Updated: September 30, 2014

# Effective Field Theory

Iain W. Stewart

EFT Course 8.851, Lecture Notes
Massachusetts Institute of Technology

PDF version:
http://courses.edx.org/c4x/MITx/8.EFTx/asset/notes_EFT.pdf


(These notes are preliminary and an updated version with further polishing will appear in the future)

# Contents


**1 Introduction** **3**

**2 Introduction to Effective Field Theory** **3**
- 2.1 Effective Field Theory Ideas . . . 3
- 2.2 Standard Model as an EFT . . . 8

**3 Tree level, Loops, Renormalization and Matching** **13**
- 3.1 A toy model . . . 13
- 3.2 Massive SM particles . . . 19

**4 Chiral Perturbation Theory** **26**
- 4.1 SU(2) ChPT . . . 26
- 4.2 SU(3) ChPT . . . 31

**5 Heavy Quark Effective Theory** **32**
- 5.1 Preliminary Treatment for Heavy Quark . . . 33
- 5.2 HQET radiative corrections . . . 36
- 5.3 Power Corrections and Reparametrization . . . 39
- 5.4 Renormalons . . . 42

**6 EFT with a Fine Tuning** **51**
- 6.1 Two Nucleon Nonrelativistic EFT . . . 51
- 6.2 Symmetries of NNEFT . . . 57
  - 6.2.1 Conformal Invariance for Nonrelativistic Field Theory . . . 57
  - 6.2.2 SU(4) spin-isospin symmetry – Wigner's SU(4) . . . 57
- 6.3 Deuteron . . . 58

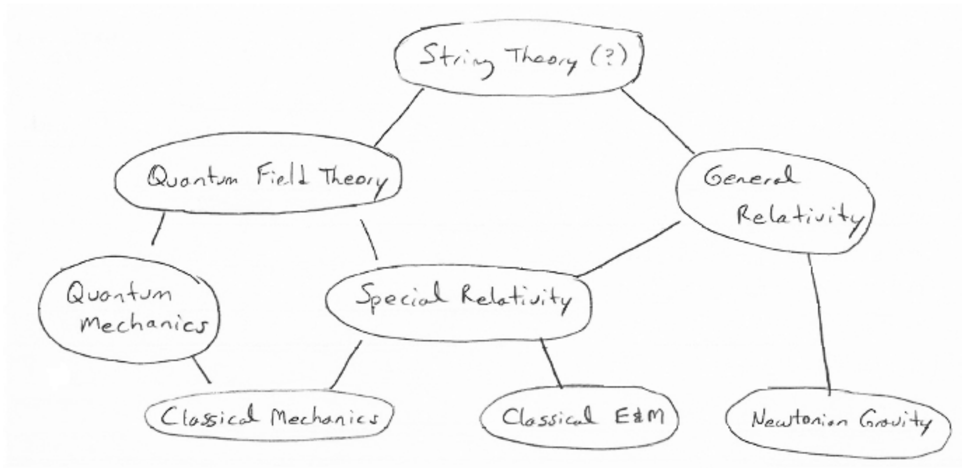
**A Introduction to the Standard Model** **60**
- A.1 U(1) gauge symmetry (QED) and SU(3) gauge symmetry (QCD) . . . 60
- A.2 The Standard Model $SU(3) \times SU(2) \times U(1)$ gauge symmetry . . . 61
- A.3 Symmetries of the Standard Model . . . 63

**B Renormalization Techniques** **65**
- B.1 Wilsonian point of view . . . 66
- B.2 Loops, Regularization, and Renormalization . . . 66
  - B.2.1 On-shell renormalization scheme . . . 66
  - B.2.2 Off-shell renormalization scheme ($\mu_R$) . . . 67
  - B.2.3 Relating schemes . . . 68
- B.3 Renormalization of QED . . . 68
  - B.3.1 QED: On-shell scheme . . . 69
  - B.3.2 QED: Off-shell momentum subtraction scheme . . . 69
  - B.3.3 QED: $MS$ scheme and $\bar{M}S$ scheme . . . 70
- B.4 QCD: renormalization, $\beta$-function, asymptotic freedom . . . 71
  - B.4.1 Background field method . . . 72
  - B.4.2 Heuristic explanation of asymptotic freedom . . . 74
- B.5 Asymptotic behavior and fixed points . . . 76

# 1 Introduction

These lecture notes provide reading material on Effective Field Theory following the course 8.851, which is taught as an advanced graduate course at MIT, and its EdX counterpart 8.EFTX.

The big picture is that there is interesting physics at all scales:




For most of your physics career you've been moving up this graph toward more and more general theories. As we move up, it becomes harder to compute (e.g. hydrogen energy Levels with quantum field theory rather than nonrelativistic quantum mechanics, elliptic orbits of planets with general relativity rather than Newtonian gravity). In this class we'll be going in the other direction - toward finding the simplest framework that captures the essential physics in a manner that can be corrected to arbitrary precision (e.g. an expansion in $v/c \ll 1$ to construct a nonrelativistic quantum field theory). This is the guiding principal of Effective Field Theory (EFT).

# 2 Introduction to Effective Field Theory

## 2.1 Effective Field Theory Ideas

To describe a physical system, the following questions should be addressed in order to design an appropriate quantum field theory, on both a technical and a physical level:

- Fields $\rightarrow$ Determine the relevant degrees of freedom.
- Symmetries $\rightarrow$ What interactions? Are there broken symmetries?
- Power counting $\rightarrow$ Expansion parameters, what is the leading order description?

These are the key concepts that arise when one wants to build an Effective Field Theory (EFT). Note that in an EFT the power counting is a very fundamental ingredient, it is just as important as something like gauge symmetry.

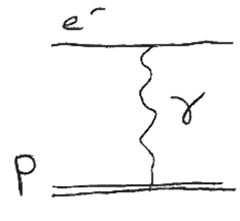
The key principle of EFT is that to describe the physics at some scale $m$, we do not need to know the detailed dynamics of what is going on at much higher energy scales $\Lambda_L \gg m$. This is good, since it allows us to focus on the relevant degrees of freedom and interactions, and therefore simplify calculations. On the other hand, this insensitivity to high energy scales implies that we must work harder (to higher precision) in order to probe short distance physics at low energies.

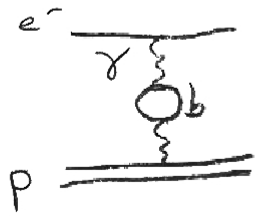
Let's exhibit some of the key concepts of an EFT with an example.

Example: We don't have to learn about bottom quarks to describe hydrogen. The hydrogen ground state binding energy is

$$E_o = \frac{1}{2} m_e \alpha^2 \left(1 + \mathcal{O}\Big(\frac{m_e^2}{m_b^2}\Big)\right) \quad , \quad \frac{m_e^2}{m_b^2} \sim 10^{-8} \tag{2.1}$$

so the correction from $b$-quarks enters as a tiny perturbation.




There is a subtlety here: $m_b$ does effect the electromagnetic coupling $\alpha$ in $\overline{\rm MS}$ since the coupling runs (e.g. $\alpha(m_W) \approx \frac{1}{128}$, $\alpha' = \alpha(0) \approx \frac{1}{137}$). More precisely, if $\alpha$ is a parameter of the Standard Model, which is fixed at high energy, then the low energy parameter $\alpha'$ that appears for hydrogen in Eq. (2.1) does depend on $m_b$. However, we can simply extract $\alpha' = \alpha(\Lambda_L^2)$ from low energy atomic physics at an energy scale $\Lambda_L$ and then use this coupling for other experiments and calculations at the same energy scale. In such an analysis no mention of $b$-quarks is required. We can summarize this by writing

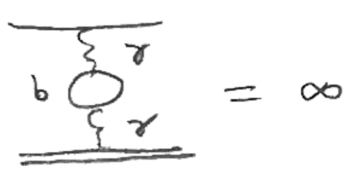
$$\mathcal{L}(p, e^-, \gamma, b; \alpha, m_b) = \mathcal{L}(p, e^-, \gamma; \alpha') + \mathcal{O}\Big(\frac{1}{m_b^2}\Big)$$

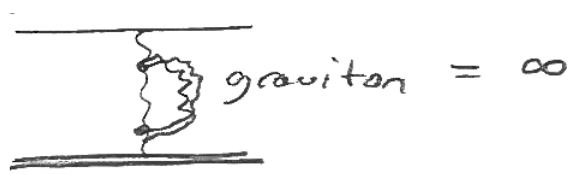
Beyond neglecting $b$-quarks and other heavy standard model particles, there are various expansions that go into our leading order description for the energy levels of hydrogen atom:

- Insensitive to quarks in the proton since $m_e\alpha \ll (\text{proton size})^{-1} \sim 200\text{MeV}$, so protons rather than quarks are the right degrees of freedom.
- Insensitive to the proton mass since $m_e\alpha \ll m_p \sim 1\,\text{GeV}$, so the proton acts like a static source of electric charge .
- A nonrelativistic Lagrangian $\mathcal{L}$ for $e^-$ suffices since $m_e\alpha \ll m_e$. Here $v_e = |\mathbf{p}_e|/m_e \sim \alpha \ll 1$.

Note that the typical momenta in the bound state are $\mathbf{p} \sim m_e\alpha$ and typical energies are $E \sim m_e\alpha^2$. The above conclusions hold despite the presence of UV divergences that appear when we consider various higher order terms induced by the above expansions.

Unregulated diagrams:



Such divergences are handled by the process of regulating and renormalizing the EFT. This procedure is needed to properly define the parametters in the EFT in the first place, and the divergences can even be exploited to track information that appears in a more complicated manner without the EFT framework.

In general, EFT's are used in two distict ways, either from the top-down or from the bottom-up:

i. Top down - Here the high energy theory is understood, but we find it useful to have a simpler theory at low energies.

- We *integrate out* (remove) heavier particles and *match* onto a low energy theory. This procedure yields new operators and new low energy couplings. More specifically, we expand the full Lagrangian as a sum of terms of decreasing relevance $\mathcal{L}_{high} \approx \sum_n \mathcal{L}_{low}^{(n)}$. The phrase "integrate out" comes from Kenneth Wilson and corresponds to explicitly integrating out the high energy field modes in the path integral formulation.
- The Lagrangians $\mathcal{L}_{high}$ and $\mathcal{L}_{low}$ will agree in the infrared (IR), but will differ in the ultraviolet (UV).
- The desired precision will tells us where to stop the expansion $\rightarrow$ how far we go with the sum on $n$.

Some examples of top-down EFT's are:

– Integrate out heavy particles, like the top quark, W, Z, and Higgs bosons from the Standard Model.

– Heavy Quark Effective Theory (HQET) for charm and bottom quarks at energies below their masses.

– Non-relativistic QCD or QED (NRQCD or NRQED) for bound states of two heavy particles.

– Soft-Collinear Effective Theory (SCET) for QCD processes with energetic hadrons or jets.

Note that for effective theories built from Quantum Chromodynamics (QCD), a separation of scales is needed to distinguish physics that is perturbative in the coupling $\alpha_s(\mu)$ evaluated at the scale $\mu = Q$ from effects that are non-perturbative in the coupling evaluated at a scale close to $\Lambda_{\rm QCD} \ll Q$.

Also note that the $\sum_n \mathcal{L}_{low}^{(n)}$ is an expansion in powers of the power counting parameter, but there are also logarithms which will appear with arguments that are the ratio of mass scales or the power counting parameter. In a perturbative EFT with a coupling like $\alpha_s$ the renormalization of $\mathcal{L}_{low}^{(n)}$ allows us to sum the large logs $\alpha_s \ln(\frac{m_1}{m_2}) \sim 1$ when $m_2 \ll m_1$. Indeed any logarithms that appear in QFT should be related to renormalization in some EFT.

ii. Bottom up - Here the underlying theory is unknown. In this bottom-up case we construct the EFT without reference to any other theory. Even if the underlying theory is known, we can also consider constructing the EFT from the bottom-up if the matching is difficult, for example if the matching would have to be nonperturbative in a coupling and hence is not possible analytically.

- Construct $\sum_n \mathcal{L}^{(n)}$ by writing down the most general set of possible interactions consistent with all symmetries, using fields for the relevant degrees of freedom.
- Couplings are unknown but can be fit to experimental or numerical data (e.g. lattice QCD)
- Desired precision tells us where to stop the expansion $\rightarrow$ How high do we go in the sum over $n$ before stopping.

Some examples of bottom-up EFT's are:

– Chiral Perturbation Theory for low energy pion and kaon interactions.

– The Standard Model (SM) of particle and nuclear physics.

– Einstein Gravity made Quantum with graviton loops.

**Comment:** The $\sum_n$ expansion is in powers, but there are also logs. Renormalization of $\mathcal{L}_{low}^{(n)}$ allows us to sum large logs $\ln\left(\frac{m_1}{m_2}\right)$ $(m_2 \ll m_1)$. It's true even when $m_1$ and $m_2$ are not masses particles - it's usually the case that logs in QFT are summed up with some EFT.

| Fermions | | mass | |
|---|---|---|---|
| quarks | $u_L, u_R$ | 1.5 - 3.3 MeV | |
| | $d_L, d_R$ | 3.5 - 6.0 | |
| | $s_L, s_R$ | 100 ± 30 | |
| | $c_L, c_R$ | 1.27 ± .09 | |
| | $b_L, b_R$ | 4.20 ± .12 | |
| | $t_L, t_R$ | 171200 ± 1000 | |
| leptons | $e_L, e_R$ | 0.511 | |
| | $\mu_L, \mu_R$ | 105.66 | |
| | $\tau_L, \tau_R$ | 1777 | |
| | $\nu_{eL}$ | $\Delta m^2_\odot \simeq 8 \times 10^{-5}\, eV^2$ | $[\nu_e \leftrightarrow (\nu_\mu, \nu_\tau)]$ |
| | $\nu_{\mu L}$ | $\Delta m^2_{atm} \simeq 2 \times 10^{-3}\, eV^2$ | $[\nu_\mu \to \nu_\tau]$ |
| | $\nu_{\tau L}$ | | |
| | sterile neutrinos $N_R$ ? | | |
| Other masses: | $m_\gamma = 0$ | $M_W = 80.42$ | |
| | $m_{gluon} = 0$ | $M_Z = 91.19$ | |

Figure 1: Fermion content of the Standard Model

## 2.2 Standard Model as an EFT

Lets look at the Standard Model of particle and nuclear physics as a bottom up EFT with $\sum_n \mathcal{L}^{(n)}_{low} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + ....$ The 0'th order is the Standard Model Lagrangian studied in MIT's 8.325 class: QFT III (refer to the supplement notes for more details). This Lagrangian already involves the relevant degrees of freedom. The gauge symmetry of the Standard Model is $SU(3)_{color} \times SU(2)_{weak} \times U(1)_Y$ with the following vector gauge boson content: (8 gluons $A^A_\mu$)× (3 weak bosons $W^a_\mu$)× (1 $U(1)$ boson $B_\mu$). The fermionic and bosonic content of SM is described in the table below (and further detail can be found from the Particle Data Group website at *http://ptg.lbl.gov*).

The question we would like to answer is *What is* $\mathcal{L}^{(1)}$*?* Before doing that lets review some things about the 0'th order term. The 0th order Lagrangian is $\mathcal{L}^{(0)} = \mathcal{L}_{gauge} + \mathcal{L}_{fermion} + \mathcal{L}_{Higgs} + \mathcal{L}_{N_R}$. Let us write down the first 2 terms explicitly ($Y$, $T^a$, $T^A$ are $U(1)$, $SU(2)$, $SU(3)$ representation):

$$\begin{aligned}
\mathcal{L}_{gauge} &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{4} W^{a\mu\nu} W^a_{\mu\nu} - \frac{1}{4} G^{A\mu\nu} G^A_{\mu\nu} \\
\mathcal{L}_{fermion} &= \sum_{\Psi_L} \bar{\Psi}_L i \not{D} \Psi_L + \sum_{\Psi_R} \bar{\Psi}_R i \not{D} \Psi_R \\
iD_\mu &= i\partial_\mu + g_1 B_\mu Y + g_2 W^a_\mu T^a + g_3 A^A_\mu T^A
\end{aligned}$$

The power counting for the SM as an EFT must be based on what we've left out: a new mass scale at the higher energy $\Lambda_{new}$. The expansion parameteter (power counting factor) should be a mass ratio of the form $\epsilon = \frac{m_{SM}}{\Lambda_{new}}$, where $m_{SM}$ is the particle mass in the SM (e.g. $m_W$, $m_Z$, $m_t$). Higher mass dimension operators $O_D$ (dimension $[O_D] = D > 4$) can be built out of SM degrees of freedom with couplings to the order of $\Lambda_{new}^{4-D}$.

Before moving on further, Let us review the meaning of renormalizability in the context of an EFT:

i. Traditional Definition - A theory is renormalizable if at any order of perturbation, divergences from loop integrals can be absorbed into a finite set of parameters.

ii. EFT Definition - A theory must be renormalizable order by order in its expansion parameters:

- This allows for an infinite number of parameters, but only a finite number at any order in $\epsilon$.

- If an $\mathcal{L}^{(0)}$ is traditionally renormalizable, it does not contain any direct information on $\Lambda_{new}$.

Next we will look at a simple example of renormalizability in an EFT. Example: Let us look at an example in a scalar field theory, in the case where mass dimension determines power counting. Consider a $d$ dimensional theory:

$$S[\phi] = \int d^d x \Big(\frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{\tau}{6!}\phi^6\Big)$$

From the definition of mass dimension, $\big[S[\phi]\big] = 0$ and $[x] = -1$. It is then straightforward to find $[\phi] = \frac{d-2}{2}$, $[m^2] = 2$, $[\lambda] = 4 - d$ and $[\tau] = 6 - 2d$. Assuming we want to study $< \phi(x_1)...\phi(x_n) >$ at large distance $x^\mu = s x'^\mu$ (controlling $s \to \infty$ while keeping $x'^\mu$ fixed - same value of $x'^\mu$ but cover more distance as $s$ goes up), then to normalize the kinetic term one can redefine the large distance scalar field to be $\phi'(x') = s^{\frac{d-2}{2}}\phi(x)$:

$$S'[\phi'] = \int d^d x' \Big(\frac{1}{2}\partial^\mu \phi' \partial_\mu \phi' - \frac{1}{2} m^2 s^2 \phi'^2 - \frac{\lambda}{4!} s^{4-d} \phi'^4 - \frac{\tau}{6!} s^{6-2d} \phi'^6\Big)$$

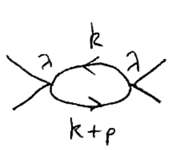
The correlation function:

$$< \phi(x_1)...\phi(x_n) >= s^{\frac{n(2-d)}{2}} < \phi'(x'_1)...\phi'(x'_2) >$$

Taking $d = 4$, as $s \to \infty$ we find $m^2$ becoming more and more important, $\lambda$ being equally important and the $\tau$ term becoming less important at large distance. Therefore, the operator $\phi^2$ is relevant since its mass dimension is $[\phi^2] < d$ as the coupling $[m^2] > 0$, the operator $\phi^2$ is marginal since its mass dimension is $[\phi^4] = d$ as the coupling $[\lambda] = 0$ and the operator $\phi^6$ is irrelevant since its mass dimension is $[\phi^2] > d$ as $[\tau] < 0$. Large distance means small momenta, therefore the energy scale decrease. If $m$ is the mass of a particle in a the theory at a high energy scale $\Lambda_E \gg m$, then the $\phi^2$ operator is a small perturbation, and in some sense can be neglected. In the low energy scale $\Lambda_E \ll m$, this term represents some non-perturbative description. Let $m \sim \Lambda_{new}$ be the mass of an unknown particle for a theory at a low energy scale $\Lambda_E \ll \Lambda_{new}$, then in terms of mass scale $m^2 \sim \Lambda_{new}^2$, $\lambda \sim \Lambda_{new}^0$ and $\tau \sim \Lambda_{new}^{-2}$. Since EFT looks toward the IR of the underlying theory, the mass term of the heavy particle will not be included. The $\phi^4$ and $\phi^6$ terms are included and they can usually be integrated out, leaving an EFT that contains only light degrees of freedom.

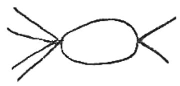
Note that relevant operators can upset power counting through kinetic terms (e.g. Higgs fine-tuning).

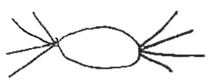
To demonstrate the ideas of traditional renormalization and EFT renormalization we will take $m = 0$ (or small $m$ such that $m^2 s^2 \sim 1$) and calculate the divergences through Feynman loop diagrams ($d = 4$ and cut-off $\Lambda$):




$$\sim \lambda^2 \int \frac{d^d k}{(k^2 - m^2 + io)((k+p)^2 - m^2 + io)} \sim \int_0^\Lambda \frac{d^d k}{k^4} \sim \ln \Lambda$$

This $\lambda\phi^4$ divergence renormalizes $\lambda$ by the counter-term  $\delta\lambda$

 $\sim \lambda\tau \ln \Lambda$ divergence renormalizes  $\tau\phi^6$

 $\sim \tau^2 \ln \Lambda$ divergence renormalizes  $\sim \phi^8$

Since $\phi^8$ is not included in $S[\phi]$ the theory is non-renormalizable in the traditional sense, but if $\tau \sim \Lambda_{new}^{-2}$ is small and $p^2\tau \ll 1$, the theory can be renormalized order by order in $\Lambda_{new}$ (to the non-positive power). From the above equation, the given scalar field theory is renormalizable up to $\Lambda_{new}^{-2}$. To have a renormalizable EFT up to $\Lambda_{new}^{-4}$, one should add a $\phi^8$ operator. In general, to include all corrections up to $\Lambda_{new}^{-r}$ (or $s^{-r}$, with $r$ non-negative), one has to consider all operators with mass dimension $\leq d + r$. This is an important relation between power counting and mass dimension.

*Although the above argument seems to be generic, can you think of what assumption might change that would lead to non-dimensional power counting? Hint: look at the properties of coordinates rescaling* $x^\mu \to sx'^\mu$

The SM Lagrangian $\mathcal{L}^{(0)}$ is renormalizable in the traditional sense, since all operators have mass dimension $\leq 4$. To get the $\mathcal{L}^{(1)}$ correction for the SM, one can add a mass dimension 5 operator $O_5$: $\mathcal{L}^{(1)} = \frac{c_5}{\Lambda_{new}} O_5$ with the $D = [O_5] = 5$ Wilson coefficient $c_5 \sim 1$ and $[c_5] = 0$ and $\Lambda_{new}$ explicit. Since nothing in $\mathcal{L}^{(5)}$ contains $\Lambda_{new}$, one is free to take $\Lambda_{new} \gg m_{SM}$. Indeed, from experimental data, $\mathcal{L}^{(1)}$ seems to give a very small corrections.

Let us now continue with the SM as an EFT and consider the corrections to $\mathcal{L}^{(0)} = \mathcal{L}^{SM}$ (e.g. for energy scale $\Lambda_E \sim m_t$). Toward the IR of the underlying theory:

$$\mathcal{L} = \mathcal{L}^{SM} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + ... = (\sim \Lambda_{new}^0) + (\sim \Lambda_{new}^{-1}) + (\sim \Lambda_{new}^{-2}) + ...$$

Assume Lorentz invariance and gauge invariance are still unbroken, then each $\mathcal{L}^{(n)}$ is Lorentz invariant and $SU(3) \times SU(2) \times U(1)$ invariant (take the Higgs vacuum expectation value to be $v = 246$ GeV, for the energy scale $\Lambda_E \gg v$ one can see the full gauge symmetry). These $\mathcal{L}^{(n)}$ should be constructed from the same degrees of freedom as $\mathcal{L}^{SM}$. Furthermore assume that no new particles are introduced at $\Lambda$. With that construction, one expects to see new physics from those corrections.

Example 1: $\mathcal{L}^{(1)} = \frac{c_5}{\Lambda_{new}} \epsilon_{ij} \bar{L}_L^{ci} H^j \epsilon_{kl} {L_L}^k H^l$ is the only $D = 5$ operator consistent with symmetry, with the Higgs doublet $H = \begin{pmatrix} h^+ \\ h^o \end{pmatrix}$ and the lepton doublet $L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$. As one can see, this Lagrangian is a singlet under $SU(3)$, $SU(2)$ and carries zero $U(1)$ hypercharge. Setting $H = \begin{pmatrix} 0 \\ v \end{pmatrix}$ gives the Majorana mass term for observed neutrinos $\frac{1}{2} m_\nu \epsilon_{ab} \nu_L^a \nu_L^b + h.c.$ with $m_\nu = \frac{c_5 v^2}{2\Lambda_{new}}$. From experimental data $m_\nu \leq 0.5(eV)$ so one expects the energy scale for new physics (new massive particle) to be around $\Lambda_{new} \geq 6 \times 10^{14}(GeV)$ as $c_5 \sim 1$. Note that the Majarana mass term in the Lagrangian violates lepton number conservation.

Example 2: $D = 6$ operators exist that violate baryon number conservation.

Example 3: With the number of leptons and baryons imposed there are 80 mass dimension 6 operators $\mathcal{L}^{(2)} = \Lambda_{new}^{-2} \sum_{i=1}^{80} c_{6i} O_{6i}$. For any observable only a few terms contribute and for any new theory at $\Lambda_{new}$ a particular pattern of $c_{6i}$ is predicted. Here's a reminder of SM charges as a reference:

| Fields | rep $SU(3)$ | rep $SU(2)$ | chg. $U(1)$ | Lorentz | |
|---|---|---|---|---|---|
| $Q_L^i = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix}$ | 3 | 2 | $1/6$ | $(\frac{1}{2}, 0)$ | spinors $\psi$ |
| $u_R^i$ | 3 | 1 | $2/3$ | $(0, 1/2)$ | |
| $d_R^i$ | 3 | 1 | $-1/3$ | $(0, 1/2)$ | |
| $L_L^i = \begin{pmatrix} \nu_L^i \\ e_L^i \end{pmatrix}$ | 1 | 2 | $-1/2$ | $(1/2, 0)$ | |
| $e_R^i$ | 1 | 1 | $-1$ | $(0, 1/2)$ | |
| $N_R^i$ | 1 | 1 | 0 | $(0, 1/2)$ | |
| $H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$ | 1 | 2 | $1/2$ | $(0, 0)$ | scalar |
| $A_A^\mu$ | 8 | 1 | 0 | $(1/2, 1/2)$ | vectors |
| $W_a^\mu$ | 1 | 3 | 0 | $(1/2, 1/2)$ | |
| $B^\mu$ | 1 | 1 | 0 | $(1/2, 1/2)$ | |

$i = 1, 2, 3$ family index $\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}$

$\begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}$

There are terms contributing to the muon anomalous magnetic moment $O_{6\omega} = \bar{L}_L \sigma^{\mu\nu} \sigma^a e_R H W^a_{\mu\nu}$ and $O_{6F} = \bar{L}_L \sigma^{\mu\nu} e_R H F_{\mu\nu}$. The contribution can be calculated to be $(g-2)_\mu =$ (contribution from $\mathcal{L}^{SM}$) $+ 4c\frac{m_\mu v}{\Lambda^2_{new}}$, and from it one extracts $\Lambda_{new} > 100(TeV)$ if $c \sim 1$. For the remaining operators, see W. Buchmuler, D. Wyler: Nucl. Phys. B268 (1986) p621-653 for details and Grzadkowski, B., et al. "Dimension-six terms in the standard model lagrangian." Journal of High Energy Physics 2010.10 (2010): 1-18 for a more up to date discussion.

