The next permutation of "acb" is "bac"
- Except the last permutation, it has a successor; except the first permutation, it has a predecessor.
- The successor of every permutation could be gotten by minimal changes on its predecessor.
- The generating algorithms of permutations are methods to generate all permutations in order.

abc   acb   bac   bca   cab   cba

Lexicographic Order

123

132

213

•Define a order relation for the given alphabet, so then we define the order of permutations are decided by comparing the characters from left to the right.

[Eg]For alphabet {1,2,3}, smaller digits are in the left, so the permutations in lexicographic order are:

123,132,213,231,312,321.

A permutation could be regarded as a string, the string could have **prefix and suffix**。

The so called *next permutation of this one* means that there's no more *other permutations* among this and the next.

This means that this one and next one should have a **common prefix** which is as long as possible, and the changes are limited to **suffixes** as short as possible.

Lexicographic Order

This requires this one and the next one have a **common prefix** as long as possible, and the changes limited to **suffixes** as short as possible.

123,132,213,231,312,321。

?

1 2 3

1 3 2

3 2 1

Scan from right to left, if it's increasing, there's no next.
Find the first decrease point.

2 3 1

2 1 3

Exchange? **Exchange with** **Larger than 1** **The smallest**

Smallest suffix

Lexicographic Order

- **[Eg]**Calculate the next permutation of 839647521
- 1 Find the first decrease pint from right to left: 4

839647521

- 2. Exchange：Find the smallest number larger than 4 in the suffix

839647521

- 3. Overturn the suffix 83965 1247

- The next permutation is: 839651247

The implementation of the algorithm is in the examples

The generation of permutations

- Simple to Complex
  - Permutations of {1}

    1

  - Permutations of {1 2}

    1 2

    2 1

  - Permutations of {1 2 3}

    1 2 3

    1 3 2

    3 1 2

    3 2 1

    2 3 1

    2 1 3

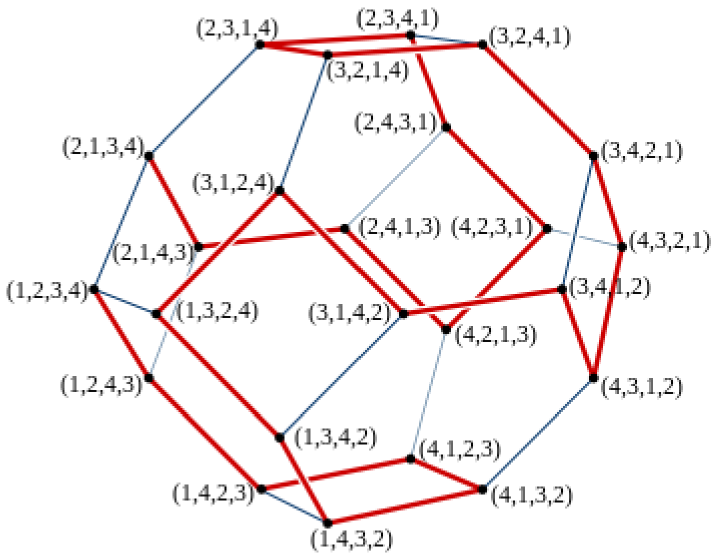
High complexity!

Waste of memory!

2 1 2

3 1 3 2 3

3 2 3 1 3



Idea of recursion：Generate permutations of {1,2…n} with permutations of {1,2,…n-1}

Generation of permutations

– Permutations of {1 2 3 4}

$\overleftarrow{1}\quad \overleftarrow{2}\quad \overleftarrow{3} \leftrightarrow \overleftarrow{4}$

$1\quad 2 \leftrightarrow 4\quad 3$

$1 \leftrightarrow 4\quad 2\quad 3$

$\overleftarrow{4}\quad 1\quad 2 \leftrightarrow 3$

$\overrightarrow{4} \leftrightarrow 1\quad 3\quad 2$

$1\quad 4 \leftrightarrow 3\quad 2$

$1\quad 3\quad 4 \leftrightarrow 2$

$1 \leftrightarrow 3\quad 2\quad 4$

Digits have moving directions

Exchange two adjacent digits each time

Exchange the largest mobile integer

Mobile Integer

- Every digit in {1,2,…,n} has a moving direction.
  - If the adjacent digit in the direction of the digit is smaller, it's a mobile integer.
  - 1 is always immobile.
  - Except when direct to the outside of the sequence, *n* is a mobile integer

$\overleftarrow{4}\ \overleftarrow{1}\ \overleftarrow{2}\ \overleftarrow{3}$

$\overrightarrow{3}\ \overleftarrow{2}\ \overleftarrow{1}\ \overrightarrow{4}$

$\overrightarrow{2}\ \overrightarrow{6}\ \overrightarrow{3}\ \overrightarrow{1}\ \overrightarrow{5}\ \overrightarrow{4}$

only 3, 5, and 6 are mobile.

