# 4 Linear Homogeneous Recurrence Relation

## 4-1 Fibonacci Rabbits

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**The delta of the $n^{th}$ month and $n\text{-}1^{th}$ month is given birth by the rabbits in $n\text{-}2$ month. So**

$$F_n = F_{n-1} + F_{n-2}$$

**In the first month there's a pair of newly-born rabbits;**
**If a pair of rabbits could give birth to a new pair every month (one male, one female);**
**New rabbits could start giving birth since the third month;**
**The rabbits never die;**
**How many rabbits would there be in the 50$^{th}$ month?**

Fibonacci number

1 1 2 3 5 8 13 21 34 55……

**OEIS: A000045**

**http://oeis.org/A000045**

**Leonardo of Pisa**

Fibonacci，Bonacci's son

Bonacci: good, natural, simple

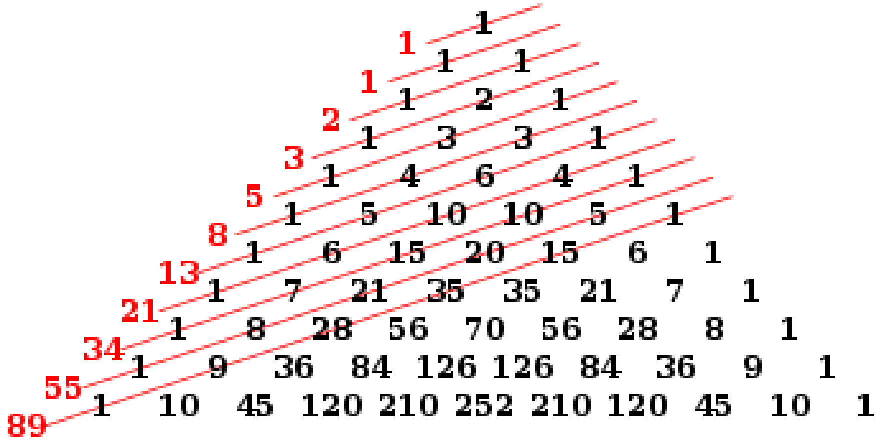
Recurrence Relation: $F(n)=F(n-1)+F(n-2) \quad n\geq 2$

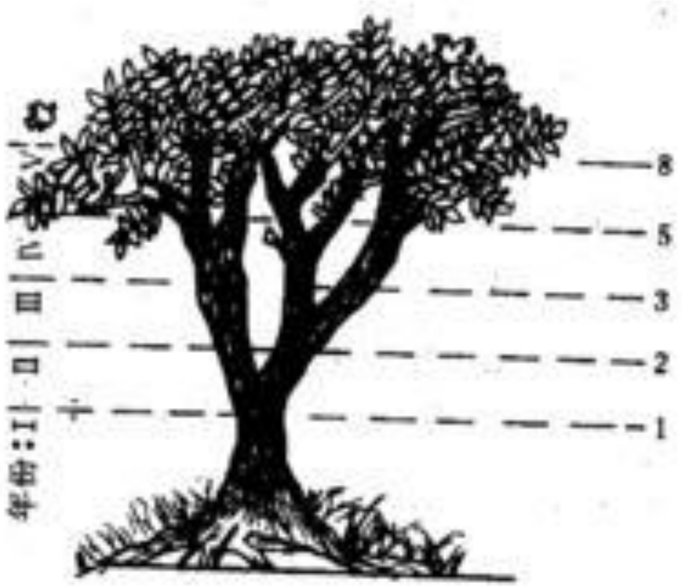
Initial values: $F(0)=0$, $F(1)=1$

- **In 1150, Indian mathematicians researched the number of arrangements to package items with length 1 and width 2 into boxes. And they described this sequence for the first time.**
- **In the western world, Fibonacci mentioned a problem about the reproduction of rabbits in Liber Abbaci in 1202.**
- Fibonacci，Leonardo 1175-1250
  - Member of the Bonacci family.
  - Travelled to Asia and Africa at 22 with his father and learned to calculate with Indian digits；
  - Played an important role in the recovery of Western Mathematics. And connected Western and Oriental mathematics.
  - G.Cardano: "We could assume that all mathematics we know except the Ancient Greek ones are gotten by Fibonacci.

# Fibonacci number

1 1 2 3 5 8 13 21 34 55……





Trillium — 3 Petals


St. Johnswort — 5 Petals


Bloodroot — 8 Petals


Black-eyed Susan — 13 Petals


Devil's Paintbrush — 21 Petals


Ox-eyed Daisy — 34 Petals


Sunflower — 55 Petals


Daisy Fleabane — 89 Petals

# Fibonacci Numbers

- The Fibonacci Quarterly founded in 1963 especially publish the newest researches on this sequence. Which includes:
  - The last digit loops every 60 numbers; the last 2 digits loops every 300 numbers; the last 3 digits loops every 1500 numbers; the last 4 digits loops every 15000 numbers; the last 5 digits loops every 150000 numbers.
  - Every 3rd number could be divided by 2. Every 4th number could be divided by 3. Every 5th number could be divided by 5. Every 6th number could be divided by 8, etc. These divisors can also construct a Fibonacci Sequence.

# **Fibonacci prime** (Sequence A005478 in OEIS)

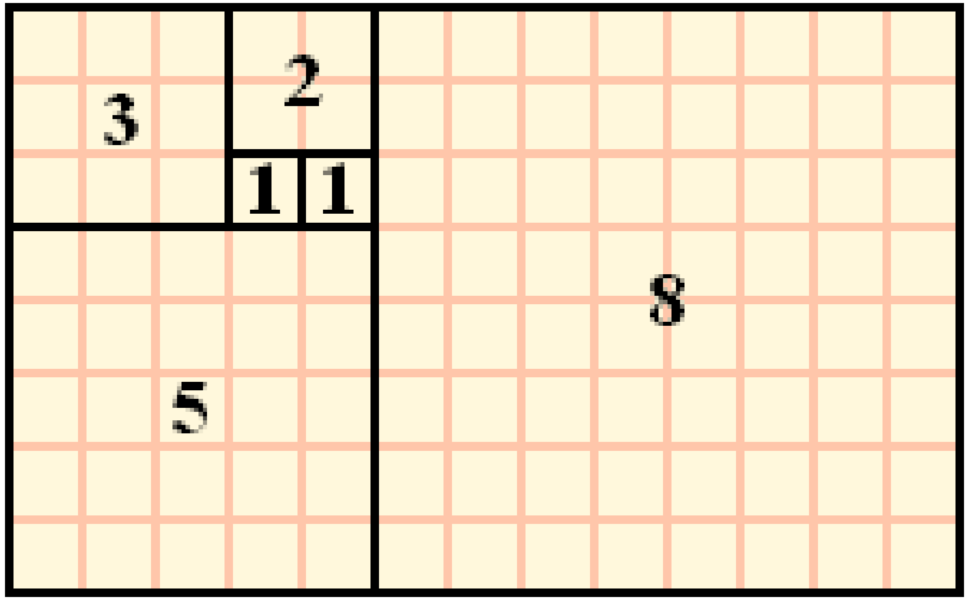
– In the Fibonacci Sequence, there are primes: 2, 3, 5, 13, 89, 233, 1597, 28657, 514229, 433494437, 2971215073, 99194853094755497,

– Except $n$ = 4, the indexes of all Fibonacci Primes are primes.

– However, not all prime index Fibonacci Numbers are primes.

– Conjecture: Are there infinite primes among Fibonacci Numbers?

– The largest known prime is the 81839$^{th}$ Fibonacci Number, which has 17103 digits.




$$F_1^{\ 2} + F^{2}_{\ 2} + \cdots + F_n^{\ 2} = F_n F_{n+1}$$

Area of the rectangle = Sum of multiple quadrates

Proof without words   vs   Logic deduction

# Recurrence Relation

$F_0 = 0, F_1 = 1, F_2 = 1 \ldots\ldots$

$F_n = F_{n-1} + F_{n-2}$

Prove the identity: $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$

**Proof:** $F_1^2 = F_2 F_1$

$$F_0 = 0, F_1 = 1, F_2 = 1 \ldots\ldots$$
$$F_n = F_{n-1} + F_{n-2}$$

# Recurrence Relation

$$F_1 + F_2 + \cdots + Fn = Fn_{+2-1}$$

Proof：

$$F_1 = F_3 - F_2$$

$$F_2 = F_4 - F_3$$

$$\ldots\ldots$$

$$F_{n-1} = F_{n+1} - F_n$$

$$+)\quad F_n = F_{n+2} - F_{n+1}$$

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$$F_1 + F_2 + \cdots + Fn = Fn_{+2-1}$$

# Recurrence Relation

$$F_0 = 0, F_1 = 1, F_2 = 1 \ldots\ldots$$
$$F_n = F_{n-1} + F_{n-2}$$

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$$

**Proof:**

$$F_1 = F_2$$
$$F_3 = F_4 - F_2$$
$$F_5 = F_6 - F_4$$
$$\ldots\ldots$$
$$+)\quad F_{2n-1} = F_{2n} - F_{2n-2}$$

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$$\therefore \quad F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$$

Detailed Expressions?

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# 4 Linear Homogeneous Recurrence Relation

## 4-2 Expressions of Fibonacci Numbers

# Magic

- There's a 80cm×80cm quadrate tablecloth. How to convert it to a 1.3m×50cm one?