When enumerating these operators, the classical equations of motion derived from $\mathcal{L}^{SM}$ can be used to reduce the number of operators - this is known as the integrating out at tree level (for more detail, see the papers mentioned above). This is obviously fine at lowest order since external lines are put on-shell in Feynman rules, and actually the same can be applied even with loops and propagators. To see this, consider the following theorems:

**i. Representation Independence Theorem**: Consider a scalar field theory and let $\phi = \chi F(\chi)$ with $F(0) = 1$ (so that one can Taylor expand the field around $\phi = 0$ with the leading term being $\phi = \chi$, which can be shown to be the 1-particle representation of quantized $\phi$ and quantized $\chi$ at the same time). Calculations of observables with $\mathcal{L}(\phi)$ and quantized field $\phi$ give the same results as with $\mathcal{L}'(\chi) = \mathcal{L}(\chi F(\chi))$ and quantized field $\chi$.

Example: Consider the $d = 4$ scalar field theory with $\eta \ll 1$ as the power counting factor:

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \lambda \phi^4 + \eta g_1 \phi^6 + \eta g_2 \phi^3 \Box \phi + \mathcal{O}(\eta^2)$$

The last term can be dropped from the equation of motion $\Box\phi + m^2\phi + 4\lambda\phi^3 + \mathcal{O}(\eta) = 0$ or by making a field redefinition $\phi \to \phi + \eta g_2 \phi^3$. The new Lagrangian is:

$$\mathcal{L}' = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \lambda' \phi^4 + \eta g_1' \phi^6 + \mathcal{O}(\eta^2)$$

Explicit computation of the 4-point and 6-point tree level Feynman graphs to $\mathcal{O}(\eta)$ with $\mathcal{L}(\phi)$ and quantized $\phi$ or $\mathcal{L}'(\chi)$ and quantized $\chi$ can be shown to give the same results. This holds even if one considers loops.

**ii. Generalized theorem** : Field redefinitions that preserve symmetries and have the same 1-particle states allow classical equations of motion to be used to simplify a local EFT Lagrangian without changing observables. For more detail regarging on-shell EFT, refer to C. Arzt: *hep-ph/9304230* and H. Georgi: Nucl. Phys. B361, p339-350 (1991).

A sketchy derivation for this theorem in a field theory with complex scalar $\phi$ can be shown as follows. Starting from $\mathcal{L}_{EFT} = \sum_n \eta^n \mathcal{L}^{(n)}$ ($\eta \ll 1$, as the power counting factor) consider removing a general first order term $\frac{1}{2}\eta T[\psi] D^2\phi$ from $\mathcal{L}^{(1)}$ that preserves symmetries of the theory, with $T[\psi]$ being a local function of various fields $\psi$ (basically, removing linear terms $D^2\phi$ in the EFT). The Green's function with sources $J$ can be obtained by functional derivatives of the partition function with respect to sources (one can see that with this approach, use of dimensional regularization is convenient):

$$Z[J] = \int \prod_i \mathcal{D}\psi_i \exp\Big(i\int d^dx\big(\mathcal{L}^{(0)} + \eta(\mathcal{L}^{(1)} - TD^2\phi) + \eta T D^2\phi + \sum_k J_k\psi_k + \mathcal{O}(\eta^2)\big)\Big)$$

Removing the term $\frac{1}{2}\eta T[\psi]D^2\phi$ is relevant to redefining the field $\phi^* = \phi'^* + \eta T$ in the path integral:

$$\begin{aligned} Z[J] \;&=\; \int \prod_i \mathcal{D}\psi'_i \frac{\delta\phi^*}{\delta\phi'^*} \exp\Big(i\int d^dx\big(\mathcal{L}^{(0)} + \frac{1}{2}\eta T(\frac{\delta\mathcal{L}^{(0)}}{\delta\phi^*} - \partial_\mu \frac{\delta\mathcal{L}^{(0)}}{\delta\partial_\mu\phi^*}) \\ &+\eta(\mathcal{L}^{(1)} - \frac{1}{2}TD^2\phi') + \frac{1}{2}\eta T D^2\phi' + \sum_k J_k\psi'_k + \frac{1}{2}J_{\phi^*}\eta T + \mathcal{O}(\eta^2)\Big) \end{aligned}$$

From here, one can see that there are 3 changes: the Lagrangian, the Jacobian and the source term $J_{\phi^*}$. The claim is that without changing the S-matrix, we can remove the change in Jacobian and the source, therefore we only need change of variable in $\mathcal{L}$:

$\delta\mathcal{L}$ needs $\phi^* + \eta T$ to transform like $\phi^*$, in order to respect the symmetries of the theory:

$$\begin{aligned} \mathcal{L} \;&=\; \frac{1}{2}(D^\mu\phi)^*(D_\mu\phi) - \frac{1}{2}m^2\phi^*\phi + (...) \\ &=\; \frac{1}{2}(D^\mu\phi')^*(D_\mu\phi) - \frac{1}{2}m^2\phi'^*\phi' + \frac{1}{2}\eta T(-D^2\phi' - m^2\phi') + (...)' \end{aligned} \tag{2.3}$$

The $-\frac{1}{2}\eta T D^2\phi'$ term from $\mathcal{L}^{(0)}$ after redefining the field cancels $\frac{1}{2}\eta T D^2\phi'$, as expected. Since the EFT Lagrangian at all orders $\eta$ contains all terms allowed by symmetries, all operators in $(...)'$ are already present in $(...)$ as the field redefinition also respects the symmetries. Thus couplings are simply redefined, and this poses no problem, since the values of couplings of an EFT aren't fixed. We therefore still have the same EFT.

The redefinition of $\phi$ differs from the original one at first order, so first order corrections of $\mathcal{L}^{(0)}$ (which are also symmetry-preserving) can be all absorbed into $\mathcal{L}^{(1)}$ couplings. $\mathcal{L}^{(1)}$ corrections go to higher orders in the Lagrangian, and terms linear in $D^2\phi$ can all be taken out from $\mathcal{L}^{(1)}$. Using the same idea, one might cancel $D^2\phi$ to any power out of $\mathcal{L}^{(1)}$ by replacing it using the equation of motion (because of the kinetic term, $D^2\phi$ should always be there in the theory). This is also relevant to redefining the fields.

Now let us turn our attention to the Jacobian. Recall that $det(\partial^\mu D_\mu) = \int \mathcal{D}c\mathcal{D}\bar{c}\exp\Big(i\int d^dx\bar{c}(-\partial^\mu D_\mu)c\Big)$ (Fadeev-Popov method) and write $\frac{\delta\phi^*}{\delta\phi'^*} = 1 + \eta\frac{\delta T}{\delta\phi'^*}$ which leads to $\bar{c}c + \eta\bar{c}\frac{\delta T}{\delta\phi'^*}c$ after including ghosts. Since the EFT is valid for the energy scale $\Lambda_E \ll \eta^{-\frac{1}{2}} (= \Lambda_{new})$, the ghosts will have mass $\sim \Lambda_{new}$ and hence decouple, just like other particles at this mass scale that were left out. Note that dropping ghosts can change the couplings.

Example: Consider $T = \Box\phi'^* + \lambda\phi'^*(\phi'^*\phi') \rightarrow \frac{\delta\phi^*}{\delta\phi'^*} = \bar{c}(1 + \eta\Box + 2\eta\lambda\phi'^*\phi)c$ and rescale $c \rightarrow c\eta^{-\frac{1}{2}}$ to have the correctly normalized kinetic term. It then becomes $\bar{c}(\eta^{-1} + \Box + 2\lambda\phi'^*\phi')c$, with the mass term of the ghost showing that it has a mass $\eta^{-\frac{1}{2}} = \Lambda_{new}$ as expected. Note that one needs a single $\phi'^*$ term in the field redefinition for this argument, which means that a term like $\phi'^* = \Box\phi^* + \lambda\phi'^*(\phi'^*\phi')$ would not be acceptable. Since ghosts always appear in loops, they can be removed like heavy particles and contribute some correction to the couplings.

We now look at the source term. Consider a Green's function of n-points scalar fields:

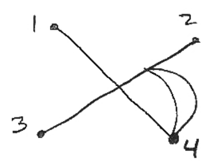
$$G^{(n)} = \langle 0| \mathcal{T}\Big(\phi(x_1)...\phi(x_n)...\Big) |0\rangle = \langle 0| \mathcal{T}\Big(\big(\phi'(x_1) + \eta T(x_1)\big)...\big(\phi(x_n) + \eta T(x_n)\big)...\Big) |0\rangle \ .$$

Here the ... on the right stand in for other fields and we use real $\phi$ for notational simplicity. The change of source can be shown to drop out of $\langle S\rangle$ from LSZ reduction (e.g. field rescaling and field renormalization cancellation, no pole, no contribution to the scattering):

$$\int \prod_i d^d x_i e^{ip_i x_i} \langle 0| \mathcal{T}\Big(\phi(x_1)...\phi(x_n)...\Big) |0\rangle \sim \Big(\prod_i \frac{i\sqrt{Z}}{p_i^2 + m_i^2 + io}\Big) \langle p_1 p_2 ...| S |p_j p_{j+1}...\rangle \Big|_{p_i^0 \rightarrow \sqrt{\mathbf{p}_i^2 + m_i^2}}$$

Example 1: Consider a scalar field theory with the field redefinition $\phi' = \phi + \eta\phi = (1 + \eta)\phi$ i.e. $T[\phi] = \phi\partial^2\phi$. The 4-point Green's function gives a prefactor $(1 + \eta)^4$ after redefining the field, and it's cancelled by the renormalization of the field $\sqrt{Z} = 1 + \eta$. This is the field redefinition and field renormalization cancellation.

Example 2: Consider a scalar field theory with the field redefinition $\phi' = \phi + \eta g_2 \phi^3$ i.e. $T[\phi] = g\phi^3$. The four point function will get extra terms, for example, the corrections coming from Feynman diagrams similar to:




$$= \eta g \langle 0| \mathcal{T}\Big(\phi'(x_1)\phi'(x_2)\phi'(x_3)\phi'^3(x_4)\Big) |0\rangle$$

Here, the $\phi^3$ doesn't give a single particle pole at $x_4$, so it has no contribution for scattering (external fields are taken on-shell), which means that the S-matrix stays the same after the field redefinition.

Example 3: Consider a scalar field theory with field redefinition $\phi = \phi' + \partial^2\phi' = \phi + (\partial^2 + m^2)\phi - m^2\phi$. The second term gives no pole and therefore not contribute to the scattering, and the 3rd term can be treated in the same way as in example 1 above.

# 3 Tree level, Loops, Renormalization and Matching

## 3.1 A toy model

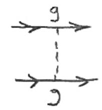
To demonstrate the ideas behind the matching technique through tree level, loops and renormalization with a simple calculation, consider a toy model with a heavy real scalar $\phi$ of mass $M$ and a light fermion $\psi$ of mass $m$. The Lagrangian (call it theory 1) can be written down as:

$$\mathcal{L}_1 = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}M^2\phi^2 + g\phi\bar{\psi}\psi$$

If the energy scale of interest $\Lambda_E \ll M$ one can integrate out the massive degrees of freedom $\phi$ and build a new theory of the light fermion $\psi$ alone. Interaction terms in the theory of fermions (call it theory 2), can be written down under the requirement of preserving the gauge symmetry:

$$\mathcal{L}_2 = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{c_1}{M}\bar{\psi}\Box\psi + \frac{c_2}{M^2}\bar{\psi}\psi\bar{\psi}\psi + \frac{c_2'}{M^2}\bar{\psi}\partial\!\!\!/\Box\psi + \frac{c_3}{M^3}\bar{\psi}\Box^2\psi + \frac{c_4}{M^4}\bar{\psi}\psi\bar{\psi}\Box\psi + ...$$

Let us look at the 4-point fermion interaction term in theory 2. One can immediately do the matching to tree level (loops contribute corrections, and one should do the matching at relevant orders) of theory 1 and find that $c_2 = -c_4 = g^2$:




$$= (ig)^2 \frac{i}{q^2 - M^2} \approx \frac{ig^2}{M^2} + \frac{ig^2 q^2}{M^4} + ...$$

Since the scalar propagator always comes with an even power contribution of $\frac{1}{M}$, we can see that $c_1 = c_3 = 0$ from tree level matching. Indeed, to match $c_1$, $c_3$ and even $c_2'$ one should go through detailed calculations with loops.

Another way to see this is using the classical equation of motion for $\phi$ to simplify the theory:

$$\phi = g\bar{\psi}\frac{1}{\Box + M^2}\psi \approx g\bar{\psi}(\frac{1}{M^2} - \frac{\Box}{M^4} + ...)\psi$$

Plug this back into the Lagrangian and we find the same results for the Wilson coefficient $c$ from matching at tree level.

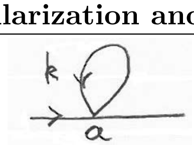
Calculations with loops require a lot more care. Let us briefly review some important concepts:

- Regularization is the technique to cut-off UV divergences in order to obtain finite results. Different regularization methods introduce different cut-off parameters (e.g. hard cut-off $\Lambda_{UV}^2$, dimensional regularization $d \to d - 2\epsilon$, lattice spacing).
- Renormalization is the technique used to pick a scheme to give definite meaning to each coefficient and operator of the QFT. It might also introduce some renormalization parameter (e.g. $\mu$ in $\overline{MS}$, $p^2 = -\mu_R^2$ for off-shell subtraction scheme, $\Lambda$ for Wilsonian). The relation between bare $a^{\text{bare}}$, renormalized $a^{\text{ren}}$ and counter-term $\delta a$ coefficients $a$ in different renormalization schemes (UV cut-off with integrated momenta $p$, $\Lambda_{UV} \leq |p|$ and $\overline{MS}$ dimensional regularization) are related:

$$a^{\text{bare}}(\Lambda_{UV}) = a^{\text{ren}}(\Lambda) + \delta a(\Lambda_{UV}, \Lambda) \quad , \quad a^{\text{bare}}(\epsilon) = a^{\text{ren}}(\mu) + \delta a(\epsilon, \mu)$$

Let us now show the difficulties with loop calculations and renormalization of coefficients:

**i. Regularization and Power Counting:** Consider in $d = 4$ the self interaction and mass correction in theory 2




which corrects the fermion mass at order $\frac{m^2}{M^2}$ by $\Delta m \sim \frac{ic_2}{M^2}\int \frac{d\!\!{}^-{}^d k(\gamma^\mu k_\mu + m)}{k^2 - m^2 + io} = \frac{c_2 m}{M^2}\int \frac{d\!\!{}^-{}^4 k_{\mathbb{E}}}{k_{\mathbb{E}}^2 + m^2}$ (using a Wick rotation from Lorentzian to Eulidean signature $k \to k_{\mathbb{E}}$). Before performing the calculation, note that the higher dimension operator (which is suppressed at the low energy scale), should give rise to a small correction. The physical part of the intergation $\int \frac{d^4 k_{\mathbb{E}}}{k_{\mathbb{E}}^2 + m^2}$ (at the very same order) at that energy scale should be insensitive to $M$ (since the contribution from that mass scale will disturb power counting), and from dimensional analysis one can guess $\int \frac{d^4 k_{\mathbb{E}}}{k_{\mathbb{E}}^2 + m^2} \sim m^2$, as the small correction must be $\Delta m \sim \frac{am^3}{M^2}$ with $k_{\mathbb{E}} \sim m$ domination. Doing the math in different regularization schemes gives:

- UV cut-off with $\Lambda_{UV} \sim M$, since one excludes the physics at around that energy scale. This regularization breaks Lorentz symmetry by imposing a hard cut-off momentum:

$$\frac{c_2 m}{M^2}\int_0^{\Lambda_{UV}} \frac{đ^4 k_{\mathbb{E}}}{k_{\mathbb{E}}^2 + m^2} = \frac{c_2 m}{(4\pi)^2}\Big(\frac{\Lambda_{UV}^2}{M^2} + \frac{m^2}{M^2}\ln\big(\frac{m^2}{\Lambda_{UV}^2}\big) - \frac{m^4}{M^2\Lambda_{UV}^2} + ...\Big)$$

  The first term in the bracket is in the power counting order of $\mathcal{O}(1)$ is dominated by $k_{\mathbb{E}} \sim \Lambda_{UV}$, which is not a correction coming out of the expected order. If one tries a normal way of absorbing the physics from energy scale $\Lambda$ to $\Lambda_{UV}$ with a piece $\frac{c_2 m}{M^2}\int_0^{\Lambda_{UV}} \frac{đk_{\mathbb{E}}}{k_{\mathbb{E}}^2+m^2}$ in the fermion mass counter-term $\delta m(\Lambda_{UV}, \Lambda)$ to improve things, terms with different orders $\frac{\Lambda^2}{m^2}$ and $\ln\big(\frac{m^2}{\Lambda^2}\big)$ will be present in 4-point fermion interaction renormalization $\langle(\bar{\psi}\psi)^2\rangle^{\mathrm{ren}}(\Lambda)$. To recover the right power counting, a counter-term should be introduced to absorb the whole $\mathcal{O}(1)$ term instead. In this regularization the power counting only applies to renormalized couplings and operators order by order, power counting breaks down (the power counting factor $\frac{m}{M}$ is incorrect because of the mass dependence of the regulator $\Lambda_{UV}$).

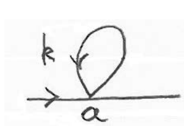
- $\overline{MS}$ with dimensional regularization $d = 4 - 2\epsilon$:

$$\frac{c_2 m}{M^2}\int_0^{\infty} \frac{đ^d k_{\mathbb{E}}}{k_{\mathbb{E}}^2 + m^2}\Big|_{d\to 4-2\epsilon}^{\epsilon,\mu} = \frac{c_2 m}{(4\pi)^2}\Big(\frac{m^2}{M^2}\big(-\frac{1}{\epsilon} + \ln\big(\frac{m^2}{\mu^2}\big) - 1\big) + \mathcal{O}(\epsilon)\Big)$$

  The first term can be absorbed into the $\overline{\mathrm{MS}}$ counter-term, and note that it can be related to a similar term in UV cut-off regularization when $\epsilon = -\frac{\Lambda_{UV}}{m^2}$. The second term inside the bracket has the same log behavior with the similar term in UV cut-off regularization if $\mu = \Lambda_{UV}$. The third term inside the bracket corresponds to the domination of the integration around the Lorentzian pole $k^2 \sim m^2$, giving a small correction $\Delta m \sim \frac{c_2 m^3}{M^2}$. The regularization does not break the power counting (the $\frac{m^2}{M^2}$ term is still there, in front of a divergent term and non-divergent terms, keeping track of orders) because the regulator doesn't depend on the mass scale $M$ (the infinitesimal dimensionless $\epsilon$), and one can still do power counting.

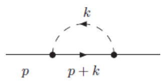
In principle any regulator is acceptable, but if one can choose the regulator to preserve symmetries (e.g. gauge invariance, Lorentz symmetry, chiral symmetry) and also preserve power counting by not yielding a mixing of terms of different orders in the expansion, the calculations become easier because, in general, operators will always mix with other operators of the same dimension and same quantum numbers (with a matrix of counter-terms). For power counting, this corresponds to using mass-independent regulators (strictly speaking, a new mass scale may still appear but in a way that doesn't directly change the power counting factor, and it's mass-independent in the sense that it doesn't see the thresholds of particles' masses in the theory). If the regulator doesn't have these desired properties (e.g. Supersymmetry is broken by dimensional regularization), one can still use counter-terms to restore symmetries and power counting, therefore simplifying the calculations.

Now Let us do the matching explicitly with 1 loop. Consider the self interaction diagram in theory 2 and absorb the mismatch with theory 1 by redefining the field and mass with $\delta Z_\psi$ and $\delta m$ (counter-terms). Note that the last term can be added into the mass counter-term, too:




$$= (1+\delta Z_\psi)\not{p} - (m+\delta m) - \frac{c_2 m^2}{16\pi^2 M^2}\big(1 - \ln\frac{m^2}{\mu^2}\big)$$

A loop diagram in theory 1 that is relevant to the 1-loop self interaction in theory 2 should be used (the calculation is taken in the limit when $M \gg p_{\mathbb{E}} \sim m$):




$$= \not{p} - m - \frac{g^2}{16\pi^2}\Big(\not{p}(\frac{1}{2}\ln\frac{M^2}{\mu^2} - \frac{1}{4} + \frac{m^2}{2M^2} - \frac{p^2}{6M^2}) + m(\ln\frac{M^2}{\mu^2} - 1 - \frac{m^2}{M^2}\ln\frac{m^2}{M^2} - \frac{p^2}{2M^2})\Big)$$

The terms in this loop calculation of theory 1 with $p^2$ and $\not{p}p^2$ from the point of view in theory 2 come from the interaction $c_1$ and $c_2'$ and therefore can be matched to give non-zero results. $c_3$ can also be matched if one expands further to $p^4$. Note that the matching shows that $c_1$ and $c_3$ are in the first order of power counting $\frac{m}{M}$, so instead of redefining them to become operators at higher order of the form $m\bar{\psi}\Box\psi$ and $m\bar{\psi}\Box^2\psi$ which would make the Wilson coefficients be of order 1, one should think of the old Wilson coefficients are zeroes plus some correction. The rest can be matched to the self-interaction calculation in theory 2, with $\delta Z_\psi$ and $\delta m$ defined as:

$$\begin{aligned}
\delta Z_\psi &= -\frac{g^2}{16\pi^2}(\frac{1}{2}\ln\frac{M^2}{\mu^2} - \frac{1}{4} + \frac{m^2}{2M^2}) = -\frac{c_2}{16\pi^2}(\frac{1}{2}\ln\frac{M^2}{\mu^2} - \frac{1}{4} + \frac{m^2}{2M^2}) \\
\delta m &= \frac{g^2}{16\pi^2}(\ln\frac{M^2}{\mu^2} - 1) = \frac{c_2}{16\pi^2}(\ln\frac{M^2}{\mu^2} - 1)
\end{aligned}$$

The final part of the matching is then $\frac{m^2}{M^2}\ln\frac{m^2}{M^2} = \frac{m^2}{M^2}\ln\frac{m^2}{\mu^2}$, giving the unphysical mass scale in dimensional regularization a physical meaning, $\mu = M$, the mass of the heavy particle in the theory.

Example: Dimensional regularization in supersymmetric field theory breaks supersymmetry, and the counter-term is usually chosen in a way that the renormalized results are supersymmetric.

**ii. More on Dimensional Regularization:**

Some useful axioms:

- Linearity (complex numbers $a$, $b$): $\int \bar{d}^d p\Big(af(p) + bg(p)\Big) = a\int \bar{d}^d p + n\int \bar{d}^d p$
- Translation (vector $q$): $\int \bar{d}^d p f(q+p) = \int \bar{d}^d p f(p)$, and also Rotation
- Scaling (complex number $s$): $\int \bar{d} f(sp)^d = s^{-d}\int \bar{d}^d p f(p)$

These 3 axioms together give a unique definition to the integration up to normalization: dimensional reqularization (see Collins p. 65 for ruther details of the proof).

In Euclidean space: $d^d p = dp p^{d-1} d\Omega_d = dp p^{d-1} d\cos\theta \sin^{d-3}\theta d\Omega_{d-1}$ with $\int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$

For spherical symmetric intergration $\bar{d}^d p = \frac{p^{d-1}2}{(4\pi)^{\frac{d}{2}}}\Gamma(\frac{d}{2})dp$

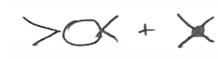
Common integration: $\int \bar{d}^d p \frac{(p^2)^\alpha}{(p^2+A)^\beta} = \frac{1}{(4\pi)^{\frac{d}{2}}} A^{\frac{d}{2}+\alpha-\beta}\frac{\Gamma(\beta-\alpha-\frac{d}{2})}{\Gamma(\beta)}\frac{\Gamma(\alpha+\frac{d}{2})}{\Gamma(\frac{d}{2})}$, $\int \bar{d}^d p(p^2)^\alpha = 0$ (see Collins p71)

Feynman's parametric integration formula: $\frac{1}{A^a B^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 dx \frac{x^{a-1}(1-x)^{b-1}}{(Ax+B(1-x))^{a+b}}$

Dimensional regularization introduces $d = 4 - 2\epsilon$, where $\epsilon > 0$ will tame the UV and $\epsilon < 0$ will tame the IR (the sign does not depend on the sign of the divergence, it's just a convention). The results are always expressed using Gamma functions of the form $\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\Big(\frac{1}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon)\Big)$.

Example: The Euclidean $I_d(q,n)$ integration with dimensional regularization:

$$I_d(q,n) = \int \frac{d^d p}{(p^2 + 2pq + m^2)^n} = \frac{(-1)^{2-n}}{(2-n)!}\frac{i\pi^{\frac{d}{2}}}{\Gamma(n)}(m^2 - q^2)^{\frac{d}{2}-n}\Big(\frac{1}{\epsilon} + \psi(3-n) + \mathcal{O}(\epsilon)\Big)$$

Note that for the diagram of the massless scalar field loop  $= \int \bar{d}^d p p^{-4} = \frac{i}{16\pi^2}(\frac{1}{\epsilon_{UV}} - \frac{i}{\epsilon_{IR}}) = 0$ at $d = 4$ ($\epsilon_{UV} = \epsilon_{IR}$), since the counter-term (always needed when there's a UV divergence) is only meant to cancel the UV divergence for physics at a high energy scale  $= -\frac{i}{16\pi^2}\frac{1}{\epsilon_{UV}}$ so  $= -\frac{i}{16\pi^2}\frac{1}{\epsilon_{IR}}$
Dimensional regularization is well-defined even with both UV and IR divergences by separating the UV and IR poles, using analytic continuation.

Example: Consider a well-defined spherical integral in the dimensional range $0 \leq d \leq d_{max}$, to extend to the lower limit $-2 \leq d \leq d_{max}$. First of all one can split these UV and IR parts by using the scale $c$:

$$\int_0^\infty dp p^{d-1} f(p^2) = \int_c^\infty dp p^{d-1} f(p^2) + \int_0^c dp p^{d-1}\big(f(p^2) - f(0)\big) + \frac{c^d}{d} f(0)$$

For $-2 \leq d < 0$, take the above equation as an analytic continuation and using dimensional regularization differently for these UV and IR parts with different values of $d$, then put them back together and get the final result after regulating the divergent poles that should be independent of the scale $c$. Take $c \to \infty$, then the integration can be simplified to $\int_0^\infty dp p^{d-1}\big(f(p^2) - f(0)\big)$.