Generating Permutations

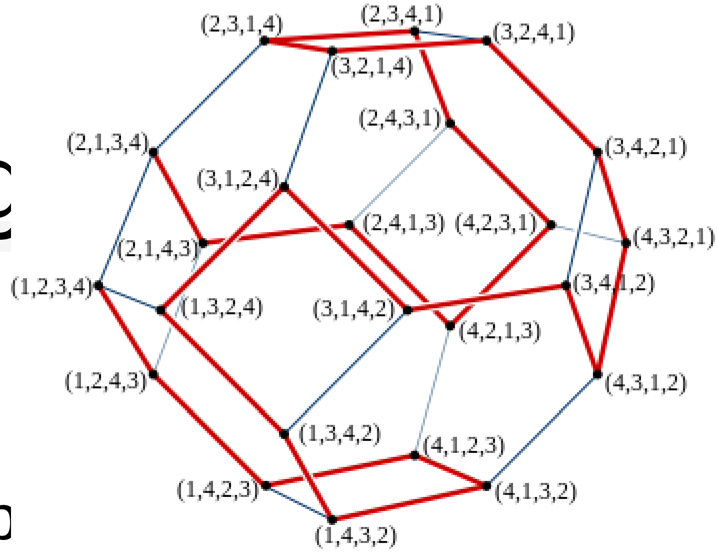
– Permutations of {1 2 3 4}

1 2 3 4
1 2 4 3
1 4 2 3
4 1 2 3
4 1 3 2
1 4 3 2
1 3 4 2
1 3 2 4

Step 0: List from small to larger, all directed to the left
Step 1: The largest mobile integer currently is 4
Step 2: Exchange 4 with its directed neighbor
…
…
Step i: When 4 reached the boundary and becomes immobile, the largest mobile integer is 3. Exchange 3 and 2, then change the direction of 4 to make it mobile again.

- Move the largest mobile integer
- After moving a digit, change directions of all larger numbers to make them mobile again.

Steinhaus–Johnson–Trotter alg



Begin with $\overleftarrow{1}\ \overleftarrow{2}\cdots\overleftarrow{n}$.

While there exists a mobile integer, do

1) find the largest mobile integer *m*.
2) switch *m* and the adjacent integer its arrow p
3) switch the direction of all integers *p* with *p*>*m*.