**0, 1, 1, 2, 3, 5, 8, 13, 21,……….**

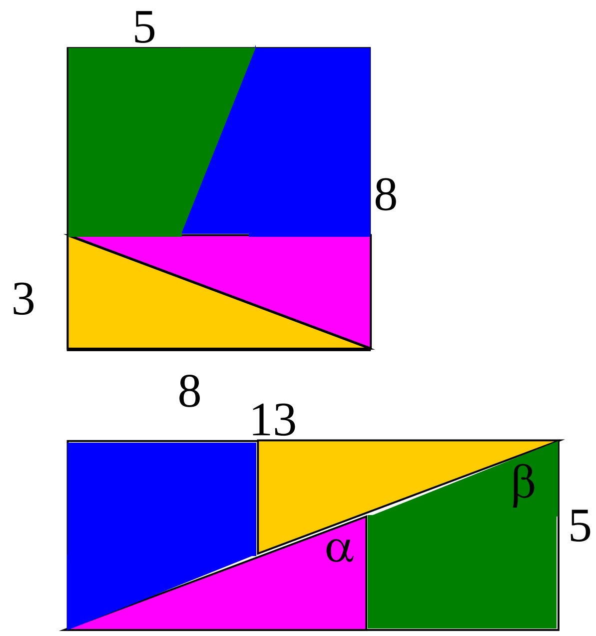
$$F(n)*F(n) - F(n-1)F(n+1) = (-1)^n$$

***n*=0,1,2**

Larger tablecloths?

F(100)=?

**Direct expressions?**




$$\tan\alpha = \frac{8}{3} \cong 2.67, \tan\beta = \frac{5}{2} = 2.5$$

# Fibonacci Recurrence

$F_0 = 0, \quad F_1 = 1, \quad \ldots\ldots$

$F_n = F_{n-1} + F_{n-2}$

Assume $\boxed{G(x) = F_1 x + F_2 x^2 + \cdots}$

$$x^3 : F_3 = F_2 + F_1$$

$$x^4 : F_4 = F_3 + F_2$$

$$+) \quad \cdots\cdots\cdots\cdots\cdots$$

$$G(x) - x^2 - x = x(G(x) - x) + x^2 G(x)$$

$$\therefore \quad (1 - x - x^2) G(x) = x$$

$$\therefore \; G(x) = \frac{x}{1-x-x^2} = \frac{x}{(1-\frac{1-\sqrt{5}}{2}x)(1-\frac{1+\sqrt{5}}{2}x)} = \frac{A}{1-\frac{1+\sqrt{5}}{2}x} + \frac{B}{1-\frac{1-\sqrt{5}}{2}x}$$

# Fibonacci Recurrence

$$\begin{cases} A+B=0 \\ \frac{\sqrt{5}}{2}(A-B)=1 \end{cases} \qquad \begin{cases} A+B=0 \\ A-B=\frac{2}{\sqrt{5}} \end{cases} \qquad A=\frac{1}{\sqrt{5}}, \quad B=-\frac{1}{\sqrt{5}}$$

$$\therefore \quad G(x)=\frac{1}{\sqrt{5}}\left[\frac{1}{1-\frac{1+\sqrt{5}}{2}x}-\frac{1}{1-\frac{1-\sqrt{5}}{2}x}\right] \quad =\frac{1}{\sqrt{5}}[(\alpha-\beta)x+(\alpha^2-\beta^2)x^2+\cdots]$$

$$\alpha=\frac{-2}{1-\sqrt{5}}=\frac{1+\sqrt{5}}{2}, \quad \beta=\frac{2}{1+\sqrt{5}}=\frac{1-\sqrt{5}}{2}$$

$$\boxed{\mathrm{F}_{\mathrm{n}}=\frac{1}{\sqrt{5}}(\alpha^{\mathrm{n}}-\beta^{\mathrm{n}})=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{\mathrm{n}}-\left(\frac{1-\sqrt{5}}{2}\right)^{\mathrm{n}}\right)}$$

$$\frac{F_n}{F_{n-1}}=\frac{1+\sqrt{5}}{2}\approx 1.618$$

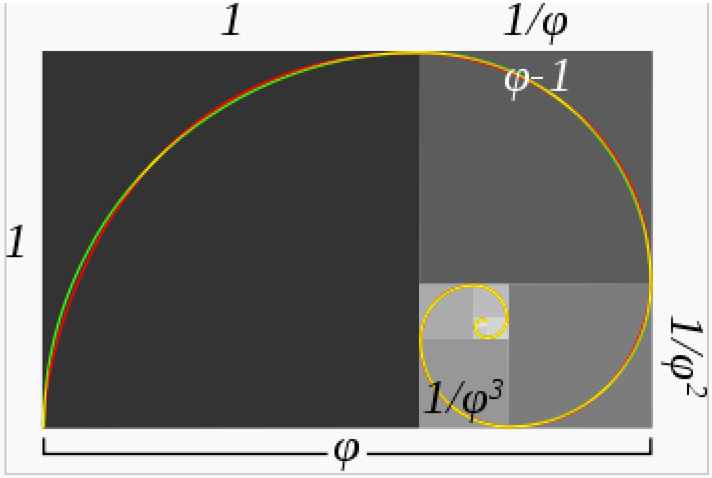
# Fibonacci Sequence

$$F_n = F_{n-1} + F_{n-2}$$

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$\frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2} \approx 1.618$$

$$\varphi = [1;1,1,1,\ldots] = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}$$

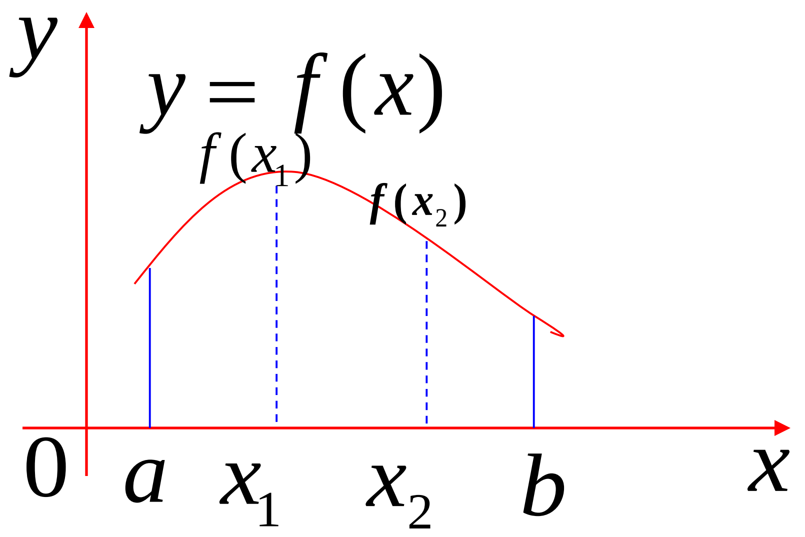



# Applications in Optimization Methods

Assume that function $f(x)$ reaches its maximum at $x =$ ξ. ， Design an optimization algorithm to find the extreme point to a certain extent within finite iterations. The simplest way is to trisect the interval (a,b).

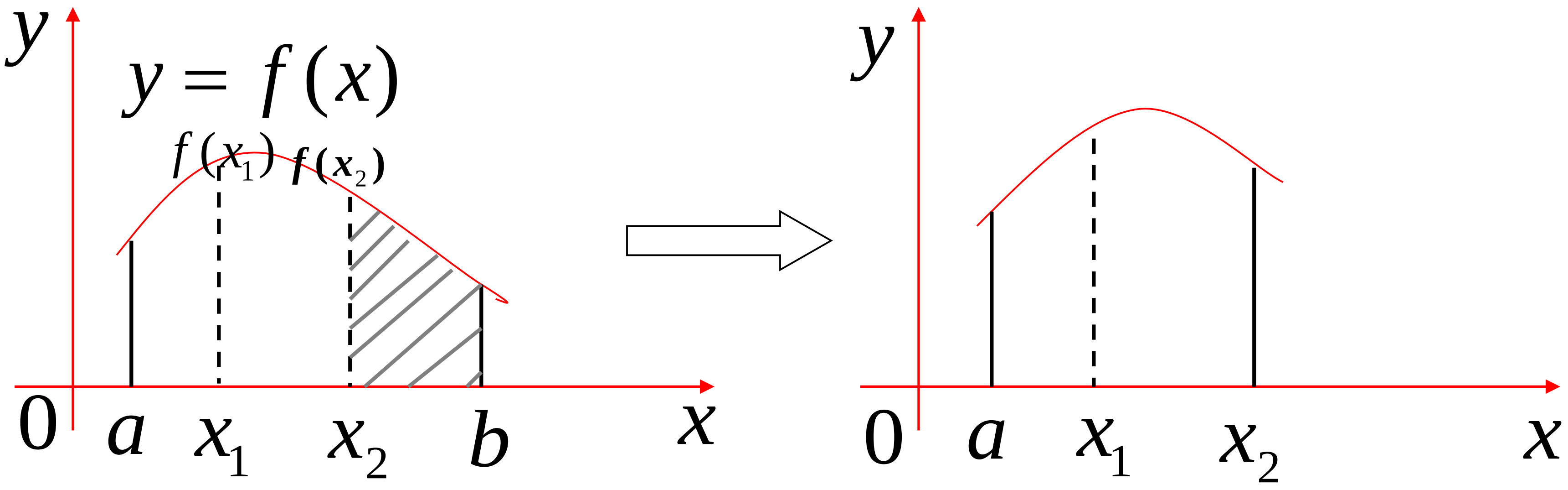
$$x_1 = a + \frac{1}{3}(b-a), \quad x_2 = a + \frac{2}{3}(b-a)$$