Now, let us look into renormalization after dimensional regularization:

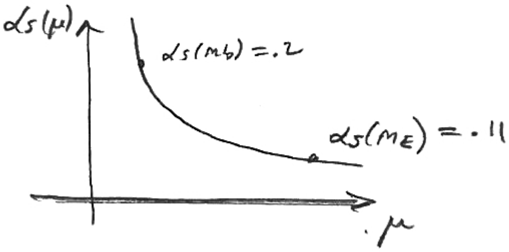
- MS scheme: a mass scale $\mu$ is introduced in order to keep any renormalized couplings dimensionless.

  Example: Consider the gauge coupling interaction $g^{\text{bare}}\bar{\psi}A\!\!\!/\psi$. At $d = 4$ one has $[g^{\text{bare}}] = \frac{4-d}{2} = \epsilon$ in dimensional regularization. In term of the dimensionless renormalized coupling and the dimensionless renormalized factor, the bare coupling should be equal to $Z_g \mu^\epsilon g(\mu)$ as $g(\mu)$ depends on the chosen mass scale $\mu$. Note that the $\mu^\epsilon$ factor is not associated with loop measure.

- $\overline{MS}$ scheme: The chosen mass scale is slightly different from the MS scheme $\mu^2 \to \mu^2 e^{\frac{\gamma_E}{4\pi}}$ so that the large universal constant is removed. The advantages of this scheme are that it preserves symmetries, it is technically easy to calculate multiple loops and often gives manifest power counting. The disadvantages are that the physical picture becomes less clear (e.g. we lose positive definiteness for renormalized quantities), it can introduce renormalons (poor convergences) at large orders in perturbation theory and it does not satisfy the decoupling theorem .

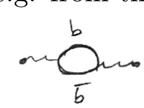
**Decoupling Theorem:** Consider building an EFT by integrating out the massive fields. If the remaining low energy theory is renormalizable and we use a physical renormalization scheme (e.g. off-shell momentum subtraction), then all effects due to heavy particles (of mass scale $M$) appear as a change in the couplings or are suppressed as $\frac{1}{M}$. Since the $\overline{MS}$ scheme is not physical, because it is mass independent (doesn't see the mass threshold), one must implement the decoupling argument of the theory by hand, removing particles of mass $M$ for $\mu leq M$.

Example: The $\overline{MS}$ scheme of QCD has $\beta(g) = \mu\frac{d}{d\mu}g(\mu) = -\frac{g^3}{16\pi^3}b_o + \mathcal{O}(g^5) < 0$ with $b_o = \frac{11}{3}c_A - \frac{4}{3}n_F T_F$. The QCD fine structure constant $\alpha_s(\mu) = \frac{g^2(\mu)}{4\pi}$ then runs as $\alpha_s(\mu) = \frac{\alpha_s(\mu_o)}{1+\alpha_s(\mu)\frac{b_o}{2\pi}\ln\frac{\mu}{\mu_o}}$ from the lowest order solution, which behaves assymptotically free. Define an intrinsic mass scale $\Lambda_{QCD}^{\overline{MS}} = \mu\exp(-\frac{2\pi}{b_o\alpha_s(\mu)})$ (by replacing $\alpha_s(\mu)$ one can show that it is independent of the choice for $\mu$) to get the nice form $\alpha_s(\mu) = \frac{2\pi}{b_o \ln(\mu/\Lambda_{QCD}^{\overline{MS}})}$, which specifies the energy scale when QCD becomes non-perturbative ($\sim 200(MeV)$). Note that $\Lambda_{QCD}^{\overline{MS}}$ depends on $b_o$ (and thus on the number of light fermionic flavors $n_F$), on the order of loop expansion for $\beta(g)$ and also on the renormalization scheme (beyond 2 loops).



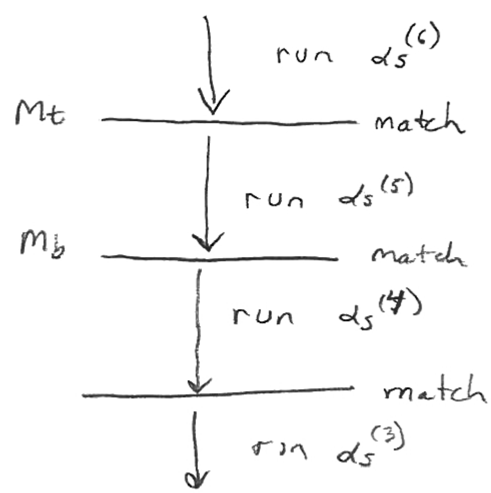

The problem comes from heavy quarks (e.g. top, bottom) contributing to $b_o$ for any $\mu$ from the point of view of the unphysical $\overline{MS}$ renormalization scheme and that contradicts the decoupling theorem at low energy scale compare to these quarks' masses. Therefore decoupling should be implemented by hand by integrating out and changing the fermion number $n_F$ (allowed in the loop, effectively) as $\mu$ gets through a quark mass threshold. Specifically, $n_F = 6$ for $\mu > m_t$, $n_F = 5$ for $m_b < \mu < m_t$ and so on.

The matching condition (perturbative diagrams and couplings) between effective theories after removing the heavy degrees of freedom should be based on the characteristics of the S-matrix (not the couplings): at the transition mass scale $m(\sim \mu_m = \mu^{(1)} = \mu^{(2)})$ the S-matrix elements with light external particles should agree between theory 1 and 2. The leading order condition for couplings (which makes them continuous at the mass threshold) can be shown to be $\alpha_s^{(1)}(\mu_m) = \alpha_s^{(2)}(\mu_m)$.

Consider matching between theories, say $n_F = 4$ and $n_F = 5$ (for the number of active quark degrees of freedom) at the bottom quark's mass threshold. Then at leading order we get $\alpha_s^{(4)}(m_b) = \alpha_s^{(5)}(m_b)$. At higher order, more complicated Feynman diagrams contribute and create the mismatch (e.g. from the effect of including the bottom quark in theory 5 gives contributions through loops of the form 

at the next order):

$$\alpha^{(4)}(\mu) = \alpha^{(5)}(\mu)\Big(1 + \frac{\alpha^{(5)}(\mu)}{\pi}\big(-\frac{1}{6}\ln\frac{\mu^2}{m_b^2}\big) + \big(\frac{\alpha^{(5)}(\mu)}{\pi}\big)^2\big(\frac{11}{72} - \frac{11}{24}\ln\frac{\mu^2}{m_b^2} + \frac{1}{36}\ln^2\frac{\mu^2}{m_b^2}\big) + ...\Big)\Big|_{\mu\sim m_b}$$




The general procedure for matching EFT's $\mathcal{L}^{(1)} \to \mathcal{L}^{(2)} \to ... \to \mathcal{L}^{(n)}$ top-down for mass thresholds $m_1 \gg m_2 \gg ... \gg m_n$ (going from higher to lower energy scale) can be summed up as follows:

1. Match the theory $\mathcal{L}^{(1)}$ at the scale $m_1$ onto $\mathcal{L}^{(2)}$ by considering the S-matrix.

2. Compute the $\beta$-function and anomalous dimension in theory 2 (which does not have particle 1) to run the couplings down from the evolution equations, then run them.

3. Match the theory $\mathcal{L}^{(2)}$ at the scale $m_1$ onto $\mathcal{L}^{(3)}$ by considering the S-matrix.

4. Compute the $\beta$-function and anomalous dimension in theory 3 (which does not have particle 2) to run the couplings down from the evolution equations, then run them.

5. Follow this procedure for any number of additional steps required.

If one is interested in the dynamics at a scale $m_{n-1} > \mu > m_n$, then one should stop at $\mathcal{L}^{(n)}$ and do the calculations for observables (e.g. matrix elements) using this Lagrangian.

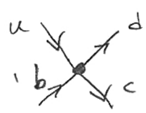
### 3.2 Massive SM particles

Usually the heavy particles t, H, W and Z are removed simultaneously from the SM. The reason for integrating them out together is because if one tries to firstly integrate out the top quark only, then $SU(2) \times U(1)$ gauge invariant of SM breaks since the top-bottom quark doublet $\begin{pmatrix} t_L \\ b_L \end{pmatrix}$ loses the top component (the problem can be solved by including Wess - Zumino terms). Also note that $\frac{m_Z}{m_t} \sim \frac{1}{2}$ is not a very good expansion parameter. The disadvantage of removing these particles at the same step is that one misses the running $m_t \to m_W$, as the analysis treats $\alpha_s(m_W) \ln \frac{m_W^2}{m_t^2}$ perturbatively.

Example: Consider the process $b \to c\bar{u}d$ at tree level with $\mathcal{L}^{SM} = \frac{g_2}{\sqrt{2}} W_\mu^+ \bar{u}_L \gamma^\mu V_{CKM} d_L + ...$ :


$$= (\frac{ig_2}{\sqrt{2}})^2 V_{cb} V_{ud}^* (\eta^{\mu\nu} - \frac{k^\mu k^\nu}{m_W^2}) \frac{-i}{k^2 - m_W^2} (\bar{c}\gamma_\mu P_L b)(\bar{d}\gamma_\nu P_L u)$$

Expand the W boson propagator to $\frac{-i\eta^{\mu\nu}}{-m_W^2} + \mathcal{O}(\frac{m_b^2}{m_W^4})$ at low energy scale $\sim m_b$:




$$= -\frac{i4G_F}{\sqrt{2}} V_{cb} V_{ud}^* (\bar{c}\gamma^\mu P_L b)(\bar{d}\gamma_\mu P_L u) \quad , \quad G_F = \frac{\sqrt{2} g_2^2}{8 m_W^2}$$

The EFT of electroweak interactions in the SM after removing t, H, W and Z is called the Electroweak Halmiltonian. The above interaction from tree level matching can be written as:

$$H_{ew} = -\mathcal{L}_{ew} = \frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* C (\bar{c}\gamma^\mu P_L b)(\bar{d}\gamma_\mu P_L u)$$

The coefficient $C$ is just equal to 1 at tree level. To go further with matching involves loops. First of all one needs to build the most general basis of operators with symmetries (complete set of structures with these degrees of freedom that can possibly occur). At the energy scale $\mu \sim m_W$ one can think of b, c, d and u as effectively massless fields to get the coefficient $C$ (which contains information about the removed mass scales) since their masses only show up in the operators, and the massless treatment makes a connection to chirality as QCD does not change the chiral nature from the original operators to the effective ones.

Example: Consider the $\bar{c}Xb$ part of a possible operator in the EFT, using chirality information we can guess the $X$ term in the middle: $P_L$ and an odd number of $\gamma$ matrices. But since 3 $\gamma$ matrices can be reduced back to 1 $\gamma$ matrix via the relation $\gamma_\alpha\gamma_\beta\gamma_\delta = g_{\alpha\beta}\gamma_\delta + g_{\beta\delta}\gamma_\alpha - g_{\alpha\delta}\gamma_\beta - i\epsilon_{\alpha\beta\delta\tau}\gamma^\tau\gamma_5$, only $\bar{c}\gamma_\mu P_L b$ is left. Also spinor and color spinor Fierzing can be used to reduce the number of operators as they are equivalent (e.g. $(\bar{\psi}_1\gamma^\mu P_L\psi_2)(\bar{\psi}_3\gamma_\mu P_L\psi_4) = (\bar{\psi}_1\gamma^\mu P_L\psi_4)(\bar{\psi}_3\gamma_\mu P_L\psi_2)$ from Fierzing with fermionic fields).

A generalization for the $D = 6$ interaction term ($C_i(\mu) = C_i\big(\frac{\mu}{m_W}, \alpha_s(\mu)\big)$) with matching includes loops and can be written down from those above arguments:

$$\mathcal{H}_{ew} = \frac{4G_F}{\sqrt{2}} V_{cb}V_{ud}^* \Big(C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu)\Big)$$

Compared to the matching at tree level, one more operator can appear from a possible arrangement of color indices:

$$O_1(\mu) = (\bar{c}^\alpha\gamma^\mu P_L b^\alpha)(\bar{d}^\beta\gamma_\mu P_L u^\beta) \quad , \quad O_2(\mu) = (\bar{c}^\beta\gamma^\mu P_L b^\alpha)(\bar{d}^\alpha\gamma_\mu P_L u^\beta)$$

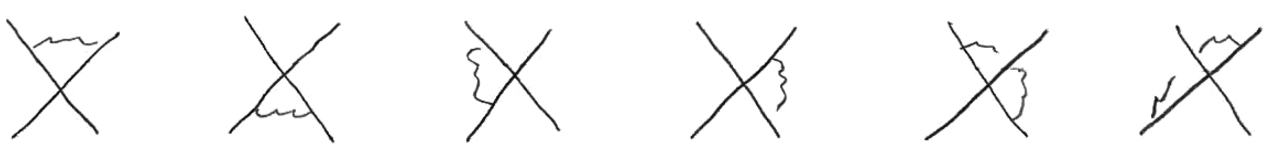
The coefficents (at $\mu = m_W$ from tree level) are:

$$C_1\big(1, \alpha_s(m_W)\big) = 1 + \mathcal{O}\big(\alpha_s(m_W)\big) \quad , \quad C_2\big(1, \alpha_s(m_W)\big) = 0 + \mathcal{O}\big(\alpha_s(m_W)\big)$$

An interesting fact about the matching is its independence from the choice for states and IR regulators as long as the same treatment is given for both theories. A clearer way to say this is that the UV matching is independent from the IR physics. Even for hadronic bound states (e.g. B, D, $\pi$), the result is valid through the use of free quark states - indeed, these stated are used because of the convenience to the matching process in calculations.

Now let us carry out the matching for $C_1$ and $C_2$ in more detail at 1-loop in the $\overline{MS}$ scheme ($d = 4-2\epsilon$). First of all, renormalize the effective field theory (assume that the dynamical contents of SM are already normalized) starting with the wavefunction $\psi = \psi^{\text{ren}} = Z_\psi^{-\frac{1}{2}}\psi^{\text{bare}}$ ($Z_\psi = 1 - \frac{\alpha_s C_F}{4\pi\epsilon}$ with $C_F = \frac{N_c^2-1}{2N_c}$). Leave out the prefactor $\frac{4G_F}{\sqrt{2}}V_{cb}V_{ud}^*$ since it will always be there so one can add it again in the end of all calculations.

The tree level matrix elements are $S_1 = \langle c\bar{u}d|\, O_1\, |b\rangle\Big|_{tree}$ and $S_2 = \langle c\bar{u}d|\, O_2\, |b\rangle\Big|_{tree}$, and diagrams with 1-loop can be calculated based on the values of $S_1$ and $S_2$.



Let's use off-shell momenta $p$ as our IR regulator and assume the external particles' masses vanish. The calculations for bare operators with 1-loop corrections produce mixing of $S_1$ and $S_2$ since gluons carry colors:

$$\begin{aligned}
\langle O_1\rangle^{\text{bare}} &= \Big(1 + 2C_F\frac{\alpha_s}{4\pi}(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^2})\Big)S_1 + \frac{3}{N_c}\frac{\alpha_s}{4\pi}(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^2})S_1 - \frac{3\alpha_s}{4\pi}(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^2})S_2 + ... \\
\langle O_2\rangle^{\text{bare}} &= \Big(1 + 2C_F\frac{\alpha_s}{4\pi}(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^2})\Big)S_2 + \frac{3}{N_c}\frac{\alpha_s}{4\pi}(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^2})S_2 - \frac{3\alpha_s}{4\pi}(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^2})S_1 + ...
\end{aligned}$$

The divergences can be killed by using wavefunction and coupling renormalization. There are 2 equivalent methods to renormalize the interactions (for more details, refer to A. J. Buras *http://arxiv.org/abs/hep-ph/9806471*):

1. Composite operator renormalization: $O_i^{\text{bare}} = Z_{ij} O_j(\psi^{\text{bare}})$ therefore $\langle O_i \rangle^{\text{bare}} = Z_\psi^{-2} Z_{ij} \langle O_j \rangle^{\text{ren}}$, with $\langle O_j \rangle^{\text{ren}}$ is a renormalized amputated Green's function

2. Renormalize coefficient: $\langle \mathcal{H} \rangle = C_i^{\text{bare}} \langle O_i(\psi^{\text{bare}}) \rangle = (Z_{ij}^{(C)} C_j^{\text{ren}})(Z_\psi^2 \langle O_i \rangle^{\text{bare}}) = C_i^{\text{ren}} \langle O_i \rangle^{\text{bare}} + (Z_\psi^2 Z_{ij}^{(C)} - \delta_{ij}) C_j^{\text{ren}} \langle O_i \rangle^{\text{bare}}$. The last term is the counter-term chosen in such a way so that the final result is $\langle \mathcal{H} \rangle = C_i^{\text{ren}} \langle O_i \rangle^{\text{ren}}$.

The relation between these 2 ways can be understood by looking at the matrix elements:

$$Z_\psi^2 Z_{ji}^{(C)} C_i^{\text{ren}} \langle O_j \rangle^{\text{bare}} = \langle \mathcal{H} \rangle = C_i^{\text{ren}} Z_\psi^2 Z_{ij}^{-1} \langle O_j \rangle^{\text{bare}} \Rightarrow Z_{ij}^{-1} = Z_{ji}^{(C)}$$

In the $\overline{MS}$ scheme, the operator-mixing renormalization matrix can be read off (after absorbing the most divergent terms $\frac{1}{\epsilon}$ into the counter-terms and leaving the matrix elements of the renormalized $\langle O_1 \rangle^{\text{ren}}$ and $\langle O_2 \rangle^{\text{ren}}$ to depend on $S_1$, $S_2$ and the renormalization scheme's parameter $\ln \frac{\mu^2}{p^2}$) from detailed calculations to be $Z = 1 + \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \begin{pmatrix} 3/N_c & -3 \\ -3 & 3/N_c \end{pmatrix}$. With this information, one can construct the anomalous dimension matrix:

1. For method 1, the anomalous dimension matrix for operators is defined as $\mu \frac{d}{d\mu} O_i = -\gamma_{ij} O_j$:

$$0 = \mu \frac{d}{d\mu} O_i^{\text{bare}} = (\mu \frac{d}{d\mu} Z_{ij}) O_j + Z_{ij} (\mu \frac{d}{d\mu} O_j) \Rightarrow \gamma_{ij} = Z_{ik}^{-1} \mu \frac{d}{d\mu} Z_{kj}$$

   Note that $\alpha_s$ also runs with $\mu \frac{d}{d\mu} \alpha_s = -2\epsilon \alpha_s + ...$, therefore $\gamma = -\frac{\alpha_s}{2\pi} \begin{pmatrix} 3/N_c & -3 \\ -3 & 3/N_c \end{pmatrix}$

2. For method 2, the anomalous dimension for the coefficients can be found from the independence of $C_i^{\text{bare}} O_i^{\text{bare}} = C_i O_i$ on the energy scale $\mu$ (we drop the "ren" notation for convenience):

$$0 = \mu \frac{d}{d\mu} (C_i^{\text{bare}} O_i^{\text{bare}}) = \mu \frac{d}{d\mu} (C_i O_i) = (\mu \frac{d}{d\mu} C_i) O_i - C_i \gamma_{ij} O_j \Rightarrow \mu \frac{d}{d\mu} C_i = C_j \gamma_{ji} = \gamma_{ij}^T C_j$$

In order to do the running, one can start by diagonalizing operators via $O_\pm = O_1 \pm O_2$ and coefficients via $C_\pm = \frac{1}{2}(C_1 \pm C_2)$ (hence at tree level $C_\pm(m_W) = \frac{1}{2}$) and get the anomalous dimensions $\gamma_+ = \gamma_{++} = -\frac{\alpha_s}{2\pi}(\frac{3}{N_c} - 3)$, $\gamma_- = \gamma_{--} = -\frac{\alpha_s}{2\pi}(\frac{3}{N_c} + 3)$ and $\gamma_{+-} = \gamma_{-+} = 0$ (for SM, $N_c = 3$). The running of coeficients at $\mu \gg \Lambda_{QCD}$ ($\beta_0 = \frac{11}{3} C_A - \frac{2}{3} n_F$) is:

$$\mu \frac{d}{d\mu} C_\pm(\mu) = \gamma_\pm\big(\alpha_s(\mu)\big) C_\pm(\mu) \Rightarrow \mu \frac{d}{d\mu} \ln C_\pm(\mu) = \gamma_\pm\big(\alpha_s(\mu)\big) \quad ; \quad \mu \frac{d}{d\mu} \alpha_s(\mu) = \beta\big(\alpha_s(\mu)\big) = -2\beta_0 \alpha_s^2(\mu)$$

Note that the anomalous dimension $\gamma$ only depends on the couplings $\alpha_s(\mu)$ because of the EFT structure in the UV region (e.g. poles, divergences). If we perform a change of variable $\mu \to \alpha_s$ (this trick can also be used at higher orders) and $\frac{d\mu}{\mu} = -\frac{1}{2\beta_0} \frac{d\alpha_s}{\alpha_s^2}$, then (running down $\mu_W \sim m_W > \mu$):

$$\ln\left(\frac{C_\pm(\mu)}{C_\pm(\mu_W)}\right) = \int_{\mu_W}^{\mu} \frac{d\mu}{\mu} \gamma_\pm = -\frac{1}{2\beta_0} \int_{\alpha_s(\mu_W)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\alpha_s^2} \gamma_\pm(\alpha_s) = a_\pm \ln\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_W)}\right) \quad ; \quad a_+ = \frac{1}{2\pi\beta_0} \quad , \quad a_- = -\frac{1}{\pi\beta_0}$$

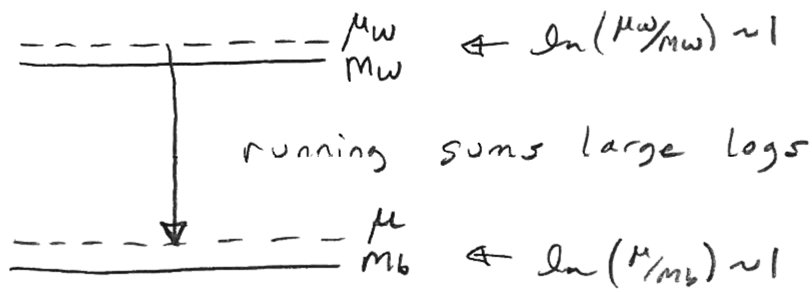
The boundary condition $C_{\pm}(\mu_W)$ is typically chosen at $\mu_W = m_W$, $2m_W$ or $\frac{m_W}{2}$. One should think of $C_{\pm}(\mu_W)$ as a fixed order series in $\alpha_s(\mu_W)$. The evolution of coefficients is:

$$C_{\pm}(\mu) = C_{\pm}(\mu_W) e^{a_{\pm} \ln\left(\frac{\alpha_s(\mu_W)}{\alpha_s(\mu)}\right)} = C_{\pm}(\mu_W) \left(\frac{\alpha_s(\mu_W)}{\alpha_s(\mu)}\right)^{a_{\pm}} \tag{3.6}$$

The decay process of interest is $b \to c\bar{u}d$ so the energy scale should be set to $\mu \sim m_b \ll m_W$. The answer $C_{\pm}(\mu)$ can be expressed as the sum of an infinite series of leading logs (LL) as $\alpha_s(\mu_W) \ln\left(\frac{\mu_W}{\mu}\right) \sim \mathcal{O}(1)$:

$$C_{\pm}(\mu) = \frac{1}{2} + ...\alpha_s(\mu_W) \ln\left(\frac{\mu_W}{\mu}\right) + ...\alpha_s^2(\mu_W) \ln^2\left(\frac{\mu_W}{\mu}\right) + ...\alpha_s^3(\mu_W) \ln^3\left(\frac{\mu_W}{\mu}\right) + ... \tag{3.7}$$

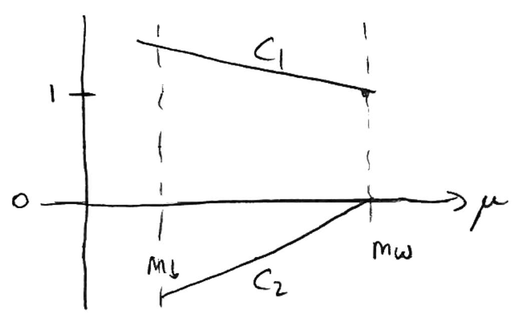
The physical picture of the running can be seen as:




The above analysis is for $\gamma_{\pm}$ and $\beta$ at the lowest order in $\alpha_s$. At higher order, the general structure for the evolution of the coefficients is $C_i(\mu) = C_j(\mu_W) U_{ji}(\mu_W, \mu)$, where $U_{ji}(\mu_W, \mu)$ is the evolution matrix. The effective electroweak Hamiltonian can therefore be written as $H_{ew} = C_j(\mu_W) U_{ji}(\mu_W, \mu) O_i(\mu)$, relating coefficients at high energy scale and operators at low energy scale. The order expansion of $C_i(\mu)$ is now receiving a correction at higher order in $\alpha_s$ (the first order is the leading log($LL$), the second order is the next leading log($NLL$) and the next order is the next next leading log($NNLL$)), which is a perturbative improvement for the renormalization:

$$C_i(\mu) = ... + ... \sum_k \alpha_s(\mu_W) \ln\left(\frac{\mu_W}{\mu}\right) + ...\alpha_s(\mu_W) \sum_k \alpha_s(\mu_W) \ln\left(\frac{\mu_W}{\mu}\right) + ...\alpha_s^2(\mu_W) \sum_k \alpha_s(\mu_W) \ln\left(\frac{\mu_W}{\mu}\right) + ...$$

At the same log order, the matching $C_i(\mu_W)$ is at 1 order diagrammatically higher than the calculation for the running of $\gamma$. At LL the matching $C_i(\mu_W)$ is at tree level while the running $\gamma$ is at 1-loop, at NLL, the matching $C_i(\mu_W)$ is at 1-loop while the running $\gamma$ is at 2-loop and at NNLL the matching $C_i(\mu_W)$ is at 2-loop while the running $\gamma$ is at 3-loop and so on.



The renormalization group flow induces the coefficient of operator $O_2$ through perturbative improvement at LL as the higher log order terms are of $\mathcal{O}(1)$, although the matching at tree level gives $C_2 = 0$. The value is $C_1(m_b) = 1.12$ and $C_2(m_b) = -0.28$.