$\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}\,\overleftarrow{4}$

$\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{4}\,\overleftarrow{3}$

$\overleftarrow{1}\,\overleftarrow{4}\,\overleftarrow{2}\,\overleftarrow{3}$

$\overleftarrow{4}\,\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}$

$\overrightarrow{4}\,\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{2}$

$\overleftarrow{1}\,\overrightarrow{4}\,\overleftarrow{3}\,\overleftarrow{2}$

$\overleftarrow{1}\,\overleftarrow{3}\,\overrightarrow{4}\,\overleftarrow{2}$

$\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{2}\,\overrightarrow{4}$

$\overleftarrow{3}\,\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{4}$

$\overleftarrow{3}\,\overleftarrow{1}\,\overleftarrow{4}\,\overleftarrow{2}$

$\overleftarrow{3}\,\overleftarrow{4}\,\overleftarrow{1}\,\overleftarrow{2}$

$\overleftarrow{4}\,\overleftarrow{3}\,\overleftarrow{1}\,\overleftarrow{2}$

$\overrightarrow{4}\,\overrightarrow{3}\,\overleftarrow{2}\,\overleftarrow{1}$

$\overrightarrow{3}\,\overrightarrow{4}\,\overleftarrow{2}\,\overleftarrow{1}$

$\overrightarrow{3}\,\overleftarrow{2}\,\overrightarrow{4}\,\overleftarrow{1}$

$\overrightarrow{3}\,\overleftarrow{2}\,\overleftarrow{1}\,\overrightarrow{4}$

$\overleftarrow{2}\,\overrightarrow{3}\,\overleftarrow{1}\,\overleftarrow{4}$

$\overleftarrow{2}\,\overrightarrow{3}\,\overleftarrow{4}\,\overleftarrow{1}$

$\overleftarrow{2}\,\overleftarrow{4}\,\overrightarrow{3}\,\overleftarrow{1}$

$\overleftarrow{4}\,\overleftarrow{2}\,\overrightarrow{3}\,\overleftarrow{1}$

$\overrightarrow{4}\,\overleftarrow{2}\,\overleftarrow{1}\,\overrightarrow{3}$

$\overleftarrow{2}\,\overrightarrow{4}\,\overleftarrow{1}\,\overrightarrow{3}$

$\overleftarrow{2}\,\overleftarrow{1}\,\overrightarrow{4}\,\overrightarrow{3}$

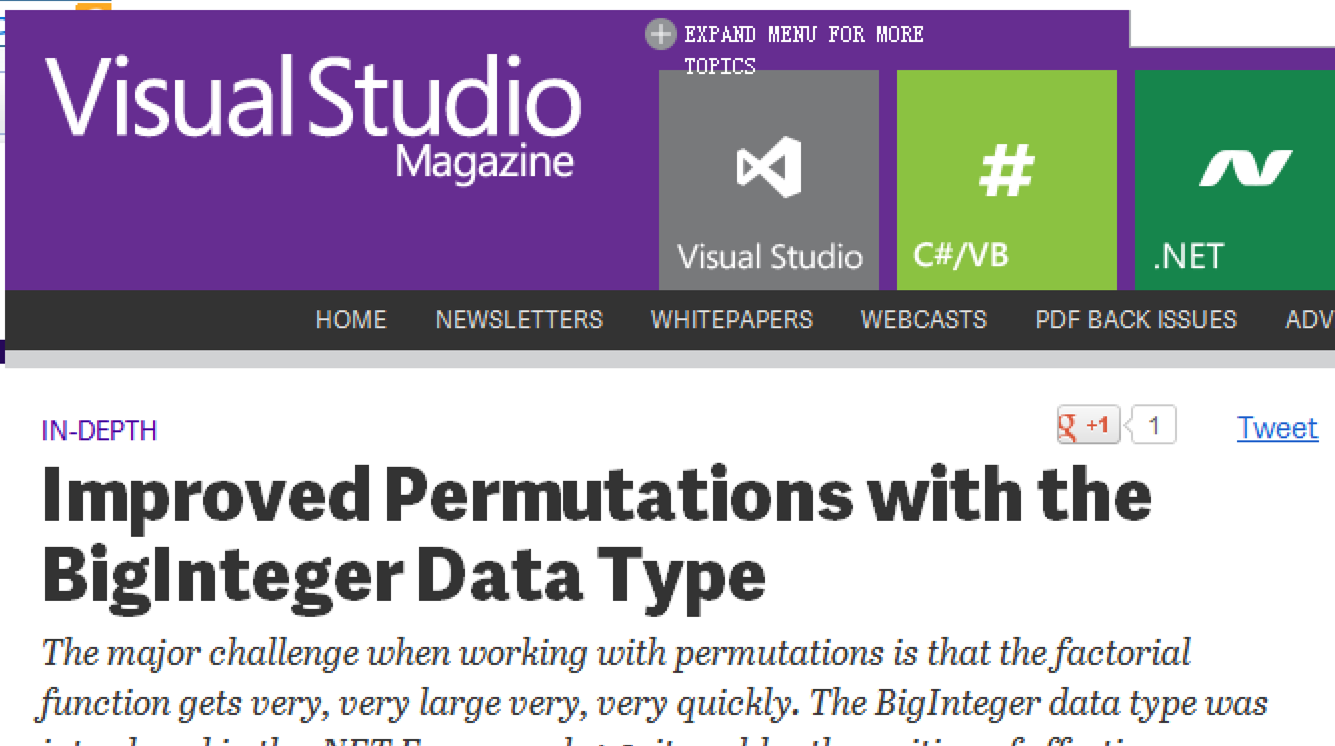
$\overleftarrow{2}\,\overleftarrow{1}\,\overrightarrow{3}\,\overrightarrow{4}$

Industrial Support

– Java

```
for(List<String> list : Permutation.of(Arrays.asList("a", "b", "c")))
    System.out.println(list);
}
```

```
for(List<Integer> list : Combination.of(Arrays.asList(1, 2, 3, 4, 5), 3))
    System.out.println(list);
}
```



Finite Fields and Their Applications
Volume 17, Issue 1, January 2011, Pages 51–67

On constructing permutations of finite fields ☆

Amir Akbary[a], Dragos Ghioca[b], Qiang Wang[c]
a Dep
b Dep
c Sch

Menemui Matematik (Discovering Mathematics)
Vol. 32, No. 2: 51– 56 (2010)