如下图：



Discuss according to the sizes of $f(1), f(2)$

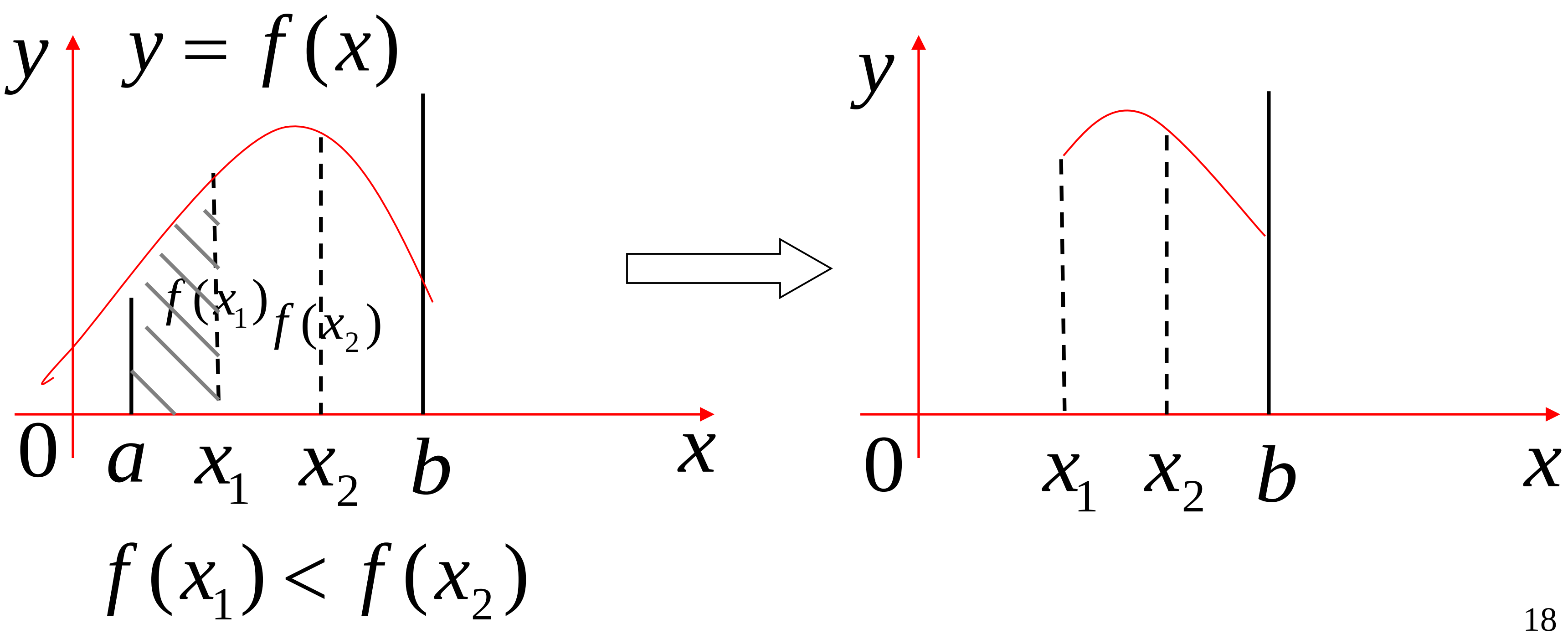
When $f(x_1) > f(x_2)$, the maximum $\xi$ must be in $(a, x_2)$ , interval $(x_2, b)$ could be removed.



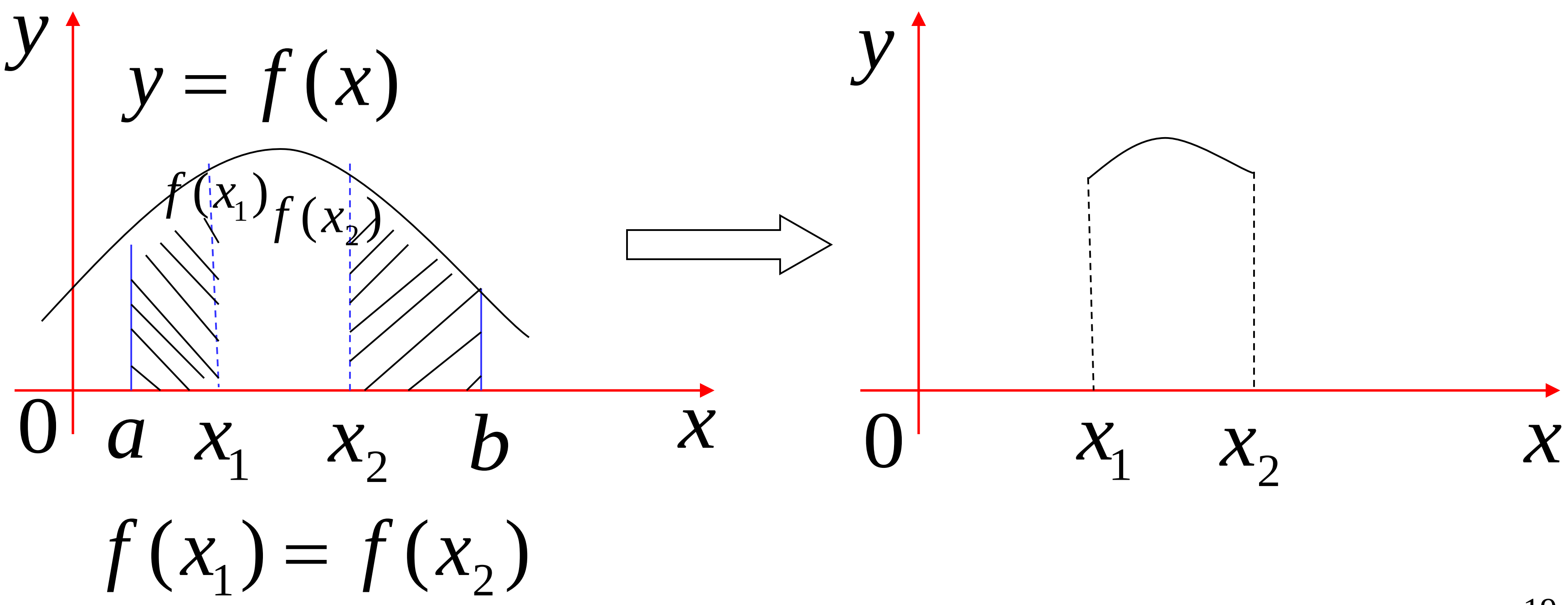

$f(x_1) > f(x_2)$

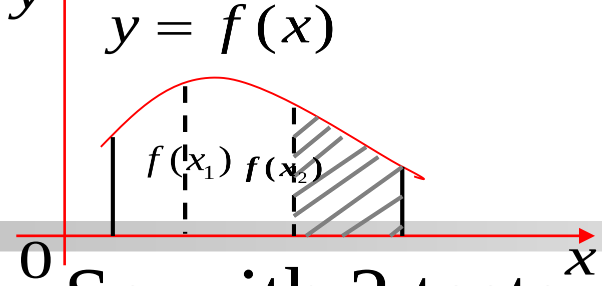
# § 2.4 Applications in Optimization Method

When $f(x_1) < f(x_2)$, the maximum $\xi$ must be in $(x_1, b)$, the range $(a, x_1)$ could be removed.



When $f(x_1) = f(x_2)$, the maximum $\xi$ must be in $(x_1, x_2)$, so both $(a, x_1)$ and $(x_2, b)$ could be removed.






So with 2 tests, at least we could reduce the range to 2/3 of the origin domain. For example $f(x_1) > f(x_2)$ and we find the maximum in $(a, x_2)$.

If continue using threefold division method, it's a fact that $x_1$ is not used. So we image to have 2 symmetrical point $x, l - x$ in (0,1) to test。

$0 \quad 1-\boldsymbol{x} \quad \boldsymbol{x} \quad 1$

If keep $(0, x)$, then we keep testing at points $x^2$, $(1-x)x$ in $(0, x)$. If

$$x^2 = (1-x)$$

Then the test at $(1-x)$ could be used again. So we save one test. We have:

$$x^2 + x - 1 = 0$$

$$\therefore \quad x = \frac{-1+\sqrt{5}}{2} = 0.618$$

0.382，0.618
0.236，0.382
0.146，0.236

$0 \quad 0.382 = (0.618)^2 \quad 0.618 \quad 1$

# Application in Optimization Method

This is the 0.618 optimization method. When finding unimodal maximums in (0,1), we could test at:

$$x_1 = 0.618, \quad x_2 = 1 - 0,618 = 0.3832$$

points. For example keep (0,0.618), as

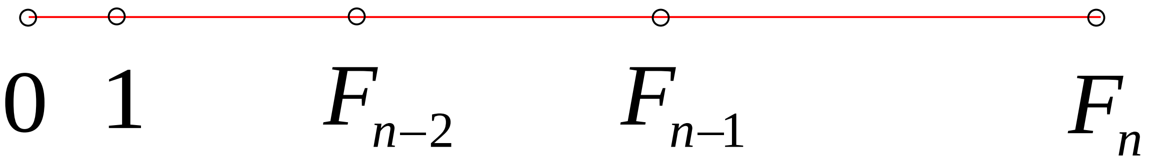
$(0.618)^2 = 0.328$, we only need to test once at

$$0.618 \times 0.328 = 0.236$$

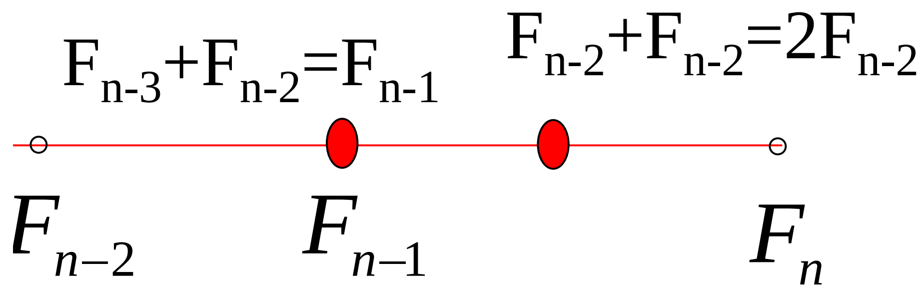
# Applications in Optimization Method

We could use Fibonacci Sequence in Optimization Method. Its difference from 0.618 method is to decide the number of tests before testing. We introduce in 2 situations.