The process $b \to c\bar{u}d$ gives the decay $\bar{B} \to D\pi$ (or in quark components $(\bar{u}b) \to (\bar{u}c)(\bar{u}d)$), and the contribution to the scattering amplitude $\langle D\pi| H_{ew} |\bar{B}\rangle$ can be written in 2 ways:

$$\langle D\pi| H_{ew} |\bar{B}\rangle = C_i(\mu_W) \langle D\pi| O_i(\mu_W) |\bar{B}\rangle = C_i(\mu) \langle D\pi| O_i(\mu) |\bar{B}\rangle$$

The first way has large logs from the terms $\sim \ln(\frac{m_W}{m_b})$ and is therefore troublesome for analysis and calculation via lattice quantum field theory. On the other hand, the second way has no large logs (these are absorbed in the coefficients and summed by the renormalization group expression) and the operators are at the same scale with the process $\sim m_b$. Physically, $C_i(\mu)$ and $O_i(\mu)$ are the right couplings and operators to use.

Now, let us compare with the full theory in the SM. The EFT is already renomalized in the $\overline{MS}$ scheme, and since the calculations in the full theory involve the weak conserved current, the UV divergences in dressing vertex and the wavefunction cancel out to give UV finite results. The 1-loop diagrams in the full theory are:

We now look at tree level $S_1$ and log terms. The full theory and the EFT (at leading order $C_1 = 1$ and $C_2 = 0$) give:

$$i\mathcal{A}^{1-\text{loop}} = \Big(1 + 2C_F\frac{\alpha_s}{4\pi}\ln(\frac{\mu^2}{-p^2})\Big)S_1 + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\ln(\frac{m_W^2}{-p^2})S_1 + ... \tag{3.8}$$

$$\langle O_1\rangle^{1-\text{loop}} = \Big(1 + 2C_F\frac{\alpha_s}{4\pi}\ln(\frac{\mu^2}{-p^2})\Big)S_1 + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\ln(\frac{\mu^2}{-p^2})S_1 + ... \tag{3.9}$$

where here the (...) contain non-logs and $S_2$ terms. The above equations are almost the same, except for a difference in $m_W^2$ and $\mu^2$. This can be understood as $m_W \to \infty$ from the point of view of the EFT, therefore the role of $m_W$ and $\mu$ are similar (cut-off). The calculations for EFT involve only triangle loops (since the W boson propagator is effectively shrunk to a point in the Feynman diagrams and all the physics at the high energy scale is absorbed into the mass scale $\mu$) so it's much easier than the full theory (moreover the $\frac{1}{\epsilon}$ term is all that is required in the EFT to compute the anomalous dimension, which is a lot more convenient than the SM). The $\ln(-p^2)$ terms are matched between these theories, which means they agree in the IR region. This check tells us that the EFT has the right degrees of freedom at the low energy scale (it's kind of obvious in the case here, but in other theories it can be non-trivial).

The difference of $\mathcal{O}(\alpha_s)$ gives the matching at 1-loop (at tree level, $i\mathcal{A} = C_i\langle O_i\rangle = S_1$):

$$0 = i\mathcal{A} - \Big(C_1\langle O_1\rangle + C_2\langle O_2\rangle\Big) = i\mathcal{A}^{1-\text{loop}} - C_1^{(1)}S_1 - \langle O_1\rangle^{\mathcal{O}(\alpha_s)} - C_2^{(1)}S_2 - ...$$

The index indicates the order in term of $\alpha_s$ as $C_1 = 1+C_1^{(1)}+...$ and $C_2 = 0+C_2^{(1)}+....$ From the terms $S_1 \ln(...)$ one can find that $C_1^{(1)} = -\frac{3}{N_c}\frac{\alpha_s C_F}{4\pi}\ln(\frac{\mu^2}{m_W^2})$. One can also show that $C_2^{(1)} = \frac{3\alpha_s C_F}{4\pi}\ln(\frac{\mu^2}{m_W^2})$ with a similar calculation for the $S_2 \ln(...)$ terms. The coefficient is independent of the IR regulator (IR region diverges as the off-shell regulator $-p^2 \to 0$). At this point it is clear to see that $\mu$ separates the UV region at scale $\sim m_W$ and the IR region at scale $-p^2 \to 0$ in the full theory from the point of view of the EFT. In other words, we can see the full theory as a cross between the EFT at large momentum and the EFT at small momentum. The EFT large momentum part $\ln(\frac{m_W^2}{\mu^2})$ is absorbed in the renormalized coeffients and the EFT small momentum part $\ln(\frac{\mu^2}{-p^2})$ is encoded in the renormalized operators with all light degrees of freedom, and together they reproduce the full $\ln(\frac{m_W^2}{-p^2})$. Note that the multiplication becomes addition through expansion at the same order in $\alpha_s$:

$$1+\alpha_s \ln(\frac{m_W^2}{-p^2}) = \Big(1+\alpha_s \ln(\frac{m_W^2}{\mu^2})\Big)\Big(1+\alpha_s \ln(\frac{\mu^2}{-p^2})\Big)$$

Order by order in $\alpha_s(\mu)$ the $\ln\mu$ terms cancel out. The $\mu$ dependence of $C_i(\mu)$ and $\langle O_i(\mu)\rangle$ should be gone since $C_i(\mu)\langle O_i\rangle = i\mathcal{A}$, which is $\mu$-independent. The result is $\mu$ independence at any $\alpha_s$ order one works with, even that $\alpha_s$ itself depends on $\mu$ (the cancellation for $\mu$ dependence of $\alpha_s(\mu)$ comes from higher orders).

Now, let us do some sketchy calculations at NLL. Ignore the mixing for simplicity $i\mathcal{A}^{\text{EFT}} = C(\mu)\langle O(\mu)\rangle$ and note that the coefficients, operators (matrix elements) at next leading order are scheme-dependent (however, note that the 1-loop anomalous dimension is scheme-independent as long as the scheme is mass-independent). The matching is:

$$i\mathcal{A} = 1+\frac{\alpha_s}{4\pi}\Big(-\frac{\gamma^{(0)}}{2}\ln(\frac{m_W^2}{-p^2})+A^{(1)}\Big) \quad , \quad i\mathcal{A}^{\text{EFT}} = C(\mu)\Big(1+\frac{\alpha_s}{4\pi}\big(-\frac{\gamma^{(0)}}{2}\ln(\frac{\mu^2}{-p^2})+B^{(1)}\big)\Big)$$

Therefore $C(\mu_W \sim m_W) = 1+\frac{\alpha_s}{4\pi}\Big(\frac{\gamma^{(0)}}{2}\ln(\frac{\mu^2}{m_W^2})+A^{(1)}-B^{(1)}\Big)$. With $\beta_1 = \frac{34}{3}C_A^2 - \frac{10}{3}C_A n_F - 2C_F n_F$, the run at NLL originates from these evolution equations:

$$\mu\frac{d}{d\mu}\ln C = \gamma(\alpha_s) = \gamma^{(0)}\frac{\alpha_s(\mu)}{4\pi}+\gamma^{(1)}\big(\frac{\alpha_s(\mu)}{4\pi}\big)^2 \quad ; \quad \mu\frac{d}{d\mu}\alpha_s(\mu) = -2\alpha_s(\mu)\Big(\beta_0\frac{\alpha(s)(\mu)}{4\pi}+\beta_1\big(\frac{\alpha(s)(\mu)}{4\pi}\big)^2\Big)$$

The trick $\mu \to \alpha_s$ gives in general $\frac{d\mu}{\mu} = \frac{d\alpha_s}{\beta(\alpha_s)}$, and an all orders solution for $C(\mu)$ is $\ln(\frac{C(\mu)}{C(\mu_W)}) = \int_{\alpha_s(\mu_W)}^{\alpha_s(\mu)} d\alpha_s \frac{\gamma(\alpha_s)}{\beta(\alpha_s)}$. For NLL, with $J = \frac{\gamma^{(0)}\beta_1}{2\beta_0^2} - \frac{\gamma^{(1)}}{2\beta_0}$ (take $\mu_W = m_W$):

$$U(m_W,\mu) = \exp\Big(\int_{\alpha_s(m_W)}^{\alpha_s(\mu)} d\alpha_s \frac{\gamma(\alpha_s)}{\beta(\alpha_s)}\Big) = \big(1+\frac{\alpha_s(\mu)}{4\pi}J\big)\Big(\frac{\alpha_s(m_W)}{\alpha_s(\mu)}\Big)^{\frac{\gamma^{(0)}}{2\beta_0}}\big(1-\frac{\alpha_s(m_W)}{4\pi}J\big)$$

Combining next order matching and NLL running:

$$C(\mu) = \Big(1+\frac{\alpha_s(\mu)}{4\pi}J\Big)\Big(\frac{\alpha_s(m_W)}{\alpha_s(\mu)}\Big)^{\frac{\gamma^{(0)}}{2\beta_0}}\Big(1+\frac{\alpha_s(m_W)}{4\pi}(A^{(1)}-B^{(1)}-J)\Big)$$

Of the EFT scheme choice, in the above expression $B^{(1)}$, $\gamma^{(1)}$, $J$, $C(\mu)$ (and also $\langle O(\mu)\rangle$) are scheme-dependent. On the other hand $\beta_0$, $\beta_1$, $\gamma^{(0)}$, $\mathcal{A}^{(1)}$, $B^{(1)}+J$ (and also $C(\mu)\langle O(\mu)\rangle$) are scheme-independent.

Example: A sketchy derivation for $B^{(1)}+J$ to be scheme-independent starts with $\langle O\rangle' = (1+\frac{\alpha_s}{4\pi}s)\langle O\rangle$ in a "primed" and a "un-primed" scheme, with $s$ being some constant. From this $Z' = (1-\frac{\alpha_s}{4\pi}s)Z$ and therefore $C' = (1-\frac{\alpha_s}{4\pi}s)C$, $B^{(1)'} = B^{(1)} + s$. Also $\gamma^{(1)'} = \gamma^{(1)} + 2\beta_0 s$, and from the scheme independence $C(\mu)\langle O(\mu)\rangle = C'(\mu)\langle O(\mu)\rangle'$ (or $C(m_W)U(m_W,\mu)\langle O(\mu)\rangle = C'(m_W)U'(m_W,\mu)\langle O(\mu)\rangle'$):

$$U'(m_W,\mu) = (1-\frac{\alpha_s(\mu)}{4\pi}s)U(m_W,\mu)(1+\frac{\alpha_s(m_W)}{4\pi}s) \Rightarrow J' = J - s$$

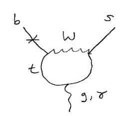
It is then clear to see how $B^{(1)'} + J' = B^{(1)} + J$, and therefore $A^{(1)} - B^{(1)} - J$ is scheme-independent.

The LL result gives $\left(\frac{\alpha_s(m_W)}{\alpha_s(\mu)}\right)^{\frac{\gamma^{(0)}}{2\beta_0}}$ scheme-independent, and that means the scheme-dependent part $\left(1+\frac{\alpha_s(\mu)}{4\pi}J\right)$ of $C(\mu)$ should be cancelled by the scheme dependence of the operator $\langle O(\mu)\rangle$ at the lower end of the evolution (at $\mu$).

Some remarks for the EFT of SM, in general:

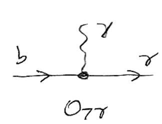
- $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is inherently 4-dimensional, and it must be treated carefully in dimensional regularization.
- In 4 dimensions, the set of 16 matrices $\{1,\gamma^5,\gamma^\mu,\gamma^5\gamma^\mu,\sigma^{\mu\nu}\}$ is a complete basis, but that's not true in general in $d$ dimensions. Additional operators are called evanescent, and in dimesional regularization one might need these operators, although they are vanish as $\epsilon \to 0$.

From the electroweak Hamiltonian, one can do some phenomenology.

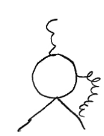
Example: Consider the $b \to s\gamma$ flavor changing neutral current process. Since this process doesn't appear at tree level, it is sensitive to loop corrections (e.g. 

). Some of the effective operators after integrating out the $t$ quark and $W$ boson contribute to this process:

$$\begin{aligned} O_{7\gamma} &= \frac{e^2}{8\pi^2} m_b \bar{s}\sigma^{\mu\nu}(1+\gamma^5)bF_{\mu\nu} \\ O_{8G} &= \frac{g}{8\pi^2} m_b \bar{s}T^a\sigma^{\mu\nu}(1+\gamma^5)bG^a_{\mu\nu} \\ O_1 &= \left(\bar{s}\gamma^\mu(1+\gamma^5)u\right)\left(\bar{u}\gamma_\mu(1-\gamma^5)d\right) \quad , \quad O_2, O_3, ..., O_{10} \end{aligned}$$

Let us go through some diagrams. At leading order 

$C^{LO}_{7\gamma} = C^{LO}_{7\gamma}(\frac{m_W}{m_t}) \approx -0.195$, then at 1-loop level 

$= 0$ (in the 't HooftVeltman scheme for $\gamma^5$) and 2-loop level  divergences give the leading order in the anomalous dimension $\gamma^{(0)}$ at first order $\alpha_s$. From all diagrams and doing the matching, the LL evolution at $\mu = m_b$ gives 50% bigger value (which means the branching ratio $Br(b \to s\gamma)$ is enhanced by $(1.5)^2 = 2.3$ times):

$$C_{7\gamma}(\mu = m_b) = (\frac{\alpha_s(m_W)}{\alpha_s(\mu)})^{\frac{16}{23}} C^{LO}_{7\gamma} + \frac{8}{3}\left((\frac{\alpha_s(m_W)}{\alpha_s(\mu)})^{\frac{14}{23}} - (\frac{\alpha_s(m_W)}{\alpha_s(\mu)})^{\frac{16}{23}}\right) C^{LO}_{8G} + \sum_{i=1}^{8} h_i (\frac{\alpha_s(m_W)}{\alpha_s(\mu)})^{a_i} C^{LO}_1 \approx -0.300$$

QCD corrections are crucial for using $b \to s\gamma$ to constrain new physics.

# 4 Chiral Perturbation Theory

Chiral Perturbation Theory (ChPT) is an example of a bottom up EFT, with non-linear symmetry representations. In this EFT, loops are not suppressed by the couplings but by powers (in the power expansion), therefore it has a non-trivial power counting (power counting theorem).

Let us briefly review QCD chiral symmetry for light quarks (approximately massless):

$$\mathcal{L}_{QCD} = \bar{\psi} i \not{D} \psi = \bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{D} \psi_R \quad ; \quad \psi_L \to L\psi_L \quad , \quad \psi_R \to R\psi_R$$

Under the unitary chiral transformation ($L$ and $R$) the physics of the theory doesn't change. In QCD, the full chiral symmetry (with both left and right) is broken (the axial symmetry is broken and only the vector remains) because of the mass terms:

| $G = (L, R) \to H$ | $\psi$ | Goldstones | Expansion |
|---|---|---|---|
| $SU(3)_L \times SU(3)_R \to SU(3)_V$ <br> 8 gen.   8 gen.   8 gen | $\begin{pmatrix} u \\ d \\ s \end{pmatrix}$ | 8 $\quad \pi, K, \eta$ | $\frac{m_{u,d,s}}{\Lambda_{QCD}} \sim \frac{1}{3}$ |
| $SU(2) \times SU(2)_R \to SU(2)_V$ <br> 3 gen.   3 gen.   3 gen. | $\begin{pmatrix} u \\ d \end{pmatrix}$ | 3 $\quad \pi$ | $\frac{m_{u,d}}{\Lambda_{QCD}} \sim \frac{1}{50}$ |

## 4.1 SU(2) ChPT

The goal for ChPT is to find an effective field theory for the Goldstones (light degrees of freedom, bound states of the quark fields in the original theory). The matching at $\Lambda_{QCD}$ is non-perturbative, so a better approach for the EFT $\mathcal{L}_\chi$ is just using the symmetry breaking pattern (and the degrees of freedom). The Wilson coeffients for operators in the EFT can be fixed through experimental data or with numerical values from lattice QFT. In the bottom up point of view, any theory with the same symmetry breaking pattern will give the same $\mathcal{L}_\chi$ but different coefficients (the high energy physics is encoded in the coefficients). For now, Let us stick with ChPT for the bound states of $u$ and down quarks, the 3 pions.

Consider a theory with a similar symmetry breaking pattern: a $SU(2)$ linear $\sigma$ model with $\pi = \sigma + i\tau^a\pi^a$ ($\tau^a$ is a Pauli matrix). The Lagrangian of the full theory is:

$$\mathcal{L}_\sigma = \frac{1}{4}\mathrm{Tr}\,(\partial^\mu \pi \partial_\mu \pi) + \frac{\mu^2}{4}\mathrm{Tr}\,(\pi^\dagger \pi) - \frac{\lambda}{16}\big(\mathrm{Tr}\,(\pi^\dagger\pi)\big)^2 + \bar{\psi}_L i\not{\partial}\psi_L + \bar{\psi}_R i\not{\partial}\psi_R - g\big(\bar{\psi}_L \pi \psi_L + \bar{\psi}_R \pi^\dagger \psi_R\big)\,,$$

where the first term is the kinetic term, the second is the mass term and the last 2 terms couple our Goldstone with a fermion. The theory has a global $SU(2)_L \times SU(2)_R$ symmetry with the transformations $\psi_L \to L\psi_L$, $\psi_R \to R\psi_R$ and $\pi \to L\pi R^\dagger$ (explicitly, $L = \exp\left(\frac{i}{2}\alpha_L^a\tau^a\right)$ and $R = \exp\left(\frac{i}{2}\alpha_R^a\tau^a\right)$). This symmetry is spontaneously broken since the potential $V = -\frac{\mu^2}{2}(\sigma^2 + \pi^a\pi^a) + \frac{\lambda}{4}(\sigma^2 + \pi^a\pi^a)^2$ has minimum at $\sigma^2 + \pi^a\pi^a = \frac{\mu^2}{\lambda}$. Take $\langle 0|\,\sigma\,|0\rangle = v = \sqrt{\frac{\mu^2}{\lambda}}$, $\tilde{\sigma} = \sigma - v$ and $\langle 0|\,\pi^a\,|0\rangle = 0$ (so the vector part of the symmetry is still unbroken). The new Larangian after the change in variables is:

$$\mathcal{L}_{\tilde{\sigma}} = \frac{1}{2}(\partial^\mu\tilde{\sigma}\partial_\mu\tilde{\sigma} - 2\mu^2\tilde{\sigma}^2) + \frac{1}{2}\partial^\mu\pi^a\partial_\mu\pi^a - \lambda v\tilde{\sigma}(\tilde{\sigma}^2 + \pi^a\pi^a) - \frac{\lambda}{4}(\tilde{\sigma}^2 + \pi^a\pi^a)^2 + \mathcal{O}(\psi)$$

The unbroken symmetry is global $SU(2)_V$ with the transformations $\tilde{\sigma} \to \tilde{\sigma}$ and $\pi^a \to V\pi^a V^\dagger$ (as $L = R$). The Goldstones $\pi^a$ are massless, the $\tilde{\sigma}$ field gets a mass $m_{\tilde{\sigma}} = 2\mu^2 = 2\lambda v^2$ and the $\psi$ field gets a mass $m_\psi = gv$. Take $v$ to be large so that there is a clear separation between the low energy degrees of freedom, the massless $\pi^a$, and the other massive degrees of freedom. The EFT of interest deals with $\pi^a$ without worrying about $\tilde{\sigma}$ and $\psi$.

Field redefinitions can be used as an organizational tool to produce a nice formulation for the EFT.

Example 1: The square root representation uses $S = \sqrt{(\tilde{\sigma}+v)^2 + \pi^a\pi^a} - v(= \tilde{\sigma} + ...)$ and $\phi^a = \frac{v\pi^a}{\sqrt{(\tilde{\sigma}+v)^2+\pi^a\pi^a}}(= \pi + ...)$ to produce the following Lagrangian:

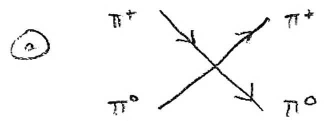
$$\begin{aligned}\mathcal{L}_{SqR} &= \frac{1}{2}(\partial^\mu S\partial_\mu S - 2\mu^2 S^2) + \frac{1}{2}\big(\frac{v+S}{v}\big)^2\big(\partial^\mu\phi^a\partial_\mu\phi^a + \frac{\phi^a\partial^\mu\phi^a\phi^b\partial_\mu\phi^b}{v^2-\phi^a\phi^a}\big)\\ &\quad -\lambda v S^3 - \frac{\lambda}{4}S^4 + \bar{\psi}i\partial\!\!\!/\psi - g\big(\frac{v+S}{v}\big)\bar{\psi}\big(\sqrt{v^2-\phi^a\phi^a} - i\phi^a\tau^a\gamma^5\big)\psi\end{aligned}$$

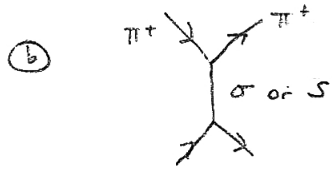
Since the expansion of $S$ and $\phi^a$ gives a term linear in $\tilde{\sigma}$ and $\pi^a$, the representation independence theorem can be used to quantize the theory and give the same results for observables.

Example 2: The exponential representation uses the same $S$ as the square root representation in addition to $\sigma + i\tau^a\pi^a = (v+S)\Sigma$ with $\Sigma = \exp\big(\frac{i\tau^a\Pi^a}{v}\big)$ to produce the following Lagrangian:

$$\begin{aligned}\mathcal{L}_{\text{Exp}} &= \frac{1}{2}(\partial^\mu S\partial_\mu S - 2\mu^2 S^2) + \frac{(v+S)^2}{4}\text{Tr}\,(\partial^\mu\Sigma\partial_\mu\Sigma^\dagger)\\ &\quad -\lambda v S^3 - \frac{\lambda}{4}S^4 + \bar{\psi}i\partial\!\!\!/\psi - g(v+S)\big(\bar{\psi}_L\Sigma\psi_R + \bar{\psi}_R\Sigma^\dagger\psi_L\big)\end{aligned}$$

Once again, we see that representation independence theorem works for the fields $S$ and $\Pi^a$. Dropping the massive $S$ and $\psi$ fields in the above Lagrangian (integrating them out in the lowest order is equivalent to simply dropping them), one gets the non-linear $\sigma$ model Lagrangian $\mathcal{L}_\chi = \frac{v^2}{4}\text{Tr}\,(\partial^\mu\Sigma\partial_\mu\Sigma^\dagger)$. This action is equivalent to the original one for low energy phenomenology of the pions. To see this, let us do a tree level calculation for the scattering process of Goldstones $\pi^+\pi^0 \to \phi^+\pi^0$ with the momentum transferred $q = p'_+ - p_+ = p_0 - p'_0$. We have 2 possible types of diagrams - direct scattering and exchange scattering:




expand in $q^2/v^2$

| | | | | |
|---|---|---|---|---|
| Linear | $-2i\lambda$ | $+$ | $(-2i\lambda v)^2 \frac{i}{q^2-m_\sigma^2}$ | $= (-2i\lambda)\left(1+\frac{2\lambda v^2}{q^2-2\lambda v^2}\right) = i q^2/v^2 + \cdots$ |
| Square Root | $i q^2/v^2$ | | $\mathcal{O}(q^4)$ | |
| Exponential | $i q^2/v^2$ | | $\mathcal{O}(q^4)$ | |
| Non-Linear $\mathcal{L}_\chi$ | $i q^2/v^2$ | | $0$ | |

The quantum theories of these 4 Lagrangia all agree at $\mathcal{O}(\frac{q^2}{v^2})$. From the above results one can see that the linear action is the most inconvenient since the derivative nature of the interactions is only seen by cancellations between graphs, and the non-linear action is the most convenient since it only has the appropriate low energy degrees of freedom $\Sigma$ and contains the derivative couplings.

Under global $SU(2)_L \times SU(2)_R$ transformation we see $S \to S$ and $\Sigma \to L\Sigma R^\dagger$ (since $\pi \to L\pi R^\dagger$), so the field $\Pi^a$ transforms non-linearly (while $\Sigma$ and $\pi^a$ transform linearly). The infinitesimal transformations of these fields are $\Pi^a \to \Pi^a + \frac{v}{2}(\alpha_L^a - \alpha_R^a) + \mathcal{O}(\Pi^2)$.

Instead of going through all of the above analysis, one can write down $\mathcal{L}_\chi$ from the start, in general. Consider a symmetry breaking from $G \to H$ and parametrize the coset $G/H$ by $\Sigma$. It is in this coset that the Goldstones transform. The transformation generator $g = (L,R) \in G$ is broken to $(V,V) = h \ni H$, and the parametrization for the coset can be thought of as $g = (g_L, g_R) = \Xi h$ (the symmetry broken part $\Xi = \Xi(x)$ can have position dependence). The transformation is $\Xi \to g\Xi h^{-1}$.

Example: For $G = SU(N)_L \times SU(N)_R$ and $H = SU(N)_V$, look at the separation $(g_L g_V, g_R g_V) = (gLg_R^\dagger, 1)(g_R g_V, g_R g_V)$ (using $g_R g_R^\dagger = 1$ as a $SU(N)$ generator). Note that $(g_R g_V, g_R g_V) \in H$, so the broken symmetry $(gLg_R^\dagger, 1)$ can be parametrized by a $SU(N)_A$ matrix $\Sigma = g_L g_R^\dagger$ which transforms as $\Sigma \to L\Sigma R^\dagger$ (this can be read off from the form of $\Sigma$).

A good way to parametrize $G/H$ is to use the components $\Pi^a(x)$ of the broken generators $t^a$ in $\Xi(x)$'s exponential form $\Xi(x) = \exp\left(\frac{it^a\Pi^a(x)}{v}\right)$. This is known as the Callan-Coleman-Wess-Zumino (CCWZ) prescription. One has the freedom to pick a choice for $t^a$.