New Recursive Circular Algorithı

Sharmila Karim[1], Zurni Om
Khairil Iskandar Othman[4], a
123 College of Art and S

Ann. Comb. 14 (2010) 3–16
DOI 10.1007/s00026-010-0042-9
Published online February 16, 2010

Annals of Combinatorics

Permutations Generated by Stacks and Deques

Michael Albert[1], Mike Atkinson[1], and Steve Linton[2]
[1]Department of Computer Science, University of Otago, PO Box 56, Dunedin 9054, New Zealand

A Two-level Algorithm for Generating Multiset Permutations

Tadao Takaoka
Department of Computer Science, University of Canterbury
Christchurch, New Zealand
y.ac.nz

ırmutations in O(1) time for eac
ı memory requirement. There a

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From Permutations to Iterative Permutations

Hoang Chi Thanh [#1], Nguyen Thi Thuy Loan [*2], Nguyen Duy Ham [^3]

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Common Permutation generators

In C++ standard library, next_permutation, prev_permutation, could generate permutations in lexicographic order.

#include <algorithm>

bool next_permutation(iterator start, iterator end);

bool prev_permutation(iterator start, iterator end);

The next_permutation() function attempts to transform the given range of elements [start,end) into the next lexicographically greater permutation $n!$ of elements. If it succeeds, it returns true, otherwise, it returns false.

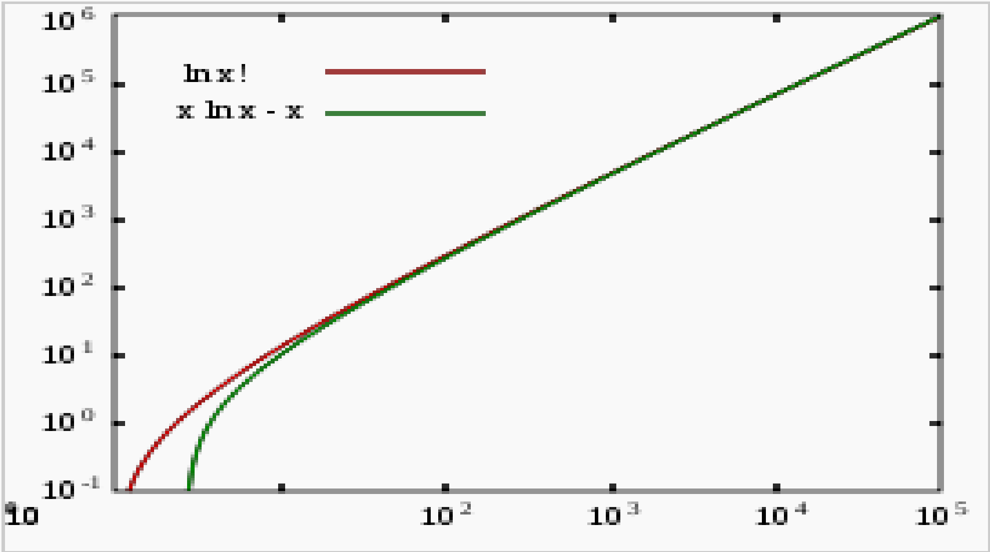
http://www.slyar.com/blog/stl_next_permutation.html

1.6 Stirling's Formula

- The value of n! increases very fast.
- The meaning of Stirling's Formula is:
- When n is large enough, it's very hard to calculate n!
- Although there are a lot of inequalities about n!, it's not enough to estimate the value of n!. Especially when n is very large, it's not accurate.
- Convert factorials to power functions so it's simpler to estimate and it's more accurate when n is larger.

$$lim \frac{n!}{\sqrt{2\pi n}(\frac{n}{e})^n} = 1 \quad n \to \infty \quad \mathrm{n}! \sim \sqrt{2\pi n}(\frac{n}{e})^n$$

Could we generate a part of the permutations? For example from #100 to #200?



The ratio of (ln $n!$) to (n ln n − n) approaches unity as n increases.

$$60! \approx \sqrt{6.28318*60}*(22.072767)^{60} \approx 8.34 \times 10^{81}$$

For the permutations of 60 characters, if we use a computer which generate 10^7 permutations per second, it would need:

$$T=60!/(365 \times 24 \times 3600 \times 10^7) = 2.65 \times 10^{67} \text{ years}$$

Summary of Week 2

- Basic counting principals
  - +-×÷
  - Independency, hidden conditions
- Permutation, Combination
  - Order matters or not?
  - With or without repetitions?
  - Indexes, bars……
- Generating algorithms
  - Lexicographic，SJT……
  - Tools

组合数学 Combinatorics

Yuchun Ma