(a) The possible testing number is some $F$n。

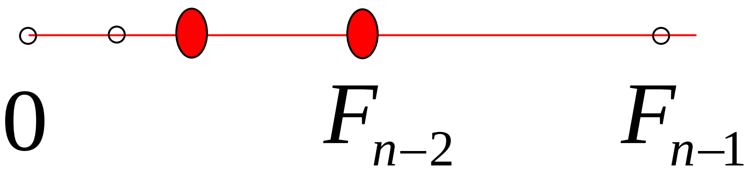


- At this point testing points are division points $F_{n\text{-}1}$ and $F_{n\text{-}2}$ .
  - If $F_{n\text{-}1}$ is better, remove the part smaller than $F_{n\text{-}2}$;
    - The remained part contains $F_n\text{-}F_{n\text{-}2} = F_{n\text{-}1}$ division points,
    - In which test pints $\mathrm{F}_{\mathrm{n-2}}$ and $\mathrm{F}_{\mathrm{n-3}}$ are correspondent to former index $\mathrm{F}_{\mathrm{n-2}}+\mathrm{F}_{\mathrm{n-2}}=2\mathrm{F}_{\mathrm{n-2}}$ and $\mathrm{F}_{\mathrm{n-3}}+\mathrm{F}_{\mathrm{n-2}}=\mathrm{F}_{\mathrm{n-1}}$. Just right, $\mathrm{F}_{\mathrm{n-1}}$ has been tested in the previous round.



- If $F_{n\text{-}2}$ is better, remove the part larger than $F_{n\text{-}1}$.
  - In the remained part there are $F_{n\text{-}1}$ division points, in the next step among test points $F_{n\text{-}2}$ and $F_{n\text{-}3}$, $F_{n\text{-}2}$ has been tested.。

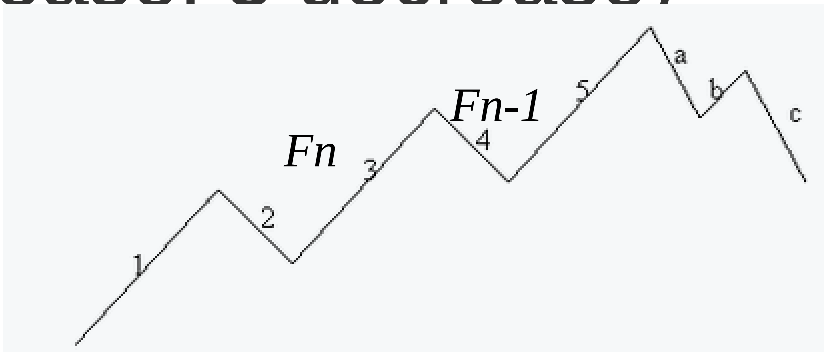
So among the $F_n$ possible tests, we could find the extreme with at most *n-1* tests.




One difference between the Fibonacci Method and 0.618 method is that Fibonacci Method could be used when the parameters are all integers. If the number of possible tests are smaller than $F_n$ but larger than $F_{n-1}$, we could add several imagine points to make it $F_n$ points. We could assume these image points to be worse than any other points without actually compare them.

# Elliott wave

A complete loop includes 8 waves (5 increase, 3 decrease)

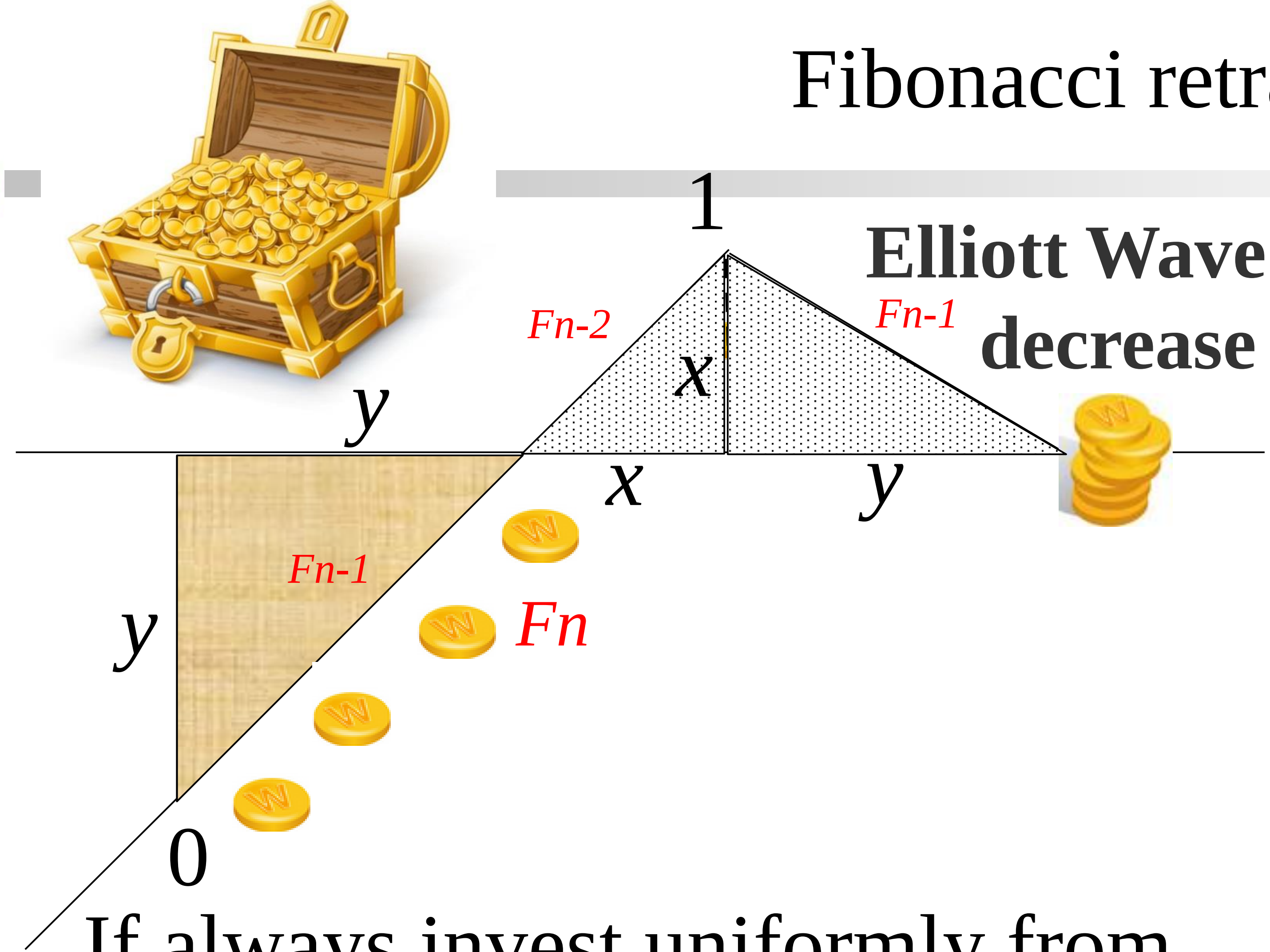



A complete period includes 8 waves, in which 5 are increasing, 3 are decreasing. They are all Fibonacci Numbers. In details we could get 34 waves and 144 waves, they are also Fibonacci Numbers.


Common retracement ratio are 0.382、0.5 and 0.618.

It mainly reflects the psychology of investors.