Example 1: Pick $t^a = \tau_L^a$, then $\Xi(x) = \left(\exp\left(\frac{i\tau^a\Pi^a(x)}{v}\right), e^0\right) = \left(\Sigma(x), 1\right)$. The transformation law satisfies:

$$\Xi' = \begin{pmatrix} \Sigma' & 0 \\ 0 & 1 \end{pmatrix} = g\Xi h^{-1} = \begin{pmatrix} L & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}^{-1} \Rightarrow V = R \ , \ \Sigma' = L\Sigma R^\dagger$$

Example 2: Pick $t^a = \tau_L^a - \tau_R^a$, then $\Xi(x) = \left(\exp\left(\frac{i\tau^a\Pi^a(x)}{v}\right), \exp\left(-\frac{i\tau^a\Pi^a(x)}{v}\right)\right)$. The transformation law satisfies:

$$\Xi' = \begin{pmatrix} \Sigma' & 0 \\ 0 & \Sigma'^{-1} \end{pmatrix} = g\Xi h^{-1} = \begin{pmatrix} L & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma^{-1} \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}^{-1} \Rightarrow \Sigma' = L\Sigma V^{-1} = V\Sigma R^{\dagger}$$

For QCD, the common convention is $v = \frac{f}{\sqrt{2}}$ so that $\Sigma(x) = \exp\left(\frac{2iM(x)}{f}\right)$ with $M(x) = \frac{\tau^a \Pi^a(x)}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Pi^0 & \Pi^1 - i\Pi^2 \\ \Pi^1 + i\Pi^2 & -\Pi^0 \end{pmatrix}$, and the chiral Lagrangian $\mathcal{L}_\chi$ becomes:

$$\begin{aligned} \mathcal{L}_\chi &= \frac{f^2}{8}\mathrm{Tr}\,(\partial^\mu \Sigma \partial_\mu \Sigma^\dagger) = \frac{f^2}{8}\mathrm{Tr}\left| \int_0^1 \exp\left(\frac{2iM}{f}s\right)\partial\left(\frac{2iM}{f}\right)exp\left(\frac{2iM}{f}(1-s)\right)\right|^2 \\ &= \frac{1}{2}\mathrm{Tr}\left|\partial M + \frac{i}{f}\{M, \partial M\} - \frac{2}{3f^2}(\partial M M^2 + M\partial M M + M^2 \partial M) + ...\right|^2 \\ &= \frac{1}{2}\mathrm{Tr}\,(\partial^\mu M \partial_\mu M) + \frac{1}{6f^2}\mathrm{Tr}\left([M, \partial^\mu M][M, \partial_\mu M]\right) + ... \\ &= \frac{1}{2}\delta^{ab}\partial^\mu \Pi^a \partial_\mu \Pi^b - \frac{\epsilon^{abe}\epsilon^{cde}}{3f^2}\Pi^a \partial^\mu \Pi^b \Pi^c \partial_\mu \Pi^d + ... \end{aligned}$$

, where the first term is the kinetic term and the second is the 4-point interaction.

The chiral symmetry is explicitly broken in the SM because of the quark mass terms $-\bar{\psi}_L M_q \psi_R - \bar{\psi}_R M_q \psi_L$, so the Goldstones become pseudo-Goldstones. If we treat the quark mass matrix as a spurion field and say $M_q$ transforms as $M_q \to L M_q R^\dagger$, the symmetry breaking terms become invariant under the transformation. By fixing this term, $M_q$ explicitly breaks the symmetry, and from the lowest order term of the mass matrix in the effective Lagrangian one can find the masses of the pseudo-Goldstons :

$$\mathcal{L}_\chi^{\mathrm{mass}} = \mu \mathrm{Tr}\,(M_q \Sigma^\dagger + M_q^\dagger \Sigma) = -\frac{2\mu}{f^2}(m_u + m_d)\Pi^a \Pi^a + ... ,$$

where we have used that $M_q = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$. These Goldstones have therefore the same mass-squared $m_\Pi^2 = \frac{4\mu}{f^2}(m_u^2 + m_d^2)$ at this level of analysis.

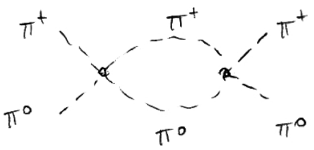
In QCD the left chiral current is calculated from $J^a_{L\mu} = \bar{\psi}\gamma_\mu P_L \tau^a \psi$ (with a similar equation for the right current), and it can be coupled with a vector field $l_\mu = \tau^a l^a_\mu$ in the Lagrangian so that $J^a_{L\mu} = -\frac{\delta \mathcal{L}_\chi}{\delta l^a_\mu}$. Similarly, with the spurion left and right vector field $l_\mu = t^a l^a_\mu$ and $r_\mu = t^a r^a_\mu$, the EFT theory can be made locally gauge invariant by using them to change the derivatives, with a specific transformation rule $l_\mu \to L(x) l_\mu L^\dagger(x) + i\partial_\mu L(x) Ł^\dagger(x)$ and $r_\mu \to R(x) r_\mu R^\dagger(x) + i\partial_\mu R(x) R^\dagger(x)$. In detail, $\partial_\mu \Sigma \to D_\mu \Sigma = \partial_\mu \Sigma + i l_\mu \Sigma - i\Sigma r^\mu$. With this trick, one can find the left and right Noether currents to be $J^a_{L\mu} = -\frac{\delta \mathcal{L}_\chi}{\delta l^a_\mu} = -\frac{if^2}{4}\mathrm{Tr}\,(t^a \Sigma \partial_\mu \Sigma^\dagger)$ and $J^a_{R\mu} = -\frac{\delta \mathcal{L}_\chi}{\delta l^a_\mu} = -\frac{if^2}{4}\mathrm{Tr}\,(t^a \Sigma^\dagger \partial_\mu \Sigma)$ respectively. Note that the axial current has the expansion $J^a_{A\mu} = J^a_{L\mu} - J^a_{R\mu} = -\frac{f}{2}\partial_\mu \Pi^a + ...$, hence the matrix element $\langle 0 | J^a_{A\mu} |\Pi^b\rangle = \frac{if}{2}\delta^{ab}p_\mu$ at tree level gives the pion decay constant to be $\frac{f}{2}$.

Next we turn to look at the Feynman rules, power counting and loops in ChPT. The lowest order Lagrangian is given by:

$$\mathcal{L}_\chi^{(0)} = \frac{f^2}{8}\mathrm{Tr}\,(\partial^\mu \Sigma^\dagger \partial_\mu \Sigma) + \mu \mathrm{Tr}\,(M_q \Sigma^\dagger + M_q^\dagger \Sigma)$$

Here, $\partial^2 \sim v_0 m_q$, so $p^2 \sim m_\pi^2$. We will expand in the power counting factor $\frac{p^2}{\Lambda_\chi^2} \ll 1$ as well as $\frac{m_\pi^2}{\Lambda_\chi^2}$. Notice that we have both a derivative and mass expansion simultaneously. $\Lambda_\chi$ is a (large) mass scale, and it is natural to choose $\Lambda_\chi \sim f$ since the 4-point pions vertex is $\sim \frac{p^2}{f^2}$ from the interaction $\frac{1}{6f^2}\mathrm{Tr}\left([M,\partial^\mu M][M,\partial_\mu M]\right)$, and $\sim \frac{\mu}{f^4}M_q(\sim \frac{m_\Pi^2}{f^2})$ from the interaction $\frac{3\mu}{6f^4}\mathrm{Tr}\left(M_q M^4\right)$. Also from experimental data we get $f \gg m_\Pi$.

Now let us take a look at one of the 1-loop diagrams (e.g.  ):




$$\sim \frac{1}{f^4}\int {\mathchar'26\mkern-12mu d}^d l \frac{(p_+ - l)^2 (l - p_+')^2}{(l^2 - m_\Pi^2)\left((l - p_+ - p_0)^2 - m_\Pi^2\right)} + \ldots \sim \frac{\{p^2, m_\Pi^2\}}{f^2}\frac{\{p^2; m_\Pi^2\}}{(4\pi f)^2} + \ldots$$

In this equation, $\{p^2, m_\pi^2\}$ indicates the power counting, as those two are equivalent. Additionally, we used dimensional regularization to preserve chiral symmetry and power counting. The $(4\pi)^2$ in the denominator is the loop factor, and since $\frac{\{p^2, m_\Pi^2\}}{f^2}$ is the order of tree level, the loops are suppressed by $\frac{\{p^2; m_\Pi^2\}}{(4\pi f)^2}$, hence $\Lambda_\chi$ is often chosen to be $4\pi f \sim 1.6$ GeV. Another choice for $\Lambda_\chi$ is the mass of $\rho$ (the lightest pseudo-Goldstones integrated out of this ChPT), with $m_\rho \sim 800$ (MeV).

In the $\overline{MS}$ scheme $[M] = 1 - \epsilon$, $[f] = 1 - \epsilon$ and $[\mu] = 2 - 2\epsilon$ and therefore with the renormalized mass scale parameter $\Lambda$ one gets $f^{\mathrm{bare}} = \Lambda^{-\epsilon}$ and $\mu^{\mathrm{bare}} = \Lambda^{-2\epsilon}\mu$ (in ChPT the loops do not renormalize the leading order Lagrangian, so counter-terms aren't needed). There's no $\Lambda$ dependence in $M_q$ and $m_\Pi$ since $\frac{\mu^{\mathrm{bare}}}{f^{\mathrm{bare}}} = \mu f^2$. The loops have UV divergences of the form $\frac{1}{\epsilon} + \ln\left(\frac{\Lambda^2}{p^2}\right)$ and $\frac{1}{\epsilon} + \ln\left(\frac{\Lambda^2}{m_\Pi^2}\right)$ which enter at $\mathcal{O}(p^4, p^2 m_\Pi^2, m_\Pi^4)$, and counter-terms for these poles should come from the Lagrangian at higher orders. These new operators are:

$$\mathcal{L}_\chi^{(2)} = L_1\left(\mathrm{Tr}\,(\partial^\mu\Sigma\partial_\mu\Sigma^\dagger)\right)^2 + L_2\mathrm{Tr}\,(\partial^\mu\Sigma\partial^\nu\Sigma^\dagger)\mathrm{Tr}\,(\partial_\mu\Sigma\partial_\nu\Sigma^\dagger) + \ldots \tag{4.2}$$

Let $\frac{f^2}{8}\chi = \mu M_q$. The equation of motion, $(\Box\Sigma)\Sigma^\dagger - \Sigma(\Box\Sigma^\dagger) - \chi\Sigma^\dagger + \Sigma\chi^\dagger + \frac{1}{2}\mathrm{Tr}\,(\chi\Sigma^\dagger - \chi^\dagger\Sigma) = 0$, can be used to remove $\Box$ terms, and with $SU(2)$ identities the Lagrangian can be further simplified (e.g. $\mathrm{Tr}\,(\partial^\mu\Sigma\partial_\mu\Sigma^\dagger\partial^\nu\Sigma\partial_\nu\Sigma^\dagger) = \frac{1}{2}\left(\mathrm{Tr}\,(\partial^\mu\Sigma\partial_\mu\Sigma^\dagger)\right)^2$). At $\mathcal{O}(p^4)$ one can include both loops with $p^4\ln\frac{\Lambda^2}{p^2}$ terms and $p^4 L_i(\Lambda)$ terms from higher order interaction to eliminate the divergence by the counter-terms $\delta L_i$ (the $\Lambda$ dependence, by construction, is cancelled between these contributions). $\Lambda$ can be thought of as a cut-off dividing UV and IR physics between the low energy physics from the $\Lambda$-dependent matrix elements of the loops (with pion fields as light degrees of freedom) and high energy physics from the coefficients $L_i(\Lambda)$. The expectation value for couplings can be guessed as $\frac{L_i(\mu)}{f^2} = \frac{1}{(4\pi f)^2}\left(a_i\ln\left(\frac{\Lambda}{\Lambda_\chi}\right) + b_i\right)$, with $a_i$ and $b_i$ encoding high energy physics. From naive dimensional analysis, because changing the mass scale $\Lambda$ moves pieces between the loops and the coefficients $L_i(\Lambda)$, one expects them to be at the same order of magnitude $a_i \sim b_i \sim 1$.

In practice one has to pick a value for $\Lambda$, and it's typically chosen at a high mass scale $\Lambda \sim m_\rho$ or $\Lambda_\chi$ so that the large logs are placed in the matrix elements instead of the coefficients, as the dimensional analysis holds for them and the power counting estimation works. Note that there is no infinite series of large logs to sum over in this EFT, since the kinetic terms don't get renormalized (when $\epsilon \to 0$).

If the regularization is a hard cut-off $\Lambda_c$, then the 1-loop diagrams involve terms $\sim \frac{\Lambda_c^4}{\Lambda_\chi^4}$ that break chiral symmetry (and therefore effectively cannot be absorbed by counter-terms in $\mathcal{L}_\chi$, which preserves chiral symmetry), terms $\sim \frac{\Lambda_c^2 p^2}{\Lambda_\chi^4}$ that break power counting (which should be suppressed as $\mathcal{O}(p^4)$ for the right power counting) and are absorbed in $\mathcal{O}(p^2)$ to restore power counting, and terms $\sim \frac{p^4 \ln \Lambda_c}{\Lambda_\chi^4}$ that can be eliminated by the counter-terms of $p^4 L_i$ (similar to dimensional regularization). There are difficulties as mentioned above, so this choice of regularization is not often chosen for ChPT.

A difference between a theory with gauge symmetry and chiral symmetry is the structure of IR physics. In ChPT, the derivative $\partial_\mu$ couplings make the IR region nicer since one usually has good $m_\Pi^2 \to 0$ and $p^2 \to 0$ limits.

ChPT can be used for phenomenology.

Example: The pion scattering process $\pi\pi \to \pi\pi$ below the inelastic thresholds can be described by a simple quantum mechanics S-matrix, which we can enumerate channel by channel. For example, $S_{lI} = e^{2\delta_{lI}}$, with $l$ is for partial wave (angular momentum state) and $I$ is for isospin. The effective range expansion for the phase-shift gives $p^{2l+1} \cot \delta_{lI} = -\frac{1}{a_{lI}} + \frac{r_{lI}^{(0)} p^2}{2} + \dots$. Detailed ChPT calculations with the direct 4-pion scattering predict the values for $a_{lI}$ (e.g. $a_{00} = \frac{7 m_\Pi^2}{16\pi f_\Pi^2}$ and $a_{02} = -\frac{m_\Pi^2}{8\pi f_\Pi^2}$, with $f_\Pi$ being the pion decay constant), which are parameter-free after $m_\Pi$ and $f_\Pi$ are known.

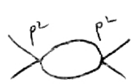
Now let us go back to the power counting of Feynman diagrams in ChPT. Consider a diagram with $N_V$ vertices, $N_I$ internal lines, $N_E$ external lines and $N_L$ loops. Expand $N_V = \sum_n N_n$ so that $N_n$ counts the number of vertices in $\mathcal{O}(p^n, m_\Pi^n)$. We use dimensional regularization, so that the power counting isn't ruined. Let us count the mass dimension ($\Lambda_\chi$ factors) for a matrix element with $N_E$ external pions:

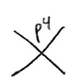
- Vertices give $\Lambda_\chi^{\sum_n N_n(4-n)}$, where the factor $(4-n)$ comes from the mass dimension of the couplings.
- $f(\sim \Lambda_\chi)$ contributions from the pion lines $\Lambda_\chi^{-2N_I - N_E}$ (internal lines is the contraction between 2 pion fields while external is 1), because each factor of pion field comes with a factor $f$ through $\frac{\Pi^a}{f}$

Topologically, one has the Euler identity to put a constrain $N_I = N_L + N_V - 1$ and this can be used to remove $N_I$ from the calculations. Hence the mass matrix elements should be $\sim \Lambda_\chi^{4-N_E-D}\{p, m_\Pi\}^D g(\frac{\{p, m_\Pi\}}{\Lambda})$, where $D$ can be solved to be $2 + \sum_n N_n(n-2) + 2N_L \geq 2$. The term $4 - N_E$ in the exponential comes from the dimensional analysis of the scattering amplitude. Adding vertices or loops always increases $D$ (more power in $\{p, m_\Pi\}$), and they correspond to a power suppression.

In conclusion, one just has to count the number of loops and vertices (momentum counting).

Example: 

$D = 2$, 

$D = 4$, 

$D = 4$

## 4.2 SU(3) ChPT

In the $SU(3)$ case one has an octet of pseudo-Goldstones (in the charge basis):

$$M_q = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \quad ; \quad M = \frac{\lambda^a \Pi^a}{\sqrt{2}} = \begin{pmatrix} \frac{\Pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \Pi^+ & K^0 \\ \Pi^- & -\frac{\Pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}$$

We can expand the mass term $\mu \mathrm{Tr}\,(M_q \Sigma^\dagger + M_q^\dagger \Sigma)$ to get the masses for the mesons: $m_{K^0}^2 = m_{\bar{K}^0}^2 = \frac{4\mu}{f^2}(m_d + m_s))$ and $m_{\Pi^\pm}^2 = \frac{4\mu}{f^2}(m_u + m_d)$. The masses of $\eta$ and $\Pi^0$ are mixed in the mass matrix (with a spin-violating off-diagonal term $\sim m_u - m_d$ that is often treated perturbatively) $M_{\Pi^0 \eta} = \begin{pmatrix} m_u + m_d & \frac{m_u - m_d}{\sqrt{3}} \\ \frac{m_u - m_d}{\sqrt{3}} & \frac{m_u + m_d + 4m_s}{3} \end{pmatrix}$.

If we ignore isospin violation via $m_u \approx m_d = m_{ud} = \frac{m_u+m_d}{2} \ll m_s$, then $m^2_{\Pi^0} \approx \frac{4\mu}{f^2}(m_u + m_d) \approx \frac{8\mu}{f^2} m_{ud}$ and $m^2_\eta \approx \frac{4\mu}{3f^2}(m_u + m_d + 4m_s) \approx \frac{8\mu}{3f^2}(m_{ud} + 2m_s)$.

Including the spurion left and right chiral current $\partial_\mu \to D_\mu$, the lowest order Lagrangian is:

$$\mathcal{L}^{(0)}_\chi = \frac{f^2}{8}\mathrm{Tr}\,(D^\mu\Sigma^\dagger D_\mu\Sigma + \chi\Sigma^\dagger + \chi^\dagger\Sigma) \quad , \quad \frac{f^2}{8}\chi = v_0 M_q$$

The momentum power counting gives $\Sigma \sim 1$, $D_\mu\Sigma \sim p$, $l_\mu \sim r_\mu \sim p$ (recall that these are the spurion source chiral currents), $\chi \sim p^2$ (which behaves like a scalar source) and $m_\Pi \sim m_K \sim p$. The next order is $\mathcal{O}(p^4)$:

$$\begin{aligned}
\mathcal{L}^{(2)}_\chi \;\; = \;\; & L_1\big(\mathrm{Tr}\,(D^\mu\Sigma D_\mu\Sigma^\dagger)\big)^2 + L_2\mathrm{Tr}\,(D^\mu\Sigma D^\nu\Sigma^\dagger)\mathrm{Tr}\,(D_\mu\Sigma D_\nu\Sigma^\dagger) + L_3\mathrm{Tr}\,(D^\mu\Sigma D_\mu\Sigma^\dagger D^\nu\Sigma D_\nu\Sigma^\dagger) \\
& + L_4\mathrm{Tr}\,(D^\mu\Sigma D_\mu\Sigma^\dagger)\mathrm{Tr}\,(\chi\Sigma^\dagger + \chi^\dagger\Sigma) + L_5\mathrm{Tr}\,\big(D^\mu\Sigma D_\mu\Sigma^\dagger(\chi\Sigma^\dagger + \chi^\dagger\Sigma)\big) + L_6\big(\mathrm{Tr}\,(\chi\Sigma^\dagger + \chi^\dagger\Sigma)\big)^2 \\
& + L_7\big(\mathrm{Tr}\,(\chi\Sigma^\dagger - \chi^\dagger\Sigma)\big)^2 + L_8\mathrm{Tr}\,(\chi\Sigma^\dagger\chi\Sigma^\dagger + \chi^\dagger\Sigma\chi^\dagger\Sigma) + L_9\mathrm{Tr}\,(L^{\mu\nu}D_\mu\Sigma D_\nu\Sigma^\dagger + R^{\mu\nu}D_\mu\Sigma^\dagger D_\nu\Sigma) \\
& + L_{10}\mathrm{Tr}\,(L^{\mu\nu}\Sigma R_{\mu\nu}\Sigma^\dagger) + H_1\mathrm{Tr}\,(L^{\mu\nu}L_{\mu\nu} + R^{\mu\nu}R_{\mu\nu}) + H_2\mathrm{Tr}\,(\chi\chi^\dagger)
\end{aligned}$$

In the above Lagrangian, $L_{\mu\nu} = \partial_\mu l_\nu - \partial_\nu l_\mu + i[l_\mu, l_\nu]$ and $R_{\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu + i[r_\mu, r_\nu]$. Similar to $SU(2)$ ChPT, the equation of motion can be used to remove $\Box\Sigma$ terms, and the $SU(3)$ relation also helps to reduce the number of operators (e.g. $\mathrm{Tr}\,(D^\mu\Sigma D^\nu\Sigma^\dagger D_\mu\Sigma D_\nu\Sigma^\dagger) = \frac{1}{2}\big(\mathrm{Tr}\,(D^\mu\Sigma D_\mu\Sigma^\dagger)\big)^2 + \mathrm{Tr}\,(D^\mu\Sigma D^\nu\Sigma^\dagger)\mathrm{Tr}\,(D_\mu\Sigma D_\nu\Sigma^\dagger) - 2\mathrm{Tr}\,(D^\mu\Sigma D_\mu\Sigma^\dagger D^\nu\Sigma D_\nu\Sigma^\dagger)$).

One can make a correspondence between the $SU(2)$ and $SU(3)$ case since they both describes pions. The heavy particles (kaons and eta) in the $SU(3)$ case can be integrated out and the coefficients can be matched between these two theories, where the kaon and eta physics will be encoded in the coefficients of the $SU(2)$ ChPT.

<u>Example:</u> An example for the matching is $2L_1^{SU(2)} + L_3^{SU(2)} = 2L_1^{SU(3)} + L_3^{SU(3)} - \frac{1}{16(4\pi)^2}\big(1 + \ln(\frac{\Lambda^2}{m_K^2})\big)$

We now quickly look at the renormalization of these operators. The renormalization of $L_i$ starts with $L_i = L_i^{\mathrm{ren}} + \delta L_i$. The counter-terms absorb divergences $\sim \frac{1}{\epsilon} - \ln(4\pi) + \gamma_E - 1$ in the $\overline{MS}$ scheme.

<u>Example 1:</u> In the $SU(2)$ case,  causes a mass renormalization of the Goldstones, shifted from the tree level answer $m_o^2 \approx 2B_o m_{ud}$ ($B_o = \frac{4\mu}{f^2}$):

$$m_\pi^2(\Lambda) = m_o^2\Big(1 - \frac{16m_o^2}{f^2}\big(2L_4^{\mathrm{ren}}(\Lambda) + L_5^{\mathrm{ren}}(\Lambda) - 4L_6^{\mathrm{ren}}(\Lambda) - 2L_8^{\mathrm{ren}}(\Lambda)\big) + \frac{m_o^2}{(4\pi f)^2}\ln(\frac{m_o^2}{\Lambda^2})\Big)$$

<u>Example 2:</u> In the $SU(3)$ case the pion decay constant gets renormalized as:

$$f_\Pi(\Lambda) = f\Big(1 - 2\zeta_\Pi - \zeta_K + \frac{16m_{ud}B_o}{f^2}L_5^{\mathrm{ren}}(\Lambda) + \frac{16B_o}{f^2}(2m_{ud} + m_s)L_4^{\mathrm{ren}}(\Lambda)\Big) \quad , \quad \zeta_i = \frac{m_i^2}{(4\pi f)^2}\ln(\frac{m_i^2}{\Lambda^2})$$

with $f$ being the parameter in the leading order Lagrangian.

# 5 Heavy Quark Effective Theory

Heavy Quark Effective Theory (HQET) is an example of an EFT where heavy particles (that are not removed) "wiggle" under the influence of light particles. It has degrees of freedom that will come with labels and encodes heavy quark symmetry (which is not apparent in QCD) with covariant representations and reparametrization invariance. In this EFT the anomalous dimensions are functions (not just numbers) and the renormalization scheme $\overline{MS}$ shows limitations that come from the scale separation for power-law terms and renormalons.

Instead of integrating out the heavy particles, they are viewed from the EFT as sources that can wiggle. Consider a heavy quark $Q$ sitting in a bound state of a meson $Q\bar{q}$ and being surrounded by light degrees of freedom $\bar{q}$, similar to what's going on in $B^0 = \bar{b}d$.


In this example, the size of the meson is $r^{-1} \sim \Lambda_{QCD} \ll m_Q$, so that a good expansion factor is $\sim \frac{\Lambda_{QCD}}{m_Q}$. To describe the fluctuation of the heavy quark due to the light quark, one needs a top down EFT that takes the low energy limit of QCD:

$$\mathcal{L}_{\text{EFT}} = \lim_{m_Q \to \infty} \mathcal{L}_{\text{QCD}} = \lim_{m_Q \to \infty} \bar{Q}(i\not{D} - m_Q)Q + ... \tag{5.1}$$

One needs to find a way to expand the Lagrangian.

## 5.1 Preliminary Treatment for Heavy Quark

In the low energy limit of QCD, consider the propagator of the heavy quark with $v^2 = 1$ and on-shell momentum $p = m_Q v$. After receiving a kick from the light degree of freedom (soft modes), the momentum wiggles $p_\mu = m_Q v_\mu + k_\mu$ ($k_\mu \sim \Lambda_{QCD} \ll m_Q$) and the off-shell part should be encoded in the effective propagator:

$$\frac{i(\not{p} + m_Q)}{p^2 - m_Q^2} = \frac{i m_Q \not{v} + m_Q + \not{k}}{2 m_Q v k + k^2} = i\left(\frac{1 + \not{v}}{2}\right)\frac{1}{vk} + \mathcal{O}\left(\frac{1}{m_Q}\right)$$

Vertices can also be expanded from the 2 legs of the heavy quarks with $\left(\frac{1+\not{v}}{2}\right)\gamma^\mu\left(\frac{1+\not{v}}{2}\right) = v^\mu\left(\frac{1+\not{v}}{2}\right)$:

$$= -ig\gamma^\mu T^a \to -igv^\mu T^a$$

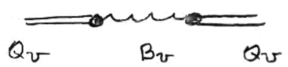
Without expanding the full Lagrangian, one can guess and write down an effective Lagrangian for those modifications from the change in the Feynman rules. This is actually the HQET Lagrangian $\mathcal{L}_{\text{HQET}} = \bar{Q}_v ivDQ_v$ with the degree of freedom $Q_v$ satisfying a projection condition $\left(\frac{1+\not{v}}{2}\right)Q_v = Q_v$.

A more direct derivation of the Lagrangian can be started by decomposing the heavy quark field:

$$Q(x) = e^{-im_Q vx}\big(Q_v(x) + B_v(x)\big) \quad ; \quad \left(\frac{1+\not{v}}{2}\right)Q_v = Q_v \quad , \quad \left(\frac{1-\not{v}}{2}\right)B_v = B_v$$

Another way to write down the projection condition is $\not{v}Q_v = Q_v$ and $\not{v}B_v = -B_v$. The derivative can be expanded as $i\not{D} = \not{v}iv \cdot D + i\not{D}_T$, with the transverse derivative $D_{T\mu} = D_\mu - v_\mu v \cdot D$ (which leaves us will the desired relation $v \cdot D_T = 0$). Then, the Lagrangian for the heavy quark becomes:

$$\begin{aligned}
\mathcal{L}_Q &= \bar{Q}(i\not{D} - m_Q)Q = (\bar{Q}_v + \bar{B}_v)e^{im_Q vx}(\not{v}ivD + i\not{D}_T - m_Q)e^{-im_Q vx}(Q_v + B_v) \\
&= (\bar{Q}_v + \bar{B}_v)e^{im_Q vx}e^{-im_Q vx}\big((\not{v} - 1)m_Q + \not{v}ivD + i\not{D}_T\big)(Q_v + B_v) \\
&= \bar{Q}_v ivDQ_v - \bar{B}_v(ivD + 2m_Q)B_v + \bar{Q}_v i\not{D}_T B_v + \bar{B}_v i\not{D}_T Q_v
\end{aligned}$$

where we have used $\not{D}_T\big(\frac{1-\not{v}}{2}\big) = \big(\frac{1+\not{v}}{2}\big)\not{D}_T$. Therefore, with only $Q_v$ as external particles, $B_v$ can be decoupled 

$\sim \frac{1}{m_Q}$ as $m_Q \to \infty$ since $B_v$ effectively have mass $\sim m_Q$. After integrating out the terms with $B_v$ fields, one has the $\mathcal{L}_{\text{HQET}}$. The field redefinitions are at tree level so the above analysis is valid to leading order in $\mathcal{O}(\frac{1}{m_Q})$ and $\mathcal{O}\big(\alpha_s(m_Q)\big)$, but one can still correctly describe the couplings to $k_\mu \sim \Lambda_{QCD} \ll m_Q$ gluons with this tree level HQET. Physically, $Q_v$ corresponds to the heavy particle and $B_v$ corresponds to the heavy antiparticle (this can be seen if we go to the rest frame with timelike $v_{r\mu} = (1, 0, 0, 0)$, then the projection $\frac{1+\not{v}_r}{2} = \frac{1+\gamma^0}{2}$ on the Dirac representation singles out the particle part and eliminates the antiparticle part of the spinor), and by choosing to pull out the phase $e^{-im_Q vx}$ one can focus on the on-shell fluctuations that are close to the particle (the opposite phase $e^{im_Q vx}$ deals with antiparticle on-shell fluctuations). Also note that when one redefines the fields, the velocity $v$ becomes the label on fields, and it is conserved by low energy QCD interactions.

In short, HQET helps to study heavy particles close to their mass-shell as one looks at the physics of the fluctuations near $m_Q$ - the antiparticle fluctuations are $2m_Q$ away, so they can be decoupled and integrated out. Hence pair creation-annihilation is not part of the theory, so the number of heavy particles is conserved, which results to a $U(1)$ symmetry for HQET that QCD (the top down origin) doesn't have - an emergent symmetry of the EFT. A generalization of that is the heavy quark symmetry in HQET:

- There is a flavor symmetry $U(N_h)$ where $N_h$ is the number of heavy quarks, since the $\mathcal{L}_{\text{HQET}}$ is blind to $m_Q$ so it does not know about flavors of the quark from QCD.
- Spin symmetry $SU(2)$, the independence of the remaining two spin components of $Q_v$, emerges because $\mathcal{L}_{\text{HQET}}$ depends on the scalar derivative $v \cdot D$ instead of the matrix derivative $\not{D}$ with spin indices; in the rest frame it can be seen from the heavy quark spin transformation $Q'_v = (1 + i\alpha_i S^i_Q)Q_v$ and $\delta\mathcal{L}_{\text{HQET}} = \bar{Q}_v[ivD, i\alpha_i S^i_Q]Q_v = 0$ (which is obvious, because $v \cdot D$ is a scalar) with $S^i_Q = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = \frac{1}{2}\gamma^5\gamma^0\gamma^i$ .
- Together these make the $U(2N_h)$ heavy quark symmetry, where $Q_v$ is fundamental with $N_h$ spin up and $N_h$ spin down degrees of freedom. This emergent symmetry has an impact on calculating observables.

The power counting in HQET is based on the power counting factor $\frac{1}{m_Q}$, as the leading order Lagrangian has no $m_Q$ and the next orders Lagrangian are suppressed by $\frac{1}{m_Q}$. From the mode expansions for the full field $Q(x) = \int \frac{d^3\mathbf{p}}{\sqrt{2E_\mathbf{p}}}\big(a^s_\mathbf{p} u^s(\mathbf{p})e^{-i\mathbf{px}} + ...\big)$, one pulls out the particle fluctuations $Q_v(x) \sim e^{-ikx}$ and gets $i\partial_\mu Q_v(x) \sim k_\mu Q_v(x)$ without a factor of $m_Q$. The variation in coordinate $x$ of $Q_v$ corresponds to the low energy fluctuations of the scale $\Lambda_{QCD}$, which is what one wants from this EFT. Looking at the sub-leading Lagrangia and the external operators, all the powers of $m_Q$ are explicit, which makes power counting easy.

There is a catch, however, and factors of $m_Q$ can hide in states. Consider a relativistic normalized state in QCD of a hadron $H$ of the form $\langle H(p')|\, H(p)\rangle = (2\pi)^3 2E_\mathbf{p}\delta^{(3)}(\mathbf{p}-\mathbf{p}')$ (spin labels are suppressed), where the state has mass dimension $-1$. The mass $m_Q$ is hiding in the physical on-shell energy $E_{\mathbf{p}} = \sqrt{m_H^2 + \mathbf{p}^2}$ (e.g. $m_B$ of $B$ mesons has the heavy quark mass $m_b$ in it). The state in HQET from the leading order $\mathcal{L}_{\text{HQET}}$ quantization for the hadron $|H(v)\rangle$ must include a different normalization as well as $\frac{1}{m_Q}$ corrections: $|H(p)\rangle = \sqrt{m_H}\big(\,|H(v,k)\rangle + \mathcal{O}(\frac{1}{m_Q})\big)$ and $\langle H(v',k')|\,H(v,k)\rangle = (2\pi)^3 2v^0 \delta_{v,v'}\delta^{(3)}(\mathbf{k}-\mathbf{k}')$, from which one can see that the state has the mass dimension $-\frac{3}{2}$. A similar treatment can also be done for Dirac spinors $u^s(\mathbf{p}) = \sqrt{m_H} u^s(v)$, where we include the $\sqrt{m_H}$ factor to cancel the $\frac{1}{\sqrt{2E}}$ in the mode expansions.

Example: The Dirac spinors are relativistically normalized so that $\bar{u}^s(\mathbf{p})\gamma_\mu u^s(\mathbf{p}) = 2p_\mu$, and normalized in HQET to satisfy $\bar{u}^s(v)\gamma_\mu u^s(v) = 2v_\mu$, which can be shown to be related by $u^s(\mathbf{p}) = \sqrt{m_H} u^s(v)$

With the heavy quark symmetry in HQET, one can do some spectroscopy. Light quarks and gluons are still described by a full $\mathcal{L}_{\text{QCD}}$ without heavy quarks. As $m_Q \to \infty$, a complicated bound state hadron $Q\bar{q}$ has the quantum number of the heavy degrees of freedom $Q$ and light degrees of freedom of $\bar{q}$, any number of $q\bar{q}$ and any number of gluons. Similar to QCD, the total angular momentum $\mathbf{J}$ is conserved (although the Lorentz boost invariance is broken), therefore it is a good quantum number ($\mathbf{J}^2 = J(J+1)$). In addition, the heavy quark spin $\mathbf{S}_Q$ is also unchanged, and that can be used to read off some extra information about the bound state by defining the light quark spin $\mathbf{S}_l = \mathbf{J} - \mathbf{S}_Q$ to get the new quantum number satisfying $\mathbf{S}_l^2 = S_l(S_l+1)$. Organizing the particles by $S_l$, one arrives at the symmetry doublet for the mesons (with $j_\pm = S_l \pm \frac{1}{2}$):

| $S_l^\pi$ | Mesons | |
|---|---|---|
| $\frac{1}{2}^-$ | $B$, $B^*$ | $j = 0, 1$ |
| $\frac{1}{2}^+$ | $B_0^*$, $B_1^*$ | $j = 0, 1$ |
| $\frac{3}{2}^+$ | $B_1$, $B_2^*$ | $j = 1, 2$ |

This is because the Lagrangian and the dynamics in HQET are independent of $S_Q$, so it can be added or subtracted from the $S_l$ to produce different $j$ that will belong to the same symmetry doublet. So heavy quark symmetry relates particles in a doublet of a given $S_l$. The same thing can be done for baryons:

| $S_l^\pi$ | Baryons | |
|---|---|---|
| $0^+$ | $\Lambda_b$ | $j = \frac{1}{2}$ |
| $1^+$ | $\Sigma_b$, $\Sigma_b^*$ | $j = \frac{1}{2}, \frac{3}{2}$ |

More predictions can be seen from the covariant representation of fields, which encodes heavy quark symmetry in objects with nice transformation properties. Consider a field $H_v^{(Q)}$ (the $Q$ index denotes the flavor) that annihilates the meson doublet for the gound state mesons:

$$H_v^Q = \Big(\frac{1+\not{v}}{2}\Big)\big(P_{v\,\mu}^{*(Q)}\gamma^\mu + iP_v^{(Q)}\gamma^5\big)$$

This is a bispinor field with indices of $Q\bar{q}$ and $H_v^Q\not{v} = -H_v^{(Q)}$ (because $v \cdot P_v^{*(Q)} = 0$). The prefactor $\big(\frac{1+\not{v}}{2}\big)$ projects out the antiparticle part of the heavy quark degrees of freedom, $P_{v\,\mu}^{*(Q)}$ is a vector field (it is replaced by the polarization $\epsilon_\mu$ after acting on the state, with $\epsilon^2 = -1$ and $v \cdot \epsilon = 0$) and $iP_v^{(Q)}\gamma^5$ is a pseudoscalar field. Under Lorentz transformation $\Lambda$, it transforms like a bispinor $H'^{(Q)}_{v'}(x') = D(\Lambda)H_v^{(Q)}(x)D^{-1}(\Lambda)$, where $v' = \Lambda v$, $x' = \Lambda x$ and $D(\Lambda)$ is the spinor Lorentz transformation. It also transforms as $(\frac{1}{2}, \frac{1}{2})$ in the heavy quark and light quark symmetry $S_Q \otimes S_l$. To see this, simply go to the rest frame $v_{r\mu} = (1,0,0,0)$ with $\Sigma^i = \frac{i}{4}\epsilon^{ijk}[\gamma^j, \gamma^k]$, then $[S_Q^i, H_{v_r}^{(Q)}] = \frac{1}{2}\Sigma^i H_{v_r}^{(Q)}$ and $[S_l^i, H_{v_r}^{(Q)}] = -\frac{1}{2}H_{v_r}^{(Q)}\Sigma^i$. Also under heavy quark spin transformation $H_v^{(Q)} \to D(R)_Q H_v^{(Q)}$ ($\delta H_v^{(Q)} = i[\alpha_i S_Q^i, H_v^{(Q)}]$), then after some Dirac algebra one finds $\delta P_{v_r} = -\frac{1}{2}\alpha_i P_{v_r}^{*\,i}$ and $\delta P_{v_r}^{*\,i} = \frac{1}{2}\epsilon^{ijk}\alpha_j P^*_{v_r k} - \frac{1}{2}\alpha^i P_{v_r}$ as these fields are mixed.

With $H_v^{(Q)}$, one can easily read off the heavy quark symmetry prediction.

Example 1: From QCD, the heavy quark decay constants of $\bar{B}$,$D$ can be guessed from just Lorentz symmetry, parity and time reversal to be $\langle 0|\,\bar{q}\gamma^\mu\gamma^5 Q\,|P(p)\rangle = -if_P p^\mu = -if_P m_P v^\mu$, and also for $\bar{B}^*$, $D^*$ to be $\langle 0|\,\bar{q}\gamma^\mu Q\,|P^*(p,\epsilon)\rangle = f_{P^*}\epsilon^\mu$ ($f_P$ has mass dimension 1 and $f_{P^*}$ has mass dimension 2). With heavy quark symmetry in HQET one can relate $P$ and $P^*$.

In HQET, these vector currents can be expressed as $\bar{q}\Gamma^\mu Q = \bar{q}\Gamma^\mu Q_v + \mathcal{O}\big(\frac{1}{m_Q}, \alpha_s(m_Q)\big)$, the matrix element in this EFT is $\langle 0|\,\bar{q}\Gamma^\mu Q_v\,|H(v)\rangle$ with $H(v)$ denoting either $P$ or $P^*$ of zero residual momentum $k$. Under the heavy quark spin transformation $Q_v \to D(R)_Q Q_v$, the current changes $\bar{q}\Gamma^\mu Q_v \to \bar{q}\Gamma^\mu D(R)_Q Q_v$. It is convenient to rewrite $\bar{q}\Gamma^\mu Q_v$ in terms of $H_v^{(Q)}$ which has $P$ and $P^*$ inside, and to preserve the same transformation law under heavy quark spin rotation, using the trick pretending $\Gamma^\mu \to \Gamma^\mu D(R)^{-1}$ so that the currents are spuriously invariant. Since $H_v^{(Q)} \to D(R)H_v^{(Q)}$, each of these spuriously invariant currents should have a single term $\Gamma^\mu H_v^{(Q)}$, because it only contains a single initial state heavy meson field. Lorentz covariance requires that the currents must have the scalar form $\mathrm{Tr}\,(X\Gamma^\mu H_v^Q) = \mathrm{Tr}\,(\Gamma^\mu H_v^Q X)$, where $X$ is a Lorentz bispinor which is generally of the form $\frac{a_0(v^2)-a_1(v^2)\not{v}}{2}$. Call $a = a_0(v^2) - a_1(v^2)$, then the trace becomes to $-iav^\mu P_v$ when $\Gamma^\mu = \gamma^\mu\gamma^5$ and $aP_v^{*\mu}$ when $\Gamma^\mu = \gamma^\mu$. One arrives at:

$$\langle 0|\,\bar{q}\gamma^\mu\gamma^5 Q\,|P(v)\rangle = -iav^\mu \quad ; \quad \langle 0|\,\bar{q}\gamma^\mu Q\,|P^*(v)\rangle = a\epsilon^\mu$$

The first prediction from HQET is that $a$ ($\sim \Lambda_{\mathrm{QCD}}^{3/2}$) must have the same values for $\bar{B}$, $D$,$\bar{B}^*$ and $D^*$. The connections between states in HQET and QCD gives $f_P = \frac{a}{\sqrt{m_P}}$ and $f_{P^*} = a\sqrt{m_P^*}$. The second prediction is the size of the decay constants for the mesons, such as $f_B \sim \frac{\Lambda_{QCD}^{3/2}}{m_B^{1/2}} \sim 180$ MeV as well as the ratio between decay constants, such as $\frac{f_B}{f_D} \sim \sqrt{\frac{m_D}{m_B}} \sim 0.6$.

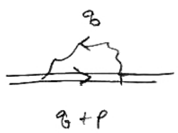
Example 2: Semileptonic decays $\bar{B} \to Dl\nu$ and $B \to D^*l\nu$ are greatly simplified in HQET, since in QCD one has to deal with 6 unnormalized form-factors while they can all be reduced to only 1 normalized form factor (Isgur-Wise function) in HQET

## 5.2 HQET radiative corrections

HQET has an interesting renormalization structure for the Lagrangian $\mathcal{L}_{\mathrm{HQET}}$ and the current $J_\mu$. This can be used to match with the top-down origin QCD, e.g. $J_\mu^{QCD} = C(\frac{\mu}{m_Q})J_\mu^{HQET} + \mathcal{O}(\frac{1}{m_Q})$.

The wavefunction renormalization $Q_v^{\mathrm{bare}} = Z_Q^{1/2}Q_v$ in HQET, using dimensional regularization (in

Feynman gauge, and using the $\overline{MS}$ scheme which introduces the extra factor $\frac{\mu^{2\epsilon}}{(4\pi e^{-\gamma_E})^\epsilon}$) becomes:

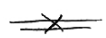



$$= -C_F g^2 \int đ^d q \frac{v^2}{q^2 v(q+p)} = -\frac{iC_F g^2}{8\pi^2} vp \frac{1}{\epsilon} + ...$$

To carry out the above integration, we used $\frac{1}{ab} = 2\int_0^\infty \frac{d\lambda}{(a+2b\lambda)^2}$ with $a = q^2$, $b = v(q+p)$ and $(t^2 - A)^2 = (q^2 + 2\lambda vq + 2\lambda vp)^2$ with $t = q + \lambda v$, $A = \lambda(\lambda - 2vp)$:

$$\begin{aligned} -C_F g^2 \int đ^d q \frac{v^2}{q^2 v(q+p)} &= (-C_F g^2) \frac{2i\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^\infty d\lambda \lambda^{\frac{d}{2}-2} (\lambda - 2vp)^{\frac{d}{2}-2} \\ &= (-C_F g^2) \frac{2i\Gamma(\epsilon)}{16\pi^2} \frac{\Gamma(\frac{3}{2}-\frac{d}{2})\Gamma(\frac{d}{2}-1)}{2\sqrt{\pi}} (-vp)^{d-3} = -\frac{iC_F g^2}{8\pi^2} vp \frac{1}{\epsilon} \end{aligned} \tag{5.4}$$

The wavefunction counter-term can be calculated - it will differ from $Z_\psi$ in QCD because of the different loop integration for heavy quark degrees of freedom compared to light quarks:



$$= i(Z_h - 1)vp \Rightarrow Z_h = 1 + \frac{C_F g^2}{8\pi^2 \epsilon}$$

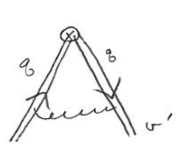
Note that there are usually 2 choices for doing the renormalization in this case: if one uses the $\overline{MS}$ scheme (as above) one only needs to keep the pole divergence, while in the on-shell renormalization scheme one gets other extra terms. The matching result from both these methods must be the same in the end.

The next step is to renormalize the local operators. Consider a heavy-to-light transition, $b \to ue^- \bar{\nu}$. To describe this process we use an operator of the form $O_\Gamma^{(0)} = \bar{q}^{(0)} \Gamma Q_v^{(0)}$ with a light quark $q$ and a heavy quark $Q_v$. The renormalized operator (grouping all the renormalized factors in the counter-term) becomes:

$$O_\Gamma = \frac{1}{Z_O} O_\Gamma^{(0)} = \bar{q}\Gamma Q_v + (\frac{\sqrt{Z_q Z_h}}{Z_O} - 1)\bar{q}\Gamma Q_v \tag{5.6}$$

By evaluating the diagram  and including wavefunction renormalization, one arrives at $Z_O = 1 + \frac{g^2}{8\pi^2\epsilon}$. The anomalous dimension is $\gamma_O = -\frac{g^2}{4\pi^2} = -\frac{\alpha_s}{\pi}$, corresponding to the renormalization group evolution below the heavy quark mass $m_Q$ (the conserved current above $m_Q$ has no evolution) since this mass is treated as UV information of this effective field theory. Note that in this calculation independence on the spin structure of $\Gamma$ originates from a HQET symmetry.

For a heavy-to-heavy transition, such as $B \to D^* e^- \bar{\nu}$ (with quark content $b \to ce^- \bar{\nu}$), the operator of interest has the form $T_\Gamma^{(0)} = \bar{Q}_{v'}^{(0)} \Gamma Q_v^{(0)}$ (hence renormalized $T_\Gamma = \bar{Q}_{v'}\Gamma Q_v + (\frac{Z_h}{Z_T} - 1)\bar{Q}_{v'}\Gamma Q_v$). The diagram calculation (in Feynman gauge) has both UV and IR divergences, and taking the external momentum of the quarks to be zero for the sake of simplicity, we get:




$$= -iC_F g^2 (vv') \int \frac{đ^d q}{q^2 (vq)(v'q)}$$

Combining this with the contribution of the wavefunction renormalization and looking at the UV behavior, with $\omega = v \cdot v'$ and $r(\omega) = \frac{\ln(\omega+\sqrt{\omega^2-1})}{\sqrt{\omega^2-1}}$, one arrives at $Z_T = 1 - \frac{g^2}{6\pi^2\epsilon}\big(\omega r(\omega) - 1\big)$, leading to a non-trivial anomalous dimension $\gamma_T = \frac{g^2}{3\pi^2}\big(\omega r(\omega) - 1\big)$. Once again, the anomalous dimension is independent of the spin structure $\Gamma$, which is easy to see from the Feynman diagram from the heavy quark symmetry, because both non-$\Gamma$ vertices and propagators don't have any spin structure characteristic. However, the result does depend on the structure of the heavy quarks' hard motion, a complication that arises from the fact that $\ln\big(\frac{m_Q}{\Lambda_{QCD}}\big)$ in QCD splits into $\ln\big(\frac{\mu}{\Lambda_{QCD}}\big)$ in HQET and $\ln\big(\frac{\mu}{m_Q}\big)$ in the Wilson coefficients, and the anomalous dimension has to sum over both of these large logs.

The Wilson current in general must depend on these indices ($\mu$, $v$, $v'$), and because it is a scalar we can write:

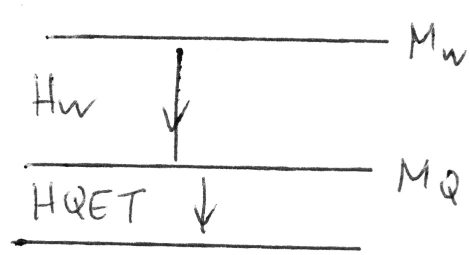
$$C(\alpha_s, \mu, m_b v^\mu, m_c v'^\mu) = C(\alpha_s, \mu, m_b^2, m_c^2, \omega = v \cdot v') \tag{5.8}$$

Example: In $B \to D^* e^- \bar{\nu}$ let $p_B^\mu = m_B v^\mu = m_{D^*} v'^\mu + q^\mu$ ($q^\mu$ is the 4-momentum transfer), hence $\omega = v \cdot v' = \frac{m_B^2 + m_{D^*}^2 - q^2}{2 m_B m_{D^*}}$, which is fixed by kinematics (the allowed kinematic range is $1 \leq \omega \leq 1.5$).

There is more interesting physics to be noted from the study of these transition operators:

- In QCD the vector current $\bar{q}_1 \gamma^\mu q_2$ is conserved for massless quarks so no anomalous dimension contribution arises (masses don't spoil this, as $\mu \gg m$). However, in HQET the scales are $\mu \lesssim m$ and therefore the currents $\bar{q}_1 \gamma^\mu Q_v$ and $\bar{Q}_{v'} \gamma^\mu Q_v$ are not conserved.
- From the results at leading log order the matching at $\mu = m_Q$ (so that $C(\mu = m_Q, ...) = 1$) yields $C_{LL}(\mu, ...) = C(m_Q) U(m_Q, \mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\right)^{-\frac{\gamma}{2\beta_0}}$ (similar to the Electroweak Hamiltonian), where $\gamma$ is a constant for heavy to light operators and $\gamma = \gamma_{LL}(\omega)$ for heavy to heavy operators.
- The corresponding $\mu$-dependence arises in HQET matrix elements, e.g. the decay constant matrix element $\langle 0 | \bar{q} \gamma^\mu \gamma^5 Q_v | P(v) \rangle = -i a(\mu) v^\mu$ is a function of $\mu$ (note that this works for a perturbation theory cut-off scale $\mu \sim 1$ GeV $\gtrsim \Lambda_{QCD}$, since $a(\mu)$ has no large logs and other complications).

With the knowledge gained from the study of the renormalization group evolution, one can move on to the matching analysis. We will use the $\overline{MS}$ scheme everywhere and consider perturbative corrections corrections at the scale $m_Q$, $\alpha_s(m_Q)$. Before matching onto HQET, we integrate out the weak interaction exchange partners to get to to QCD with $\mathcal{H}_{ew}$. This is then matched to HQET by considering perturbative corrections in the heavy quark mass scale and integrating out the heavy degrees of freedom below their mass scale. A pictorial view of how $\mathcal{H}_{ew}$ of QCD is matched onto HQET is:




Consider a matrix element in QCD with Electroweak Halmintonian $\mathcal{H}_{ew}$ (setting the light quark momentum to zero for simplicity). The finite residue factor coming from UV corrections of the light quark wavefunction (in $\overline{MS}$) is $R^{(q)}$ and of the heavy one is $R^{(Q)}$, and $v_1(\mu)$ comes from the vertex correction:

$$\left\langle q(0,s')\right| \bar{q}\gamma^\mu Q \left|Q(p,s)\right\rangle = \sqrt{R^{(q)}R^{(Q)}}\bar{u}(0,s')\big(\gamma^\mu + \alpha_s(\mu)v_1^\mu\big)u(p,s) \tag{5.9}$$

This will be matched with the following matrix element in HQET, with a finite residue factor of UV corrections of the light quark wavefunction being $R^{(q)}$ and of the heavy quark wavefunction being $R^{(h)}$, and also with a vertex correction of HQET $v_1^{HQET}$:

$$\left\langle q(0,s')\right| \bar{q}\Gamma Q_v \left|Q(v,s)\right\rangle = \sqrt{R^{(q)}R^{(h)}}\bar{u}(0,s')\big(1 + \alpha_s(\mu)v_1^{HQET}\big)\Gamma u(v,s) \tag{5.10}$$

Note that $v_1^{HQET}$ is independent of the spin structure in $\Gamma$, while $v_1$ is not. Indeed, the vector current in QCD matched onto HQET gives 2 currents, $C_1^{(v)}\bar{q}\gamma^\mu Q_v$ and $C_2^{(v)}\bar{q}v^\mu Q_v$ (this can be easily seen from the fact that in QCD the index $v$ is internal while in HQET it's external; also note that a current of the form $\bar{q}\sigma^{\mu\nu}v_\nu Q_v$ is reducible).The final results are:

$$C_1^{(v)} = 1 + \frac{\alpha_s(\mu)}{\pi}\Big(\ln\big(\frac{m_Q}{\mu}\big) - \frac{4}{3}\Big) \quad , \quad C_2^{(v)} = \frac{2\alpha_s(\mu)}{3\pi} \tag{5.11}$$

Example: There's a nice trick to arrive at the above results. Let us pick an IR regulator to make the effective theory as simple as possible (one has the freedom to choose this regulator, as the Wilson coefficients and the anomalous dimensions will not depend on the specific choice): use dimensional regularization for both UV and IR divergences in the $\overline{MS}$ scheme. With that, all HQET graphs with on-shell external momenta scale as $\big(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}\big)$. The UV divergent piece ($\sim \frac{1}{\epsilon_{UV}}$) gets removed by the counter-term and there's no finite term leftover in $\overline{MS}$. One is then left in the end with $\frac{1}{\epsilon_{IR}}$ . This simplifies the matching process, since the IR divergence of the full QCD with $\mathcal{H}_{ew}$ must match. To be precise, the UV renormalized QCD graphs (dimensional regularization with similar IR regulator) gives $\frac{1}{\epsilon_{IR}}(...) + \ln\big(\frac{\mu}{m_Q}\big)(...) + (...)$, and the first term cancels when one subtracts with HQET. The matching is then just the second term. Hence, without any calculation in HQET, the matching can still be read-off, if one trusts that the IR behavior should be matched - which should be the case when the effective field theory is done right.