# Fibonacci retracement




**Elliott Wave increase part $F_n$, decrease part $F_{n-1}$.**

Profit= Loss

**Profit: $y^2/2$**

**Loss: $(x+y)x/2$**

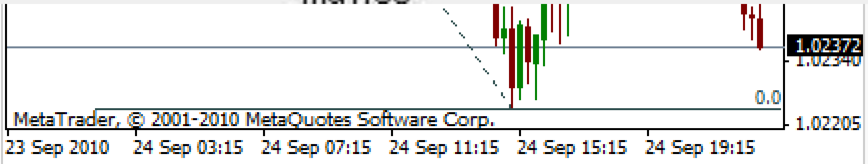
**$x+y=1$**

**$x=0.382$**

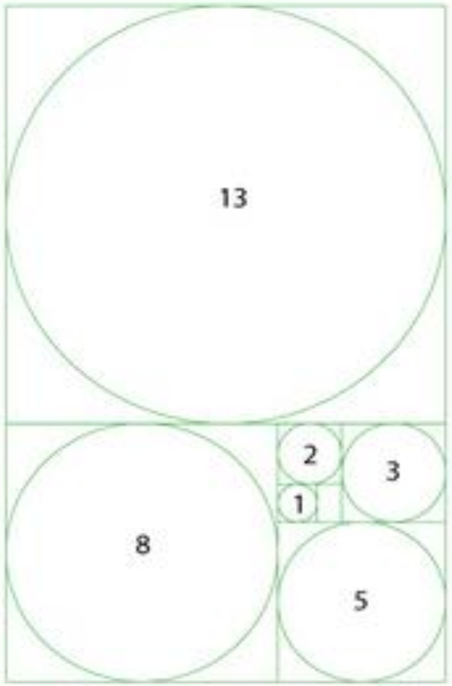
If always invest uniformly from 0 to 1, and then it decreases by x

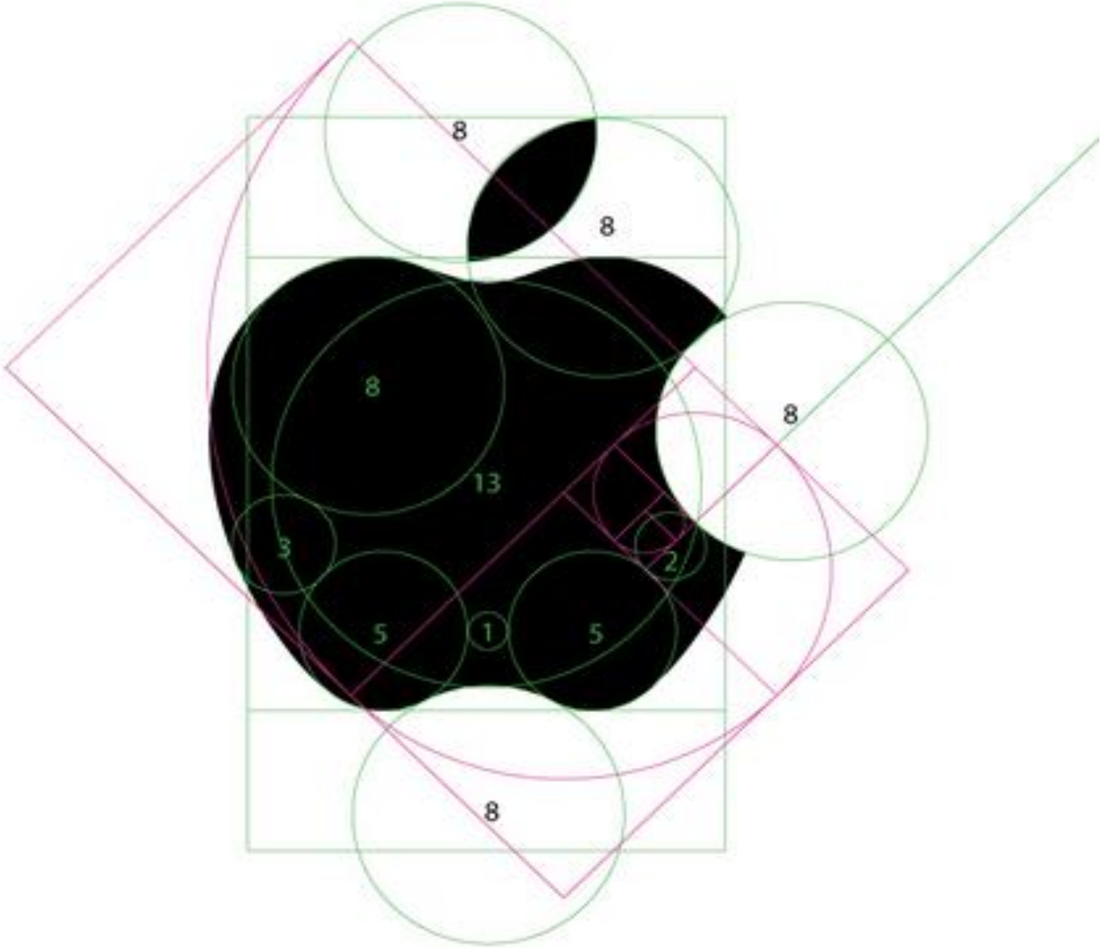
The sustentation $x$=?

# Fibonacci retracement


Fibonacci retracement levels shown on the USD/CAD currency pair. In this case, price retraced approximately 38.2% of a move down before continuing.


13
2
3
1
8
5


8
8
8
13
8
3
2
5
1
5
8

# 4 Linear Homogeneous Recurrence Relation

## 4-3 Linear Homogeneous Recurrence Relation

组合数学 Combinatorics

$F_n$-$F_{n-1}$-$F_{n-2}$=0 $h(n) – 3h(n-1)+2h(n-2)=0$

- Summary
  - Linear summation
  - RHS = 0
  - Coefficients are constants

**Def** If sequence $\{a_n\}$ satisifies:

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0,$$

$$a_0 = d_0, a_1 = d_1, \cdots, a_{k-1} = d_{k-1},$$

$C_1, C_2, \cdots, C_k$ and $d_0, d_1, \cdots, d_{k-1}$ are constants, $C_k \neq 0$, so this expression is called a k^th-order linear homogeneous recurrence relation of $\{a_n\}$.

$$h(n) = 2h(n-1) + 1, \quad h(1) = 1$$

$$\boldsymbol{a_n = a_{n-1} + a_{n-2}\, a_{n-3} \quad a_{n-1} = a_{n-2} = a_{n-3} = 1}$$

# Fibonacci Recurrence

$F_n=F_{n-1}+F_{n-2}$ $F_0=0, F_1=1$

Assume $\boxed{G(x)=F_1x+F_2x^2+\cdots}$ $\therefore\quad (1-x-x^2)G(x)=x$

$$\therefore\ G(x)=\frac{x}{1-x-x^2}=\frac{x}{\boxed{(1-\frac{1-\sqrt{5}}{2}x)(1-\frac{1+\sqrt{5}}{2}x)}}=\frac{A}{1-\frac{1+\sqrt{5}}{2}x}+\frac{B}{1-\frac{1-\sqrt{5}}{2}x}$$

*Factoring?*

$$(1-ax)^{-1}=1+ax+a^2x^2+\ldots..$$

$$(1-x-x^2)=(1-\frac{1-\sqrt{5}}{2}x)(1-\frac{1+\sqrt{5}}{2}x)$$

**(Factor Theorem)** If $a$ is a root of linear polynomial $f(x)$, which means $f(a)=0$, then polynomial $f(x)$ has a factor $x-a$.

**We need factor $(1-ax)$,**

If $a$ is a root of linear polynomial $f(x^{-1})$, which means $f(a)=0$, then polynomial $f(x^{-1})$ has a factor $x^{-1}-a=(1-ax)/x$.

$$(1-ax)^{-1}=1+ax+a^2x^2+\ldots..$$

$$(1-x-x^2)=(1-\frac{1-\sqrt{5}}{2}x)(1-\frac{1+\sqrt{5}}{2}x)$$

$(1-x-x^2)=x^2((x^{-1})^2-x^{-1}-1)=x^2((m)^2-m-1)$

Let $m=x^{-1}$ $\mathbf{C(m)=m^2-m-1=(m-\alpha)(m-\beta)}$

*Substitute* $m=x^{-1}$ *in, get* $F_n=F_{n-1}+F_{n-2}$

$F(x)=x^2(x^{-1}-\alpha)(x^{-1}-\beta)=(1-\alpha x)(1-\beta x)$

$$\alpha=\frac{-2}{1-\sqrt{5}}=\frac{1+\sqrt{5}}{2},$$

$$\beta=\frac{2}{1+\sqrt{5}}=\frac{1-\sqrt{5}}{2}$$

# Linear Homogeneous Recurrence Relation

- The recurrence expression of Fibonacci Sequence

$F_0 = 0, \quad F_1 = 1,$ $\quad \therefore \quad G(x) = \dfrac{x}{\boxed{1-x-x^2}} = \dfrac{x}{(1-\alpha x)(1-\beta x)}$

$\boxed{F_n = F_{n-1} + F_{n-2}}$

*Denominator becomes* $F(x) = x^2((x^{-1})^2 - x^{-1} - 1) = x^2((m)^2 - m - 1)$ Let $m = x^{-1}$

$C(m) = \boxed{m^2 - m - 1} = (m-\alpha)(m-\beta)$ $\quad \alpha = \dfrac{1+\sqrt{5}}{2}, \quad \beta = \dfrac{1-\sqrt{5}}{2}$

*Substitute* $m = x^{-1}$ *in, get*

$F(x) = x^2(x^{-1} - \alpha)(x^{-1} - \beta) = (1 - \alpha x)(1 - \beta x)$

$$G(x) = \frac{x}{(1-\frac{1-\sqrt{5}}{2}x)(1-\frac{1+\sqrt{5}}{2}x)} = \frac{A}{1-\frac{1+\sqrt{5}}{2}x} + \frac{B}{1-\frac{1-\sqrt{5}}{2}x}$$

- Recurrence expression of Hanoi Tower

$h(n) - 2h(n-1) = 1$ $\quad H(x) = \dfrac{x}{(1-x)(1-2x)} = \dfrac{x}{1-3x+2x^2}$