## 5.3 Power Corrections and Reparametrization

At the lowest order HQET is realized and used for perturbative calculations. It's natural to take further steps by going into the physics at higher order in the power counting expansions (series in $\frac{1}{m_Q}$). Recall that the Lagrangian supports integrating out $B_v$ at tree-level ($\not{v}Q_v = Q_v$ and $\not{v}B_v = -B_v$):

$$\mathcal{L}_Q = \bar{Q}_v ivDQ_v - \bar{B}_v(ivD + 2m_Q)B_v + \bar{Q}_v i\not{D}_T B_v + \bar{B}_v i\not{D}_T Q_v \quad , \quad \delta_{\bar{B}_v}\mathcal{L}_Q = 0 \Rightarrow B_v = \frac{i\not{D}_T Q_v}{ivD + 2m_Q} \tag{5.12}$$

Performing a field redefinition from the equation of motion:

$$\mathcal{L}_Q = \bar{Q}_v\Big(ivD + i\not{D}_T\frac{1}{ivD + 2m_Q}i\not{D}_T\Big)Q_v = \bar{Q}_v ivDQ_v - \frac{1}{2m_Q}\bar{Q}_v\not{D}_T{}^2 Q_v + ... = \mathcal{L}_Q^{(0)} + \mathcal{L}_Q^{(1)} + ... \tag{5.13}$$

The above result for the first order Lagrangian $\mathcal{L}_Q^{(1)}$ can be further simplified with $\not{D}_T{}^2 = D_T^2 + \frac{g}{2}\sigma^{\mu\nu}G_{\mu\nu}$ (the commutation relation of gauge derivatives is $[D_\mu, D_\nu] = igG_{\mu\nu}$):

$$\mathcal{L}_Q^{(1)} = -\bar{Q}_v\frac{D_T^2}{2m_Q}Q_v - g\bar{Q}_v\frac{\sigma^{\mu\nu}G_{\mu\nu}}{4m_Q}Q_v \tag{5.14}$$

*The first term is the kinetic part that breaks flavor symmetry and the second one is the magnetic moment part ($\sim \sigma\mathbf{B}$) that kills the flavor and spin symmetries. Can you see that? Hint: think about what information is stored inside $m_Q$ and $\sigma^{\mu\nu}$.*

There's another way to arrive at the effective field theory from the bottom-up point of view. Since the above method is based on tree-level results (classical physics), which means that loop corrections (quantum physics) are not included, some operators might be missed since they are vanishing at tree-level. The more general procedure (also more correct) is to write down all possible operators allowed by symmetry:

- Power counting: powers of $\frac{1}{m_Q}$ are made explicit, outlining the mass dimensions of the field content needed.
- Gauge symmetry: taking into account the gauge derivative $D_\mu$.
- Discrete symmetry (the symmetries of QCD if the famous $\theta$-term is dropped): charge conjugation ($C$), parity ($P$) and time reversal ($T$).
- Realization of Lorentz symmetry in HQET: part of the Lorentz group is broken. For the rest frame, $v = (1, 0, 0, 0)$, we can see that the part of the Lorentz generator $M_{\mu_T \nu_T}$ that is transverse with respect to $v$ (purely rotation $M_{12}$, $M_{23}$, $M_{13}$) is preserved while the others $v^\mu M_{\mu\nu_T}$ (purely boost $M_{01}$, $M_{02}$, $M_{03}$) are not. Indeed, introducing $v$ means having a preferred frame, therefore the full Lorentz symmetry should be broken. However, there's a hidden symmetry on $v$ itself (order by order in power counting) in this effective field theory that restores Lorentz symmetry at low energy by *reparametrization invariance (RPI).*

Let us take a closer look at the last statement in the above list, considering how much freedom is allowed when choosing $v$. A heavy quark 4-momentum is split into 2 pieces $p_Q^\mu = m_Q v^\mu + k^\mu$ quite randomly (one can move pieces back and forth between these 2, as long as they don't violate power counting). This can be realized as an invariant under $v^\mu \to v^\mu + \frac{\epsilon^\mu}{m_Q}$ and $k^\mu \to k^\mu - \epsilon^\mu$ with $\epsilon \sim \Lambda_{QCD}$ (let's think of it as infinitestimal). Also, there's a constraint coming from $v^2 = 1$, which means that $\epsilon \cdot v = 0$ as one has 3 degrees of freedom stored in $\epsilon$. Under this transformation, fields become:

$$\not{v} Q_v(0) = Q_v(0) \to (\not{v} + \frac{\not{\epsilon}}{m_Q})(Q_v + \delta Q_v) = Q_v + \delta Q_v \Rightarrow \delta Q_v = \frac{\not{\epsilon}}{2m_Q} Q_v \tag{5.15}$$

Summing up, RPI is realized through the following transformation:

$$v^\mu \to v^\mu + \frac{\epsilon^\mu}{m_Q} \quad , \quad Q_v(x) \to e^{i\epsilon x}(1 + \frac{\not{\epsilon}}{2m_Q}) Q_v(x) \tag{5.16}$$

Note that the extra phase $e^{i\epsilon x}$ in $Q_v$ is nothing but the change of $i\partial_\mu \to i\partial_\mu - \epsilon^\mu$ ($k^\mu \to k^\mu - \epsilon^\mu$), and this restores the Lorentz invariance of the original symmetry under a small boost ($\epsilon \sim \Lambda_{QCD} \ll m_Q$), which is exactly the region of validity for the EFT of interest.

Now we consider $\frac{1}{M_Q}$ operators for $\mathcal{L}_Q^{(1)}$ from a bottom-up point of view. While in general, there might be some radiative correction hidden inside $c_K$ and $c_F$ at this order there are no missing operators from the tree-level field redefinition approach. So, we have:

$$\mathcal{L}_Q^{(1)} = -c_K \bar{Q}_v \frac{D_T^2}{2m_Q} - c_F g \bar{Q}_v \frac{\sigma^{\mu\nu} G_{\mu\nu}}{4m_Q} Q_v \tag{5.17}$$

RPI puts some requirements on the effective theory, but since this phase is only leading order change, hence $\mathcal{L}_Q^{(0)}$ is invariant at order $m_Q^0$ since $v\epsilon = 0$. At $\mathcal{L}_Q^{(1)}$, RPI gives a mixing piece, since the RPI-realized transformation gives:

$$\mathcal{L}_Q^{(0)} \to \mathcal{L}_Q^{(0)} + \delta\mathcal{L}_Q^{9))} = \bar{Q}_v(1+\frac{\not{\epsilon}}{2m_Q})e^{-i\epsilon x} i(v+\frac{\epsilon}{m_Q})De^{i\epsilon x}(1+\frac{\not{\epsilon}}{2m_Q})Q_v \quad , \quad \delta\mathcal{L}^{(0)} = \bar{Q}_v\frac{i\epsilon D}{m_Q}Q_v \tag{5.18}$$

Going through this transformation gives some change to $\mathcal{L}_Q^{(1)}$ at this order:

$$\delta\mathcal{L}_Q^{(1)} = -c_K\bar{Q}_v\frac{i\epsilon D_T}{m_Q}Q_v \Rightarrow c_K = 1 \tag{5.19}$$

For the symmetry not to be violated, $c_K = 1$ indeed is true to all order in $\alpha_s$ as long as the renormalization scheme and the chosen regulator don't break RPI. However, there's no constraint for $c_F$ and it will run as $c_F(\mu) = \left(\frac{\alpha_s(m_Q)}{\alpha_s(\mu)}\right)^{\frac{C_A}{\beta_0}}$ (where $C_A$ is the non Abelian adjoint Casimir number $C_A = N = 3$).

RPI can in general also be used to get information about power suppressed operators operators by relating the Wilson coefficients in subleading order to those of the leading order currents. Considering mass corrections as a simple instantiation of the above statement – the mass $m_H$ of a heavy meson $H$ which contains a heavy quark $m_Q$, then one can guess order by order $m_H = m_Q + \bar{\Lambda} + \mathcal{O}(\frac{1}{m_Q})$, where $\bar{\Lambda}$ is just some $\mathcal{O}(1)$ contribution. To see the physical meaning of these pieces, remember that the effective Lagrangian can be written as $\mathcal{L} = \mathcal{L}_{HQET}^{(0)} + \mathcal{L}_{QCD}^{\text{light}} + \sum_{n=1}\mathcal{L}_Q^{(n)}$. The (non-perturbative) $\bar{\Lambda}$ piece originates from $\mathcal{L}_{QCD}^{\text{light}}$, and by finding the corresponding Halmintonian $\mathcal{H}^{(0)}$, we can write $\bar{\Lambda} = \frac{\langle H|\mathcal{H}^{(0)}|H\rangle}{\langle H|H\rangle}$, where $|H\rangle$ is the heavy meson eigenstate of the theory. Since at that order $\mathcal{H}^{(0)}$ has no $m_Q$-dependence, $\bar{\Lambda}$ is also independent of $m_Q$ (not only that, but also the spin structure, e.g. $B$ and $B^*$, and the flavor, e.g. $B$ and $D$, cannot be seen yet). That subleading mass contribution has a universal value, it only depends on $S_l^\pi$ ($S_l$ is the spin of light degrees of freedom mentioned before, and $\pi$ is just a parity indication). $\mathcal{L}^{(1)}$ and higher-order Lagrangians are used to describe the mass corrections at higher order $\mathcal{O}(\frac{1}{m_Q})$. Consider the next lowest order:

$$\mathcal{H}^{(1)} = -\mathcal{L}^{(1)} = \bar{Q}_v\frac{D_T^2}{2m_Q}Q_v + C_F g\bar{Q}_v\frac{\sigma^{\mu\nu}G_{\mu\nu}}{4m_Q}Q_v \tag{5.20}$$

Taking the matrix elements in the rest frame, with $v_r = (1,0,0,0)$, gives 2 parameters which we will call $\lambda_1$ and $\lambda_2$:

$$2\lambda_1 = -\langle H|\,\bar{Q}_{v_r}D_T^2 Q_{v_r}\,|H\rangle \quad , \quad 16\mathbf{S}_Q\mathbf{S}_l\lambda_2(m_Q) = C_F(\mu)\,\langle H|\,g\bar{Q}_{v_r}\sigma^{\mu\nu}G_{\mu\nu}Q_{v_r}\,|H\rangle \tag{5.21}$$

Notice that $\lambda_1$ is $m_Q$-independent and $\lambda_2$ contains information about time reversal properties.

*You should do the manipulation to relate* $\mathbf{S}_Q\mathbf{S}_l$ *to* $J^2$*,* $S_Q^2$ *and* $S_l^2$*. Hint:* $\mathbf{S}_Q\mathbf{S}_l = \frac{J^2-S_Q^2-S_l^2}{2}$

From their mass dimesions, one could guess that $\lambda_1, \lambda_2 \sim \Lambda_{QCD}^2$ ($\lambda_2$ has $m_Q$-dependence logarithmically $\mathcal{O}\big(\ln{(m_Q)}\big)$ only), and these are non-perturbative parameters which contain more dynamical information than $\bar{\Lambda}$.

Example: Some mass corrections order by order in the same $S_l^\pi$-multiplet:

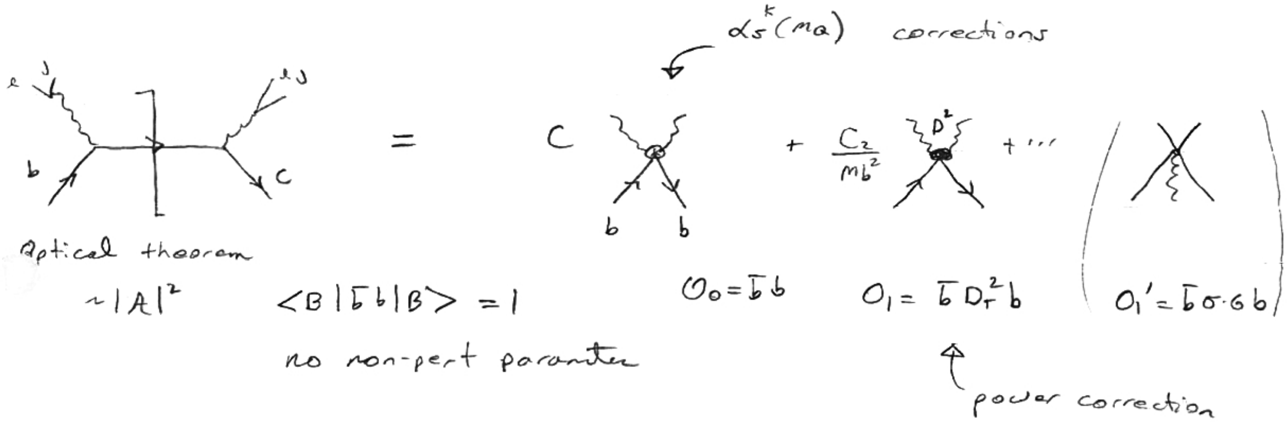
$$m_B = m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} - \frac{3\lambda_2(m_b)}{2m_b} \quad , \quad m_{B^*} = m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} + \frac{\lambda_2(m_b)}{2m_b} \quad , \tag{5.22}$$

$$m_D = m_c + \bar{\Lambda} - \frac{\lambda_1}{2m_c} - \frac{3\lambda_2(m_c)}{2m_c} \quad , \quad m_{D^*} = m_c + \bar{\Lambda} - \frac{\lambda_1}{2m_c} + \frac{\lambda_2(m_c)}{2m_c} \tag{5.23}$$

Certain combinations can be formed to cancel dependence on these power corrections: $\bar{m}_P = \frac{3m_P + m_P}{4}$ (where $P = B, D$) is independent of $\lambda_2$. Phenomenologically, we see $m_{B^*}^2 - m_B^2 \approx 0.49\ \mathrm{GeV}^2 \approx 4\lambda_2(m_b)$ and $m_{D^*}^2 - m_D^2 \approx 0.55\ \mathrm{GeV}^2 \approx 4\lambda_2(m_c)$. Hence experimentally $\frac{\lambda_2(m_c)}{\lambda_2(m_b)} = 1.12$ agrees relatively well with the theoretically leading logs RGE $\frac{\lambda_2(m_c)}{\lambda_2(m_b)} = \left(\frac{\alpha_s(m_c)}{\alpha_s(m_b)}\right)^{1/3} = 1.17$ with number of light flavors $n_f = 3$.

We can derive an important phenomenological prediction using this effective field theory. Consider a simple class of $B$-decay which are semileptonics. We divide these into exclusive decays, which make a transition between meson states, (e.g. $B \to D l \bar{\nu}$ or $B \to D^* l \bar{\nu}$) and inclusive decays where we allow transitions to any charm states (e.g. $B \to X_c l \bar{\nu}$ with $X_c = D, D^*, D\pi, D\pi\pi\pi$). Exclusive decay has form-factors for the currents between states. Our heavy quark symmetry reduces many form-factors to a single one with no $\frac{1}{m_Q}$ corrections, and this can be used to measure $V_{cb}$ – using analytical methods or lattice QCD. Inclusive decay has an Operator Product Expansion (OPE) constrained by HQET (the leading order corrections enter only at $\mathcal{O}(\frac{1}{m_Q^2})$ with dependence merely on $\lambda_1$ and $\lambda_2$). Let's focus on the inclusive OPE, starting from the triplet differential inclusive spectrum $B \to X_c l \bar{\nu}$ decay rate $\frac{d\Gamma}{dq^2 dE_l dm_X^2}$ with $q = p_l + p_{\bar{\nu}} = p_B - p_X$.

For this analysis, we will try to carry out the expansion in $\frac{\Lambda_{QCD}}{m_Q}$ for this process, summing over all intermediate $X_c$ states (allowing connection between partonic calculations and hadronic calculations through probability conservation). At leasing order, the process is $b \to c l \bar{\nu}$ with $\mathcal{O}(\alpha_s(m_Q))$ corrections, schematically and diagrammatically, the OPE can be described as:




Unsurprisingly, $C = C(\frac{\mu}{m_Q}, \alpha_s(m_Q), ...)$ is identified with the $b \to c l \bar{\nu}$ decay rate (even after including loop corrections). This OPE requires that kinematic variables are hard (i.e. $\sim m_Q$) or integrated over a region of phase space $\sim m_Q$, e.g. $\int_0^{m_B^2} dm_X^2$. If one restricts the phase space close to the edges it will introduce a new scale into the problem, and will mean that our theory cannot still be described as HQET with $m_Q \gg \Lambda_{QCD}$. At next-to-leading order we don't see any $\mathcal{O}(\frac{\Lambda_{QCD}}{m_Q})$ correction (this can be derived using the equation of motion $iv \cdot D h_v = 0$). At NNLO just $\lambda_1$, $\lambda_2$ show up in the spectrum. Experimentally, the OPE is phenomenologically very successful (e.g. $|V_{cb}| = (41.6 \pm 0.6) \times 10^{-3}$ fits).

## 5.4 Renormalons

Using the knowledge gained from the $B$-decay process, we can explore ambiguities in our perturbative series known as renormalons.

In many discussions of renormalization, there is freedom in defining the perturbative series simultaneously with Lagrangian parameters (like masses) or matrix elements (like $\lambda_1$, $\lambda_2$) – in the $\overline{MS}$ scheme it's the freedom to adjust the cut-off by choosing $\mu$ to separate perturbative and non-perturbative physics. The problem is that a poor choice of power separation can have a non-trivial impact (such as matrix elements being overwhelmed with UV physics or Wilson coefficients being IR sensitive via a hidden power-law) coming from the asymptotic structure of higher orders of the perturbation series, leading to poor convergence $\sum_n \alpha_s^n$ on one hand and irreducible uncertainty in the meaning of parameters on the other (troubles extracting the non-perturbative parameters as UV physics and IR physics are not divided correctly; for example, a parameter is doubled when one goes to the next order in perturbation series). Renormalon techniques help to quantify these problems. In other words, poor choices are plagued by renormalons ("bad objects people hate").

Example: Let's look at $b \to ue\bar{\nu}$ at lowest order (the up quark is treated as massless), think about it inclusively like $B \to X_u e\bar{\nu}$ in order to have physical sense. The decay rate as one sets $\mu = m_b$ is:

$$\Gamma(b \to ue\bar{\nu}) = \frac{G_F^2|V_{ub}|^2}{192\pi^3} m_b^5 \Big(1 + \kappa_1 \frac{\alpha_s(m_b)}{\pi}\epsilon + \kappa_2 \frac{\alpha_s(m_b)^2}{\pi^2}\epsilon^2 + \mathcal{O}(\epsilon^3)\Big) \tag{5.24}$$

$\epsilon = 1$ is just a power counting indication to help keep track of the order of $\alpha_s$. There are different choices one can use to define the bottom quark mass $m_b$, and that changes the perturbative series:

- Pole scheme: $\Gamma = \frac{G_F^2|V_{ub}|^2}{192\pi^3}(m_b^{(\text{pole})})^5(1 - 0.17\epsilon - 0.13\epsilon^2 + ...)$, correction at 2-loop pretty much the same size with 1-loop.
- $\overline{MS}$ scheme: $\Gamma = \frac{G_F^2|V_{ub}|^2}{192\pi^3}(\bar{m}_b)^5(1 + 0.30\epsilon + 0.19\epsilon^2 + ...)$, a little bit better but still not working for a clear separation for corrections at 1-loop and 2-loop level.
- 1S scheme: $\Gamma = \frac{G_F^2|V_{ub}|^2}{192\pi^3}(m_b^{(1S)})^5(1 - 0.115\epsilon - 0.035\epsilon^2 + ...)$, at least one can be happier with this compared to the previous schemes.

To convert results between different schemes (and see how poorly convergent the pole scheme is compared to other schemes), note that:

$$m_b^{(\text{pole})} = \bar{m}_b(m_b)\Big(1 + \frac{4}{3}\frac{\alpha_s(m_b)}{\pi}\epsilon + 13\frac{\alpha_s^2}{\pi^2}\epsilon^2 + ...\Big) = \bar{m}_b(m_b)(1 + 0.09\epsilon + 0.06\epsilon^2 + ...) \ , \tag{5.25}$$

$$m_b^{(\text{pole})} = m_b^{(1S)}(1 + 0.011\epsilon + 0.016\epsilon^2 + ...) \tag{5.26}$$

The lesson from this example is that the choice of mass scheme has a big impact on the perturbative series for the decay rate. Why do some choices work while others don't? This question will be answered soon.

Considering the meaning of those renormalization mass schemes:

- $m_b^{(\text{pole})}$ is physically a poor choice since there's no pole, thanks to confinement as the notion of a pole in a quark propagator is only meaningful in the perturbative sense (and is ill-defined non-perturbatively) – it is very ambiguous with an ambiguity in the mass scale $\Delta m_b^{(\text{pole})} \sim \Lambda_{QCD}$ (to see this, one has to study about the renormalon). In the HQET setup one has to use the questionable physical parameters $m^{(\text{pole})}$ with the field redefinition prefactor $e^{-im^{(\text{pole})}vx}$ as the expansion about the mass-shell. Other renormalization mass schemes can be implemented in HQET as $m^{(\text{pole})} = m + \delta m$, which will require a new operator $-\delta m\bar{Q}_v Q_v$ in the Lagrangian.

- $\bar{m}_b$ is also not good, from the point of view of HQET. It can be seen from power counting why $\overline{MS}$ isn't good with HQET, parametrically and numerically. This choice introduces a new operator $-\delta m_b \bar{Q}_v Q_v$ in our Lagrangian, where $\delta m_b = \bar{m}_b \alpha_s(m_b)$. The $\alpha_s(m_p) \approx 0.2$ doesn't give enough suppression to this term and we are essentially introducing a Lagrangian term which is $\mathcal{O}(m_b)$, which is very bad from the point of view of our power counting.

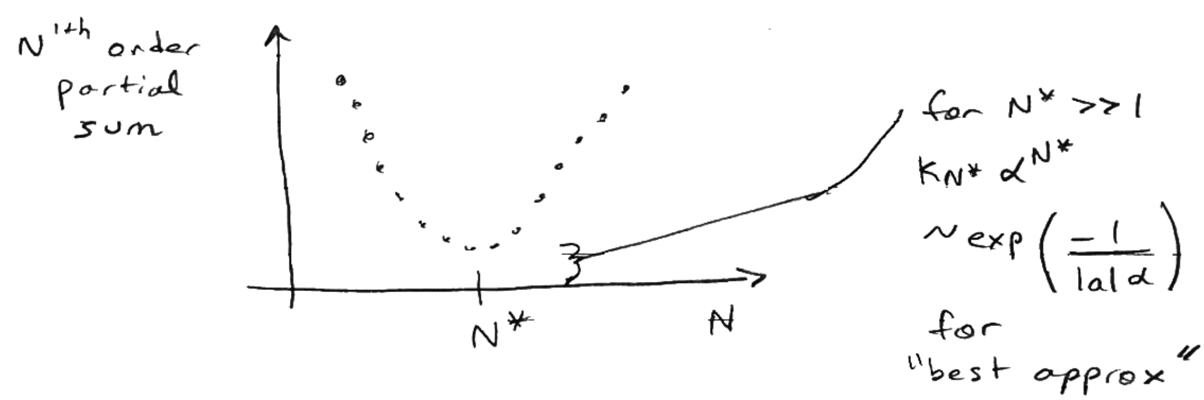
- $m_b^{(1S)}$ (defined as 1/2 of the mass of the $b\bar{b}$ bound state in perturbation theory) is the best choice out of the given three. Indeed, $\delta m = m^{(1S)}\alpha_s^2 \sim \Lambda_{QCD}$ is numerically good as the suppression is acceptable. It is however not totally alright as it grows parametrically with $m_b$ and still ruins the power counting.

- In general, a more "fancy" choice can lead to $\delta m = R\alpha_s$ with chosen $R \sim \Lambda_{QCD}$, which will then be good both parametrically and numerically.

Let's define the renormalon in a more mathematical way. First we will need a quick review. QFT perturbative series are usually not convergent but rather asymptotic series. An asymptotic series (denoted by $\doteq$) is defined by

$$f(\alpha) \doteq \sum_{n=-1}^{\infty} f_n \alpha^{n+1} \qquad \text{if and only if} \qquad |f(\alpha) - \sum_{n=-1}^{N} f_n \alpha^{n+1}| < \kappa_{N+2}\alpha^{N+2}$$

for some number $\kappa_{N+2}$. In QFT it's typical that $f_n \sim n! a^n$ as $n \to \infty$, then for any fixed $\alpha \ll 1$, no matter how small, the truncation error can grow as $\kappa_N \sim N! a^N$, hence the series has zero radius of convergence in $\alpha$ (the analytical behavior is that the series will decrease until $N$ reaches $N_* = |a|^{-\alpha}$, and then will start to grow up again). Still, even if the series is asymptotic one can make use of it (let's come back to this story later). In perturbative QED and QCD, one doesn't usually get over the first few terms in the expansion series, and it works well in QED since the growing behavior happens at a very high loop level. For QCD however the turnover happens already at 3-loop order (therefore one needs to be extra careful even at 2-loop level). The best thing one can do is to stop the series at $N_*$. That's actually not a bad thing in the sense that the mistake one made by stopping there can be characterized by $\kappa_{N_*}\alpha^{N_*} \sim e^{-|a|^{-\alpha}}$ if $N_* \gg 1$ (one will never be able to see the correction needed to get to the correct value in perturbative series because $e^{-|a|^{-\alpha}}$ doesn't have a perturbative expansion). The bad behavior at $N > N_*$ and the small gap to zero at $N = N_*$ are related to power corrections.




How poorly convergent a series is can be classified with a parameter we call $a$. To go deeper into more detail, one might want to work in a different functional basis. Here that is accomplished by a Borel transformation $f(\alpha) \leftrightarrow F(b)$:

$$f(\alpha) = \int_0^\infty dbe^{-b/\alpha}F(b) \quad , \quad F(b) = f_{-1}\delta(b) + \sum_{n=0}^{\infty} \frac{1}{n!} f_n b^n \tag{5.27}$$

Note that inserting the $\frac{1}{n!}$ factor makes the convergence better (improved convergence prefactor). For a convergent series $\sum_n f_n \alpha^{n+1}$ one can get back the same $f(\alpha)$ from the inverse transform. For a divergent series where $F(b)$ and the inverse transform exist it's reasonable to use the inverse transform to define $f(\alpha)$.