$h(n-1) - 2h(n-2) = 1$ Subtract and get $\boxed{C(x) = x^2 - 3x + 2}$

$\boxed{h(n) - 3h(n-1) + 2h(n-2) = 0}$

*Roots of* $C(x) = 0$ *are 1 and 2*

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0,$$

# Linear Homegeneous Recurrence Relation

Assume $G(x)$ is a generating function of $\{a_n\}$:

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$$

$$x^k (a_k + C_1 a_{k-1} + C_2 a_{k-2} + \cdots + C_k a_0) = 0$$

$$x^{k+1} (a_{k+1} + C_1 a_k + C_2 a_{k-1} + \cdots + C_k a_1) = 0$$

$$\cdots\cdots$$

$$x^n (a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k}) = 0$$

$$\cdots\cdots$$

Adds up both sides of these equations, get

$$G(x) - \sum_{i=0}^{k-1} a_i x^i + C_1 x \left( G(x) - \sum_{i=0}^{k-2} a_i x^i \right)$$

$$+ \cdots + C_k x^k G(x) = 0$$

# Linear Homegeneous Recurrence Realtion

$$\left(1+C_1x+C_2x^2+\cdots+C_kx^k\right)G(x)=\sum_{j=0}^{k-1}C_jx^j\sum_{i=0}^{k-1-j}a_ix^i$$

Let $P(x)=\sum_{j=0}^{k-1}C_jx^j\sum_{i=0}^{k-1-j}a_ix^i$, the order of polynomial $P(x) \le k-1$.

$$\boxed{(1-ax)^{-1}=1+ax+a^2x^2+.....}$$

$$G(x)=\frac{P(x)}{\left(1+C_1x+\cdots+C_kx^k\right)}$$

$$a_n+C_1a_{n-1}+C_2a_{n-2}+\cdots+C_ka_{n-k}=0,$$

$$\Downarrow$$

$$\boxed{C(m)=m^k+C_1m^{k-1}+\cdots+C_{k-1}m+C_k}$$

$m=x^{-1}$

$$C(m)=(m-a_1)^{k_1}(m-a_2)^{k_2}\cdots(m-a_i)^{k_i}$$

$$k_1+k_2+\cdots+k_i=k$$

$$=\frac{P(x)}{(1-a_1x)^{k_1}(1-a_2x)^{k_2}\cdots(1-a_ix)^{k_i}}$$

# Linear Homogeneous Recurrence Relation

$F_n$-$F_{n-1}$-$F_{n-2}$=0 $\qquad$ $h(n) – 3h(n-1)+2h(n-2)=0$

$x^2$-$x$-1=0 $\qquad$ $x^2$-3$x$+2=0

**Def** if sequence $\{a_n\}$ satisfies:

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0,$$

$$a_0 = d_0, a_1 = d_1, \cdots, a_{k-1} = d_{k-1},$$

$C_1, C_2, \cdots C_k$ and $d_0, d_1, \cdots d_{n-1}$ are constants, $C_k \neq 0$, then this expression is called a k^th^-order linear homogeneous recurrence relation of $\{a_n\}$.

$$\boxed{C(x) = x^k + C_1 x^{k-1} + \cdots + C_{k-1} x + C_k}$$

Characteristic Polynomial

$F_n$-$F_{n\text{-}1}$-$F_{n\text{-}2}$=0 $h(n) - 3h(n\text{-}1)+2h(n\text{-}2)=0$

**Def** if sequence $\{a_n\}$ satisfies:

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0,$$

$$a_0 = d_0, a_1 = d_1, \cdots, a_{k-1} = d_{k-1},$$

$C_1, C_2, \cdots C_k$ and $d_0, d_1, \cdots d_{n-1}$ are constants, $C_k \neq 0$, then this expression is called a k$^{th}$-order linear homogeneous recurrence relation of $\{a_n\}$.

$$\boxed{C(x) = x^k + C_1 x^{k-1} + \cdots + C_{k-1} x + C_k}$$

Characteristic Polynomial

# Linear Homegeneous Recurrence Relation

Now we discuss the calculation by situations

（1）Characteristic Polynomial $C(x)$ has distinct real roots

Assume $C(x)=(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)$

$G(x)$ could be simplified as

$$G(x)=\frac{l_1}{1-\alpha_1 x}+\frac{l_2}{1-\alpha_2 x}+\cdots+\frac{l_k}{1-\alpha_k x}$$

$$=\sum_{i=1}^{k}\frac{l_i}{1-\alpha_i x}$$

$$\boxed{a_n=\sum_{i=1}^{k} l_i\alpha_i^n}$$

$$\boxed{(1-ax)^{-1}=1+ax+a^2x^2+\ldots..}$$

$l_1, l_2, \cdots, l_k$ could be solved by

$$\begin{cases} l_1+l_2+\cdots+l_k=d_0 \\ l_1\alpha_1+l_2\alpha_2+\cdots+l_k\alpha_k=d_1 \\ \cdots\cdots \\ l_1\alpha_1^{k-1}+l_2\alpha_2^{k-1}+\cdots+l_k\alpha_k^{k-1}=d_{k-1} \end{cases}$$

in which $d_0, d_1 \ldots d_{k-1}$ are initial values of $a_n$

# Linear Homegeneous Recurrence Relation

- Fibonacci Sequence

$$F_0 = 0, \quad F_1 = 1,$$

$$\boxed{F_n = F_{n-1} + F_{n-2}}$$

**Characteristic Polynomial**

$C(x) = x^2 - x - 1 = (x-\alpha)(x-\beta)$

$$\alpha = \frac{1+\sqrt{5}}{2}; \beta = \frac{1-\sqrt{5}}{2}$$

$$F_n = A\alpha^n + B\beta^n$$

$$F_0 = 0, \quad F_1 = 1$$

$$\begin{cases} A + B = 0 \\ \frac{\sqrt{5}}{2}(A-B) = 1 \end{cases}$$

Linear Homogeneous Recurrence Relation

**Def** If sequence $\{a_n\}$ satisfies:

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0,$$

$$a_0 = d_0, a_1 = d_1, \cdots, a_{k-1} = d_{k-1},$$

$C_1, C_2, \cdots C_k$ and $d_1, d_2, \cdots d_{n-1}$ are constants.

**Characteristic Polynomial**

$$C(x) = x^k + C_1 x^{k-1} + \cdots + C_{k-1}x + C_k$$

1）Characteristic polynomial has distinct real roots, k different real roots

$$C(x) = (x-a_1)(x-a_2)\cdots(x-a_k)$$

$$a_n = l_1 a_1^n + l_2 a_2^n + \cdots + l_k a_k^n$$

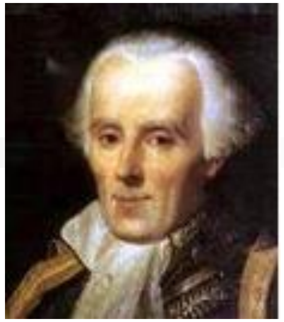
$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

# Generating Function

**Def 2-1** For sequence $a_0, a_1, a_2\ldots$, construct function

$$G(x)= a_0+a_1x+a_2x^2+\ldots,$$

$G(x)$ is the generating function of $a_0, a_1, a_2\ldots$



Laplae
1812 AD

***Seems to be functions but it's actually a mapping***

Berelli
1705- AD

Euler
1764- AD

**Recurrence** $a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0,$

$$\boxed{C(x) = x^k + C_1 x^{k-1} + \cdots + C_{k-1}x + C_k}$$

**Integer Segmentation**

**Generating function $G(x)$ as a bridge** $a_n = l_1 a_1^n + l_2 a_2^n + \cdots + l_k a_k^n$

# Linear Homogeneous Recurrence Relation

**Def** If sequence $\{a_n\}$ satisfies:

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0, \quad (2-5-1)$$

$$a_0 = d_0, a_1 = d_1, \cdots, a_{k-1} = d_{k-1}, \quad (2-5-2)$$

$C_1, C_2, \cdots C_k$ and $d_0, d_1, \cdots d_{k-1}$ are constants

**Characteristic Polynomial** $C(x) = x^k + C_1 x^{k-1} + \cdots + C_{k-1} x + C_k$

1）Characteristic polynomial has $k$ distinct real roots

$$C(x) = (x - a_1)(x - a_2)\cdots(x - a_k)$$

$$a_n = l_1 a_1^n + l_2 a_2^n + \cdots + l_k a_k^n$$

In which $l_1, l_2, \cdots l_k$ are undetermined coefficients.

$$(1+x)^{\alpha}=\sum_{n=0}^{\infty}\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n \qquad \alpha\in R \qquad \sum_{k=0}^{\infty}(k+1)x^k=\frac{1}{(1-x)^2}$$

- <u>Characteristic Polynomial has multiple roots</u>
- Eg $\quad a_n-4a_{n-1}+4a_{n-2}=0,\ \ a_0=1, a_1=4$
- Generating Function Method

$$x^2: a(2)=4a(1)-4a(0)$$
$$x^3: a(3)=4a(2)-4a(1)$$
$$+)\qquad \cdots\ \cdots\cdots\cdots\cdots$$

$$A(x)=\frac{1}{1-4x+4x^2}$$

$$A(x)=\frac{1}{1-4x+4x^2}=\frac{1}{(1-2x)^2}$$

$$=(1-2x)^{-2}=\sum_{k=0}^{\infty}C(k+1,k)2^k x^k$$

$$=\sum_{k=0}^{\infty}(k+1)2^k x^k$$

$$\boxed{a_n=(n+1)2^n}$$

Characteristic Equation Method

Characteristic Equation: $x^2$-4$x$+4=($x$-2)$^2$

Generating Function Form: $A(x)=\dfrac{ax+b}{(1-2x)^2}$

Partial Fractions: $A(x)=\dfrac{A}{(1-2x)}+\dfrac{B}{(1-2x)^2}$

$$a_n=A\times 2^n+B(n+1)2^n=(A'+Bn)2^n$$

$$a_0=A'=1,\quad a_1=(1+B)2=4$$

$$A'=1,\quad B=1$$

$$\boxed{a_n=(n+1)2^n}$$

$$G(x) = \sum_{i=1}^{t}\sum_{j=1}^{k_i}\frac{A_{ij}}{(1-\alpha_i x)^j}$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty}\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n \qquad \alpha \in R$$

## （2） Characteristic Polynomial $C(x)$ has multiple roots

Assume $\beta$ is a k-multiple root of C(x), it could be simplified as $\sum_{j=1}^{k}\frac{A_j}{(1-\beta x)^j}$

$x^n$'s coefficients $a_n = \sum_{j=1}^{k} A_j \binom{j+n-1}{n}\beta^n$, in which

$$\binom{j+n-1}{n} = \binom{j+n-1}{j-1}$$

is a *j-1*-order polynomial of n. So $a_n$ is the product of $\beta$ and a *k-1*-order polynomial of n. The term related to the solution of recurrence relation is:

$$(A_0 + A_1 n + \cdots + A_{k-1}n^{k-1})\beta^n$$

in which $A_0, A_1, \cdots, A_{k-1}$ are $k$ undetermined coefficients.

- Eg $a_n - 4a_{n-1} + 4a_{n-2} = 0, \quad a_0 = 1, a_1 = 4$

Characteristic Equation is： $x^2 - 4x + 4 = (x-2)^2$

$$a_n = (A_1 + A_2 n)2^n$$

$$a_0 = A_1 = 1$$

$$a_1 = (1 + A_2)2 = 4, \qquad A_2 = 1$$

$$a_n = (1 + n)2^n$$

**Distinct real roots** **Multiple real roots** **Conjugate complex roots?**

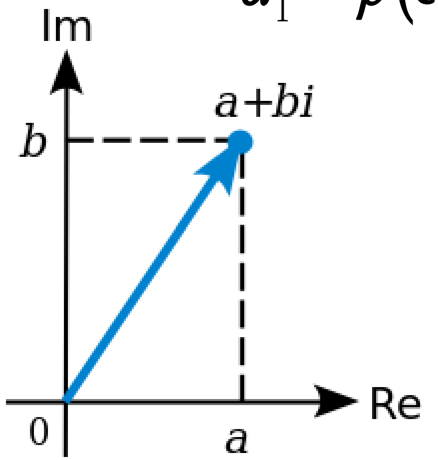
$$x^2 - x + 1 = 0$$

# Conjugate Complex Roots

- Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- When $b^2$-4$ac$<0, there's no real root, two complex roots.

$$x_{1,2} = \frac{-b \pm i \times \sqrt{4ac - b^2}}{2a}$$

$$\alpha_1 = \rho(\cos\theta + i\sin\theta), \alpha_2 = \overline{\alpha_1} = \rho(\cos\theta - i\sin\theta)$$




Trigonometrical form of complex number $z = a + bi$:

$z = \rho(cos\theta + i\ sin\theta)$

$$\rho = \sqrt{a^2 + b^2}$$

（3）Characteristic Polynomial $C(x)$ has conjugate complex roots

Assume that $a_1, a_2$ are a pair of conjugate complex roots of $C(x)$.

$$\alpha_1 = \rho(\cos\theta + i\sin\theta), \alpha_2 = \overline{\alpha_1} = \rho(\cos\theta - i\sin\theta)$$

In $\dfrac{A_1}{1-\alpha_1 x} + \dfrac{A_2}{1-\alpha_2 x}$ the coefficient of $x^n$ is:

$$A_1\alpha_1^n + A_2\alpha_2^n$$

$$
\begin{aligned}
& \boldsymbol{A_1\alpha_1^n + A_2\alpha_2^n} \\
& = (A_1 + A_2)\rho^n \cos n\theta + i(A_1 - A_2)\rho^n \sin n\theta \\
& = A\rho^n \cos n\theta + B\rho^n \sin n\theta
\end{aligned}
$$

In which $A = A_1 + A_2, \quad B = i(A_1 - A_2)$

When calculating in reality, we could solve the conjugate complex roots at first, then calculate undetermined coefficients $A, B$ to avoid the intermediate complex number calculations.

$$A_1\alpha_1^n + A_2\alpha_2^n = A\rho^n \cos n\theta + B\rho^n \sin n\theta$$

- Eg $a_n = a_{n-1} - a_{n-2}, a_1 = 1, a_2 = 0$

Characteristic equation： $x^2 - x + 1 = 0$

$$x = \frac{1 \pm \sqrt{-3}}{2} = \cos\frac{\pi}{3} \pm i \sin\frac{\pi}{3} = e^{\pm\frac{\pi}{3}i}$$

$$a_n = A_1 \cos\frac{n\pi}{3} + A_2 \sin\frac{n\pi}{3}$$

$$a_1 = \frac{1}{2}A_1 + \frac{\sqrt{3}}{2}A_2 = 1$$

$$a_2 = -\frac{1}{2}A_1 + \frac{\sqrt{3}}{2}A_2 = 0$$

$$\Rightarrow \quad A_1 = 1; \qquad A_2 = \frac{\sqrt{3}}{3}$$

$$a_n = \cos\frac{n\pi}{3} + \frac{\sqrt{3}}{3}\sin\frac{n\pi}{3}$$

# Summary of Linear Recurrence Relation

According to the non-zero roots of C($x$)

1) k distinct non-0 real roots $C(x)=(x-a_1)(x-a_2)\cdots(x-a_k)$

$$a_n = l_1 a_1^n + l_2 a_2^n + \cdots + l_k a_k^n$$

In which $l_1, l_2, \cdots, l_k$, are undetermined coefficients.

2)A pair of conjugate complex root $\alpha_1 = \rho e^{i\theta}$ and $\alpha_2 = \rho e^{-i\theta}$ :

$$\boldsymbol{a_n = A\rho^n \cos n\theta + B\rho^n \sin n\theta}$$

In which A，B are undetermined coefficients.

3)Has root $\alpha_1$ with multiplicity of k.

$$(A_0 + A_1 n + \cdots + A_{k-1} n^{k-1})\alpha_1^n$$

In which $A_0, A_1, \cdots, A_{k-1}$ are $k$ undetermined coefficients.

# 4 Linear Homogeneous Recurrence Relation

## 4-4 Applications

组合数学 Combinatorics

# Linear Homogeneous Recurrence Relation

Eg：Solve $S_n = \sum_{k=0}^{n} k$

$$S_n = 1 + 2 + 3 + \cdots + (n-1) + n$$

$$S_{n-1} = 1 + 2 + 3 + \cdots + (n-1)$$

$$\therefore \quad S_n - S_{n-1} = n$$

Similarly $S_{n-1} - S_{n-2} = n-1$

Subtract, get $S_n - 2S_{n-1} + S_{n-2} = 1$

Similarly $S_{n-1} - 2S_{n-2} + S_{n-3} = 1$

$$\therefore \quad S_n - 3S_{n-1} + 3S_{n-2} - S_{n-3} = 0$$

$$S_0 = 0, \quad S_1 = 1, \quad S_2 = 3$$

$$\therefore \quad S_n - 3S_{n-1} + 3S_{n-2} - S_{n-3} = 0$$

$$S_0 = 0, \quad S_1 = 1, \quad S_2 = 3$$

Corresponding Characteristic Equation is

$$m^3 - 3m^2 + 3m - 1 = (m-1)^3 = 0$$

$m = 1$ is a 3-multiple root

$$\therefore \quad S_n = (A + Bn + Cn^2)(1)^n = A + Bn + Cn^2$$

$$S_0 = 0, \quad \therefore \; A = 0$$

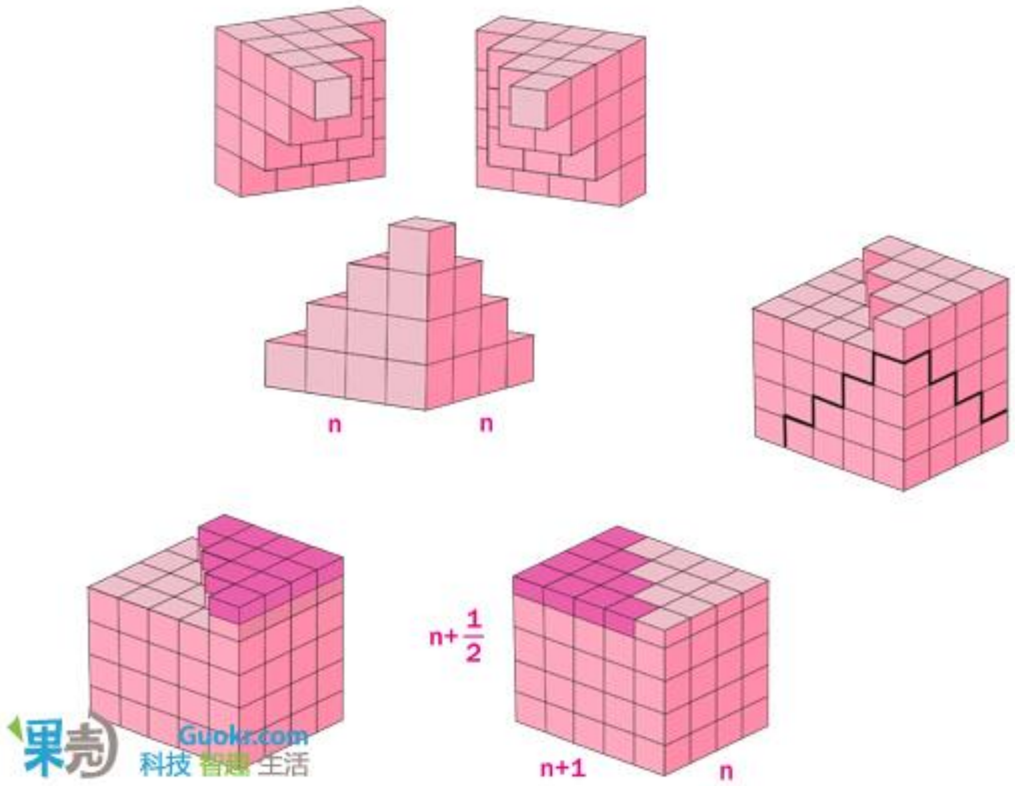
$$\boldsymbol{S_1 = 1, \quad B + C = 1}$$

$$\boldsymbol{S_2 = 3, \quad 2B + 4C = 3, \quad \therefore \; B = C = \frac{1}{2}}$$

So $\quad S_n = \frac{1}{2}n + \frac{1}{2}n^2 = \frac{1}{2}n(n+1)$

This proves $\quad 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$

$$1^2+2^2+3^2+\cdots+n^2=\frac{1}{3}n\left(n+\frac{1}{2}\right)(n+1)$$



# Linear Homogeneous Recurrence Relation

Eg2： Calculate $S_n = \sum_{k=0}^{n} k^2$

$S_n = 1 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 \qquad \therefore \quad S_n - S_{n-1} = n^2$

$S_{n-1} = 1 + 2^2 + 3^2 + \cdots + (n-1)^2$ Similarly $S_{n-1} - S_{n-2} = (n-1)^2$

Subtract, get $S_n - 2S_{n-1} + S_{n-2} = 2n - 1$

Similarly $S_{n-1} - 2S_{n-2} + S_{n-3} = 2(n-1) - 1$

Subtract, get $S_n - 3S_{n-1} + 3S_{n-2} - S_{n-3} = 2$

Similarly $S_{n-1} - 3S_{n-2} + 3S_{n-3} - S_{n-4} = 2$

$$\therefore \quad S_n - 4S_{n-1} + 6S_{n-2} - 4S_{n-3} + S_{n-4} = 0$$

$$S_0 = 0, \quad S_1 = 1, \quad S_2 = 5, \quad S_3 = 14$$

$$\therefore \quad S_n - 4S_{n-1} + 6S_{n-2} - 4S_{n-3} + S_{n-4} = 0$$

$$S_0 = 0, \quad S_1 = 1, \quad S_2 = 5, \quad S_3 = 14$$

Correspondent characteristic equation is:

$$r^4 - 4r^3 + 6r^2 - 4r + 1 = (r-1)^4 = 0$$

$r = 1$ is a 4-multiple root

$$\therefore \quad S_n = (A + Bn + Cn^2 + Dn^3)(1)^n$$

As $S_0 = 0, S_1 = 1, S_2 = 5, S_3 = 14$ we have a equation group about A、B、C、D:

$$\begin{cases} A = 0 \\ B + C + D = 1 \\ 2B + 4C + 8D = 5 \\ 3B + 9C + 27D = 14 \end{cases}$$

$$1^2+2^2+3^2+\cdots+n^2=\frac{1}{3}n\left(n+\frac{1}{2}\right)(n+1)$$

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{n \ge i > j \ge 1} (x_i - x_j).$$

=(4-3)(4-2)(4-1)(3-2)(3-1)(2-1)

=12

$$\begin{cases} A=0 \\ B+C+D=1 \\ 2B+4C+8D=5 \\ 3B+9C+27D=14 \end{cases}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix} = 12$$

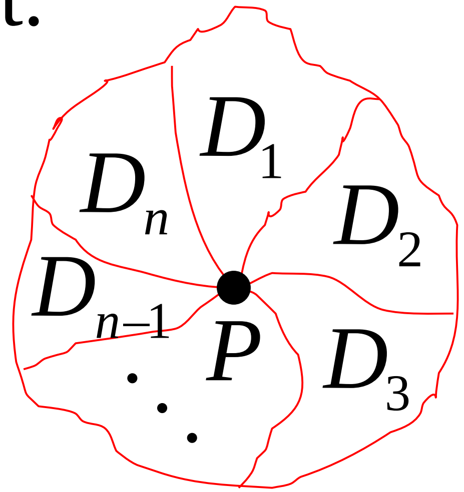
$$B = \frac{1}{12}\begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 8 \\ 14 & 9 & 27 \end{vmatrix} = \frac{1}{6}$$

$$C = \frac{1}{12}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 8 \\ 3 & 14 & 27 \end{vmatrix} = \frac{1}{2}$$

$$D = \frac{1}{12}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 3 & 9 & 14 \end{vmatrix} = \frac{1}{3}$$

## Applications of generating function and recurrence relation

**Eg:** There's a point P on the plane. It's the cross of n fields $D_1, D_2, \cdots D_n$. Color these $n$ fields with $k$ colors. We require the color of two adjacent areas to be different. Calculate the number of arrangements.

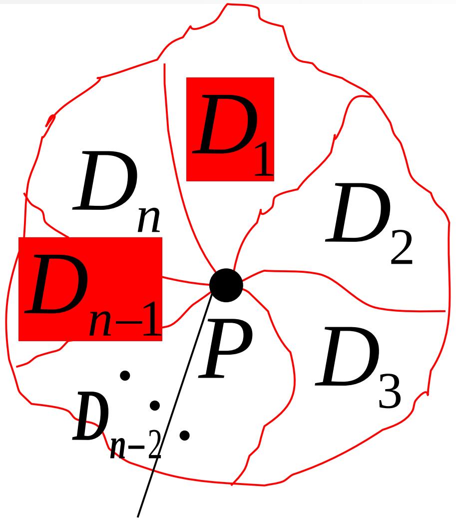



Let $a_n$ be the number of arrangement to color these areas. There are 2 situations:

# Applications of generating function and recurrence relation

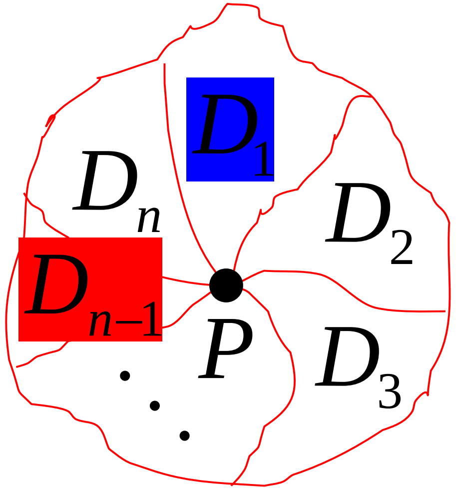
*(1)* $D_1$ and $D_{n-1}$ have the same color；

$D_n$ has $k$-1 choices, which is all colors except the one used by $D_1$ and $D_{n-1}$; the arrangements for $D_{n-2}$ to $D_1$的 are one-to-one correspondent to the arrangements for *n-2* areas. $(k-1)a_{n-2}$




(2) $D_1$ and $D_{n-1}$ have different colors.

$D_n$ has $k$-2 choices; the arrangements from $D_1$ to $D_{n-1}$ are one-to-one correspondent to the arrangements for *n-1* areas. $(k-2)a_{n-1}$




$$\therefore\ a_n = (k-2)a_{n-1} + (k-1)a_{n-2},$$

$$a_2 = k(k-1),\ a_3 = k(k-1)(k-2).$$

# Applications of generating function and recurrence relation

$$\therefore\ a_n = (k-2)a_{n-1} + (k-1)a_{n-2},$$

$$a_2 = k(k-1),\ a_3 = k(k-1)(k-2).$$

$$a_1 = 0, a_0 = k.$$

$$x^2 - (k-2)x - (k-1) = 0,$$

$$x_1 = k-1,\quad x_2 = -1.$$

$$a_n = A(k-1)^n + B(-1)^n.$$

$$\begin{cases} A + B = k, \\ (k-1)A - B = 0. \end{cases} \qquad \begin{cases} A = 1, \\ B = k-1. \end{cases}$$

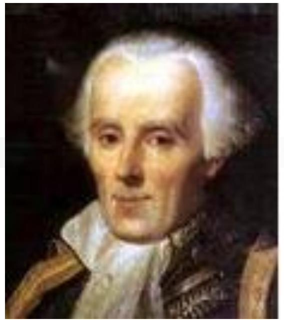
$$\therefore\quad a_n = (k-1)^n + (k-1)(-1)^n,\ n \geq 2.$$

$$a_1 = k.$$

**Def 2-1** For sequence $a_0, a_1, a_2\ldots$, construct a function

$$G(x)= a_0+a_1x+a_2x^2+\ldots,$$

Then $G(x)$ is called the generating function of $\boldsymbol{a_0, a_1, a_2\ldots\ldots}$



Laplace
1812 AD

Generating functions are a hanger to hang a serires of numbers.
— Herbert