Example: Consider the following series $\sum_{n=0}^{\infty}(-1)^n\alpha^{n+1}$, which does not converge for $\alpha > 1$. One way to calculate it is to use analytic continuation for $\alpha < 1$ since in that case the series converges and the summation gives $\frac{\alpha}{\alpha+1}$. The other way is to use Borel transformation to get a well-behaved function:

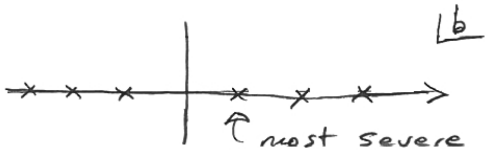
$$F(b) = \sum_{n=0}^{\infty} \frac{(-b)^n}{n!} = e^{-b} \to f(\alpha) = \int_0^\infty dbe^{-b/\alpha}e^{-b} = \frac{\alpha}{\alpha+1} \tag{5.28}$$

This integration is perfectly well-defined for large $\alpha$.

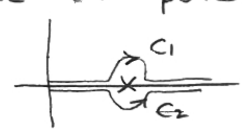
However, there are cases where the inverse transform doesn't exist (the $F(b)$ integral cannot be done) then the integrand can give information about the severity of the singularity causing the divergence.

Example: Take $f_n = a^{-n}(n+k)!$, then $F(b) = \frac{k!}{(1-\frac{b}{a})^{k+1}} + ...$ (keeping the most singular piece) has a pole-like structure at $b = a$, and this pole is called $b = a$-renormalon. If $a < 0$, the integration contour is positive and the pole is on the other side so there's no problem as the inverse transform exists (UV renormalon). If $a > 0$ the pole is on the integration contour then inverse transform doens't exist anymore (IR renormalon).

The location of the pole in Borel $b$-space tells us the severity of the singularity. The most severe pole is the one closest to the origin in the positive part of the axis (for the above example, it can be seen as follows: as $b$ becomes small then so does $a$ at pole position, hence $f_n$ goes up).

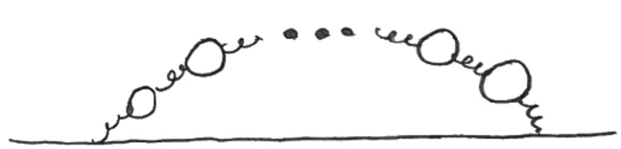



ambiguity is given by residue of pole: $C_1 - C_2$




The ambiguity can be characterized by doing the integration above or below the pole (corresponding to circling the pole) which gives the residue. In other words, a nonzero residue is the indication for an integration ambiguity.

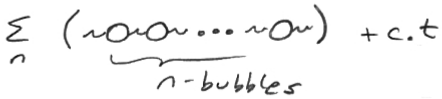
Consider the pole mass versus the $\overline{MS}$ mass. Let's pick a particular subset of Feynman diagrams, the sum of bubble diagrams $\Sigma_{\text{bubbles}}$ (which is easy to calculate), to demonstrate renormalons in these 2 mass schemes:



Indeed, the bubbles sum diagram is unique in any order of perturbation theory that gives gauge invariant contributions to the flavor/color structure of the theory, and it has the most power of $n_f$ (the number of active fermions running in each bubble). The fundamental ingredient, the bubble (in Landau gauge, QCD) is:

$$= \frac{-i}{p^2}\Big(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\Big)\delta^{ab}\Big(\frac{\beta_0\alpha_s}{4\pi}\Big)\ln\Big(\frac{-\mu^2 e^{\bar{c}}}{p^2}\Big) \quad ; \quad \bar{c} = \frac{5}{3} \quad , \quad \beta_0 = -\frac{2}{3}n_f + \frac{11}{3}C_A$$

The geometric series summation of the bubble chain gives:




$$= G^{\mu\nu}_{\text{bubbles}}(p,\alpha_s) = \frac{-i}{p^2}\Big(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\Big)\delta^{ab}\sum_{n=0}^{\infty}\Big(\frac{\beta_0\alpha_s}{4\pi}\Big)^n \ln^n\Big(\frac{-\mu^2 e^{\bar{c}}}{p^2}\Big)$$

One can do the Borel transformation to variable $u$ (which is just a rescaling of the Borel variable $b$) with $\left(\frac{\beta_0\alpha_s}{4\pi}\right)^{n+1} \to \frac{u}{n!}$ $(n > 0)$. Taking into account the vertices at the 2 ends of each bubble chain (extra $g^2 = 4\pi\alpha_s = \frac{16\pi}{\beta_0^2}\left(\frac{\beta_0\alpha_s}{4\pi}\right)$) we get:

$$\Big[g^2 G^{\mu\nu}_{\text{bubbles}}\Big](p,u) = \frac{-i}{p^2}\Big(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\Big)\delta^{ab}\frac{16\pi^2}{\beta_0}\sum_{n=0}^{\infty}\frac{u^n}{n!} = \frac{i}{(-p^2)^{2+u}}(p^2 g^{\mu\nu} - p^\mu p^\nu)\delta^{ab}\frac{16\pi^2}{\beta_0}(\mu^2 e^{\bar{c}})^u \quad (5.31)$$

Sticking this bubble chain back (acts like a modified gluon propagator with a different Feynman rule) to the bubble sum diagram, we can calculate $\Sigma_{\text{bubbles}}$ in terms of the $\overline{MS}$ mass, $\bar{m}$ (canceling the uninteresting $\frac{1}{\epsilon_{UV}}$ poles – these come from the log-divergent piece $u = 0$, while $u < 0$ and $u > 0$ probes the power law divergences through $\frac{1}{(-p^2)^{2+u}}$). After we have gotten $\bar{m}$, the pole mass can be found by looking at the topology of the propagator $\frac{i}{\not{p} - \bar{m} - \Sigma(p,\bar{m})}$ with $\Sigma(p,\bar{m}) = \bar{m}\Sigma_1(p^2,\bar{m},\alpha_s) + (\not{p} - \bar{m})\Sigma_2(p^2,\bar{m},\alpha_s)$:

$$\Big(\not{p} - \bar{m} - \Sigma(p,\bar{m})\Big)\Big|_{p^2=(m^{(\text{pole})})^2} = 0 \Rightarrow (m^{(\text{pole})})^2 = \not{p}^2 = \bar{m}^2\Big(\frac{1-\Sigma_2+\Sigma_1}{1-\Sigma_2}\Big)^2 \to m^{(\text{pole})} = \bar{m}(1 + \Sigma_1 + ...) \quad (5.32)$$

Using these guidelines, let's do the bubble sum diagram in Borel space ($G^{\mu\nu}_{\text{bubbles}}(k,\alpha_s) \to G^{\mu\nu}_{\text{bubbles}}(k,u)$):

$$\Sigma_{\text{bubbles}}(p^2,\bar{m},u) = \Sigma_1(p^2,\bar{m},u) = i^3 C_F \int d\!\!{}^-{}^d k \frac{\bar{u}(p)\gamma_\mu(\not{p} + \not{k} + \bar{m})\gamma_\nu u(p)}{(p+k)^2 - \bar{m}^2}\Big[g^2 G^{\mu\nu}_{\text{bubbles}}\Big](k,u) \quad (5.33)$$

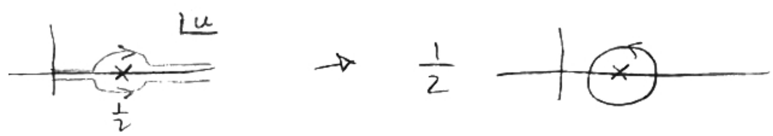
Using $\frac{1}{a^n b} = \frac{\Gamma(n+1)}{\Gamma(n)}\int_0^1 dx \frac{x^{n-1}}{\left(ax+b(1-x)\right)^{n+1}}$, the relation between the pole and $\overline{MS}$ mass can be read-off easily:

$$m^{(\text{pole})} = \bar{m}\left(\delta(u) - \frac{C_F}{6\pi\beta_0}\Big(\frac{\mu^2 e^{\bar{c}}}{\bar{m}^2}\Big)^u \frac{6(1-u)\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} + ...\right) \quad (5.34)$$

The $\delta(u)$ part comes from transforming the factor 1 ($m^{(\text{pole})} = \bar{m}$ at lowest order) to Borel space, and the omitted terms contain both the terms where there is pole structure of the form $\frac{1}{u}$ rendering it regular at $u = 0$ and terms that regular terms in u, which are not needed for this analysis. From the $\Gamma$-function structure, the pole that is closest to zero (the strongest pole) is at $u = \frac{1}{2}$ which this corresponds to the $u = \frac{1}{2}$ renormalon (this renormalon has $\left(\frac{1}{2}\right)^{-n} n!$ growth). We can further simplify this equation to:

$$m^{(\text{pole})} = \bar{m}\frac{C_F}{6\pi\beta_0}\Big(\frac{\mu}{\bar{m}}e^{\frac{\bar{c}}{2}}\Big)\frac{2}{u - \frac{1}{2}} + ... \to B(u) \quad (5.35)$$

The inverse Borel transformation $B(\alpha_s) = \int_0^\infty du e^{-u\frac{4\pi}{\beta_0\alpha_s(\mu)}} B(u)$ has an ambiguity, thanks to the renormalon sitting at $u = \frac{1}{2}$ (the ambiguity is realized by analytical continuation above or below the pole), which averages out to be a half of the residue. Specifically, $\Delta m^{(\mathrm{pole})} \approx \frac{C_F}{3\beta_0} e^{\frac{\bar{c}}{2}} \Lambda_{QCD}$:



Example: The ambiguity can be calculated as follows and can be found to be $\sim \Lambda_{QCD}$ as expected:

$$\Delta m^{(\mathrm{pole})} \approx \frac{1}{2}\mathrm{Res}[B(u)]\Bigg|_{u=\frac{1}{2}} = \frac{1}{2}(2i\pi)\oint_{u=\frac{1}{2}} duB(u) = \frac{C_F}{3\beta_0}e^{\frac{\bar{c}}{2}}\left(\mu e^{-\frac{2\pi}{\beta_0\alpha_s}}\right) = \frac{C_F}{3\beta_0}e^{\frac{\bar{c}}{2}}\Lambda_{QCD} \tag{5.36}$$

Since $m^{(\mathrm{pole})}$ has this ambiguity, one should avoid this choice of mass scheme.

Some interesting observations:

- The ambiguity doesn't depend on the use of $\bar{m}$, it's strictly associated to $m^{(\mathrm{pole})}$.
- The ambiguity is $\mu$-independent in $\alpha_s$-space, as $\Delta m^{(\mathrm{pole})} \sim \Lambda_{QCD}$. However, the residue of the pole seems to be $\mu$-dependent in Borel space. When one expresses the decay rate $\Gamma(b \to ue\bar{\nu}) = (m_b^{(\mathrm{pole})})^5(1 + ...\alpha_s + ...\alpha_s^2 + ...)$ in terms of $\bar{m}_b$, then the $u = \frac{1}{2}$ poles in $(m^{(\mathrm{pole})})^5$ and in the expansion series $(1 + ...\alpha_s + ...\alpha_s^2 + ...)$ actually cancel each other, as long as they are both expanded in the same $\alpha_s(\mu)$ order by order (also, the Borel variables have the same meaning). The bubble chain trick indeed works well for the expansion series.
- These poles are artifacts from splitting up the physics at different energy scales, so in general, they always cancel for observables.
- To cure the ambiguity, one has to introduce a new energy scale $R$. In general a scheme change gives:

$$m^{(\mathrm{pole})} = m(R) + \Delta m \quad , \quad \Delta m = R\sum_{n=1}^{\infty}\sum_k a_{nk}\ln^k\left(\frac{\mu}{R}\right)\left(\frac{\alpha_s(\mu)}{4\pi}\right)^n \tag{5.37}$$

  $m(R)$ can be chosen to be free of renormalons, if $\Delta m$ properly subtracts the pole mass renormalons. In the $\overline{MS}$ mass scheme, $R = \bar{m}(\mu = \bar{m})$; while in the interaction-based 1S mass scheme, $R = \alpha_s(\mu)m^{(1S)}$ (inverse Bohr radius). Still, $R$ can be considered a free parameter (floating cut-off) and the ambiguity is generally $R$-independent. $R$ sets the scale for absorbing the IR fluctuations (causing the instability by dressing up the pole mass) together with the pole mass – point particle mass, in a familiar sense – to yield a well defined mass $m(R)$.

Example: With $R$, one can define the MSR mass scheme by using the $\overline{MS}$ scheme and taking the only non-zero coefficients to be $a_{n0} = \bar{a}_{n0}$ (from $\overline{MS}$ value) and take $\mu = R$, so one has a well-defined mass with $m^{(\mathrm{pole})} = m^{(MSR)}(R) + R\sum_{n-1}^{\infty}\bar{a}_{n0}\left(\frac{\alpha_s(R)}{4\pi}\right)^n$. This scheme is good for doing physics where one wants to absorb things up to that floating cut-off (which pretty much decouples from the mass threshold). Note that, in a more general scheme based on floating $R$ and $\overline{MS}$ scheme, there might be 2 running scales $\mu$ and $R$, and while $\mu$ is needed for the log divergences cut-off, $R$ is needed for the power law divergences cut-off.

Before moving on, let's review about how one defines $\Lambda_{QCD}$ at higher orders in resummation (LL, NLL, ...). In $\overline{MS}$, the $\beta$-function from contributions at all orders is:

$$\beta(\alpha_s) = \frac{\partial \alpha_s(\mu)}{\partial \ln \mu} = -2\alpha_s(\mu)\sum_{n=0}^{\infty}\beta_n\Big(\frac{\alpha_s(\mu)}{4\pi}\Big)^{n+1} \quad , \quad \mu = R \to \ln\frac{R_1}{R_0} = \int_{R_0}^{R_1} dR = \int_{\alpha_s(R_2)}^{\alpha_s(R_1)} \frac{d\alpha_s}{\beta(\alpha_s)} \tag{5.38}$$

Let $t = -\frac{2\pi}{\beta_0\alpha_s(R)}$, then $\ln\frac{R_1}{R_0} = \int_{t_0}^{t_1} dt\widehat{b}(t) = G(t_1) - G(t_0)$ with the Laurent series $\widehat{b}(t) = 1 + \frac{\widehat{b}_1}{t} + \frac{\widehat{b}_2}{t^2} + ....$ The higher Laurent coefficients are $\widehat{b}_1 = \frac{\beta_1}{2\beta_0^2}$, $\widehat{b}_2 = \frac{\beta_1^2-\beta_0\beta_2}{4\beta_0^4}$, $\widehat{b}_3 = \frac{\beta_1^3-2\beta_0\beta_1\beta_2+\beta_0^2\beta_3}{8\beta_0^6}$, and so on. The integral $G(t)$ can be easily evaluated, and $\Lambda_{QCD} = R_1 e^{G(t_1)} = R_0 e^{G(t_0)}$:

$$\Lambda_{QCD} = \mu \exp\left( -\frac{2\pi}{\beta_0\alpha_s(\mu)} + \widehat{b}_1 \ln\Big(\frac{2\pi}{\beta_0\alpha_s(\mu)}\Big) + ...\right) \tag{5.39}$$

The whole expression is $\mu$-independent.

Moving on, let's treat $R$ like a variable which parametrizes a mass scheme. Similar to the renormalization group equation (RGE) from the running of $\mu$ in the $\overline{MS}$ scheme, it is expected that by flowing with the floating cut-off $R$ one also has another kind of RGE (known as $R$-RGE), since varying the scale $R$ in the MSR scheme is much like varying $\mu$ in the $\overline{MS}$ scheme. For simplicity set $\mu = R$:

$$\frac{\partial m^{(\text{pole})}}{\partial \ln R} = 0 = \frac{\partial m(R)}{\partial \ln R} + R\gamma_R\big(\alpha_s(R)\big) \quad , \quad \gamma^R\big(\alpha_s(R)\big) = \frac{\partial \Delta m(R)}{\partial R} = \sum_{n=0}^{\infty}\gamma_n^R\Big(\frac{\alpha_s(R)}{4\pi}\Big)^{n+1} \tag{5.40}$$

By setting $\mu = R$, this perturbative series avoids $\ln\left(\frac{\mu}{R}\right)$ that could be large if $\mu \gg R$. The $R$-RGE is simply $\frac{\partial m(R)}{\partial \ln R} = -R\gamma^R\big(\alpha_s(R)\big)$ (interesting fact: the power of $R$ in the RHS is actually related to the position of the renormalon). This $R$-RGE can be solved as a well-defined integral:

$$m(R_1) = m(R_0) + \Lambda_{QCD}\int_{t_1}^{t_0} dt\gamma^R(t)\frac{\partial e^{-G(t)}}{\partial t} \tag{5.41}$$

The only potential issue with this integral can come from $t = 0$, but one never gets there, as it corresponds to the Landau pole where the coupling blows up. The evolution $R_0 \to R_1$ yields new well-defined $m(R_1)$ which absorbs different amounts of IR fluctuations.

Example: Consider the LL solution with $\gamma^R(\alpha_s) = \gamma_0^R\frac{\alpha_s}{4\pi}$, $\gamma^R(t) = -\frac{\gamma_0^R}{2\beta_0}\frac{1}{t}$ and $G(t) = t$, then:

$$m(R_1) = m(R_0) + \Lambda_{QCD}^{(0)}\frac{\gamma_0^R}{2\beta_0}\int_{t_1}^{t_0}\frac{e^{-t}}{t} = m(R_0) + \frac{\gamma_0^R}{2\beta_0}\Big(\Gamma(0,t_1) - \Gamma(0,t_2)\Big)\Lambda_{QCD}^{(0)} \tag{5.42}$$

The incomplete $\Gamma$-function has asymptotic behavior when we expand about $\alpha_s = 0$ and $t = +\infty$. In detail, $\Gamma(0,t)\Lambda_{QCD}^{(0)} = -2R\sum_{n=0}^{\infty}2^n n!\Big(\frac{\beta_0\alpha_s}{4\pi}\Big)^{n+1}$, with the $2^n = \left(\frac{1}{2}\right)^{-n}$ pre-factor corresponding to the $u = \frac{1}{2}$ renormalon. The difference between these 2 asymptotic series (for $t_0$ and $t_1$) is actually a convergent series.

Example: Here we do the incomplete $\Gamma$-function expansion to see the explicit renormalon cancellation:

$$\begin{aligned} m(R_1) - m(R_0) &= -\frac{\gamma_0^R}{2\beta_0}R_1\sum_{n=0}^{\infty}\Big(\frac{\beta_0\alpha_s(R_1)}{2\pi}\Big)^{n+1} n!\Big(1 - \frac{R_0}{R_1}\sum_{k=0}^{n}\frac{1}{k!}\ln^k\frac{R_1}{R_0}\Big) &(5.43)\\ &= -\frac{\gamma_0^R}{2\beta_0}R_1\sum_{n=0}^{\infty}\Big(\frac{\beta_0\alpha_s(R_1)}{2\pi}\Big)^{n+1}\sum_{k=n+1}^{\infty}\frac{n!}{k!}\ln^k\frac{R_1}{R_0} &(5.44)\end{aligned}$$

This expression is a convegent series, renormalon-free with summation of $\ln \frac{R_1}{R_0}$ logs (with power). Hence one can connect the physics at scales $R_1 \gg R_0$ in a renormalon free fashion, which is not possible in general with the $\mu$-RGE of the $\overline{MS}$ scheme. Indeed, for phenomenology, it's often useful to switch scheme back and forth between $\overline{MS}$ and MSR.

Let's look at all higher order generalizations, say, up to N$^k$LL order ($k^{th}$-next leading log) using the series expansion form $\gamma^R(t)\frac{\partial e^{-G(t)}}{\partial t} = \frac{e^{-t}(-t)^{-\widehat{b}_1}}{t}\sum_{j=0}^{\infty} s_j \frac{1}{(-t)^j}$ (the coefficients $s$ are related to higher order $\widehat{b}_{>1}$; e.g. $s_0 = \tilde{\gamma}^R$, $s_1 = \tilde{\gamma}_1^R - (\widehat{b}_1 + \widehat{b}_2)\tilde{\gamma}_0^R$ with $\tilde{\gamma}_k^R = \frac{\gamma_k^R}{(2\beta_0)^{k+}}$). With this form, we end up with:

$$\Big[m(R_1) - m(R_2)\Big]^{N^kLL} = \Lambda_{QCD}^{(k)} \sum_{j=0}^{k} s_j (-1)^j e^{i\pi\widehat{b}_1}\Big(\Gamma(-\widehat{b}_1 - j, t_1) - \Gamma(-\widehat{b}_1 - j, t_0)\Big) \tag{5.45}$$

These imcomplete $\Gamma$-function subtractions are convergent. To show that $\Big[m(R_1) - m(R_2)\Big]^{N^kLL}$ is well-behaved, note that the anomalous dimension $\gamma^R\big(\alpha_s(R)\big)$ of $\Delta m(R)$ is also free of a $\Delta m \sim \Lambda_{QCD}$ renormalon:

$$\gamma_0^R = a_1 \quad , \quad \gamma_1^R = a_2 - 2\beta_0 a_1 \quad , \quad \gamma_2^R = a_3 - 4\beta_0 a_2 - 2\beta_1 a_1 \quad , \quad ... \quad , \quad \gamma_n^R = a_{n+1} - 2n\beta_0 a_n + ... \tag{5.46}$$

where we've defined $a_n \equiv a_{n0}$. For simplicity, let's look at the value for $a_n$ for the bubble sum, then in a given $\gamma_n^R$ one has $a_{n+1} \sim n!(2\beta_0)^n$, $a_n \sim (n-1)!(2\beta_0)^{n-1}$, ... and these growths cancel in the anomalous dimension.

Aside from phenomenological studies, the our scale $R$ can also be used for probing renormalons even in cases when the bubble sum doesn't work (e.g. no fermion loops) or other types of renormalons (e.g. ones that cannot be seen through fermions). The mechanism for finding renormalons is to use $R$-RGE. Recall that $m(R_1) - m(R_0) = \Lambda_{QCD}\int_{t_1}^{t_0} dr \gamma_R(t)\frac{d}{dt}e^{-G(t)}$ and notice that $t_0$ and $t_1$ are negative and far from $t = 0$ so things are nicely convergent, as $R$ is usually chosen to be larger than $\Lambda_{QCD}$. Consider $R_0 \to 0$ as $m(R_0) \to m^{(\text{pole})}$, $t_0 = -\ln\frac{R_0}{\Lambda_{QCD}} \to +\infty$. The integration will have to pass through the Landau pole, which introduces a new ambiguity (although near the Landau pole the series becomes non-perturbative, one can still find a path in the $t$-complex plane so that things can be treated perturbatively). At LL order:

$$m(R_1) - m^{(\text{pole})} = -\Lambda_{QCD}\frac{\gamma_0^R}{2\beta_0}\int_{t_1}^{\infty} dt\frac{e^{-t}}{t} = \int_0^{\infty} du F(u) e^{-u\frac{4\pi}{\beta_0\alpha_s(R_1)}} \quad ; \quad u = \frac{1}{2} - \frac{t}{2t_1} \quad , \tag{5.47}$$

Where it can be shown that the Borel integral factor $F(u) \sim \frac{1}{u-\frac{1}{2}}$. The Landau pole effectively becomes a Borel pole at $u = \frac{1}{2}$, therefore it makes it a little clearer that this pole corresponds to non-perturbative physics in the IR region. One can formally generalize this to all orders. This probe for the renormalon without using bubbles sum, known as the sum rule for renormalon, consists of formally taking the Borel transformation to all orders. The residue of the $u = \frac{1}{2}$ pole can be found to be $P_{1/2} = \sum_{k=0}^{\infty}\frac{s_k}{\Gamma(1+\widehat{b}_1+k)}$, and $P_{1/2} \neq 0$ means that there is a renormalon at $u = \frac{1}{2}$.

To use renormalon technology in phenomenology, let's take a look at renormalons in the OPE. Consider an OPE in the familiar $\overline{MS}$ scheme:

$$\sigma = \bar{C}_0(Q,\mu)\bar{O}_0(\mu) + \bar{C}_1(Q,\mu)\frac{\bar{O}_1(\mu)}{Q} + ... \quad ; \quad \frac{\Lambda_{QCD}}{Q} \ll 1 \quad , \quad \bar{C}_0(Q,\mu) = 1 + \sum_{n=1}^{\infty} b_n\Big(\frac{\mu}{Q}\Big)\Big(\frac{\alpha_s(\mu)}{4\pi}\Big)^n \tag{5.48}$$

where $\sigma$ is a dimensionless observable, $\overline{C}_0$ and $\overline{C}_2$ are dimensionless $\overline{MS}$ Wilson coefficients, $\overline{O}_0$ is a $\overline{MS}$ matrix element with no mass dimension and $\bar{O}_1$ is also a $\overline{MS}$ matrix element but has mass dimension 1

Since this is calculated in the $\overline{MS}$ scheme, $b_n\left(\frac{\mu}{Q}\right) = \sum_k b_{nk} \ln^k\left(\frac{\mu}{Q}\right)$. It's good from the computational point of view since there are only logs and no $\frac{\mu}{Q}$ power law terms, it naturally satisfies Lorentz and gauge invariance, and is also simple enough for multiloop calculations. However, the $\overline{MS}$ scheme has a sensitivity to renormalons, generically an IR ambiguity in $\Delta\bar{C}_0 \sim \frac{\Lambda_{QCD}}{Q}$ and a UV ambiguity in $\Delta\bar{O}_1 \sim \Lambda_{QCD}$ with a $u=1$ renormalon. Consider a toy integral $\sigma \sim \int_0^\infty d^{d-3}k \frac{f(k^2,\Lambda_{QCD}^2)}{(k^2+Q^2)^{1/2}}\mu^{2\epsilon}$, then the separation in the $\overline{MS}$ scheme with a high energy $k\sim Q$ piece and a low energy $\frac{1}{Q}$-expansion piece becomes:

$$\sigma \sim \mu^{2\epsilon}\int_0^\infty d^{d-3}\frac{f(k^2,0)+...}{(k^2+Q^2)^{/2}} + \mu^{2\epsilon}\int_0^\infty d^{d-3}kf(k^2,\Lambda_{QCD}^2)(\frac{1}{Q}+...) \sim \bar{C}_0(Q,\mu)\bar{O}_0 + \bar{C}_1\frac{\bar{O}_1(\mu)}{Q}\,. \tag{5.49}$$

Note that the identification with the OPE is $\bar{O}_0 = \bar{C}_1 = 1$. The $\overline{MS}$ scheme separates short and long distance physics for logs correctly, but for powers it relies on setting the scale of integration to zero, which is forced from the very definition of dimensional regularization and the scheme itself. This treatment leaves residual sensitivity to power divergences from including the wrong regions of momentum space in the integrals, which results in renormalons. In the Wilsonian picture (different separation), the toy model will be cut-off by an explicit scale, as shown here :

$$\sigma \sim \int_{\Lambda_f}^\infty dk\frac{f(k^2,0)+...}{(k^2+Q^2)^{1/2}} + \int_0^{\Lambda_f} dkf(k^2,\Lambda_{QCD}^2)\Big(\frac{1}{Q}+...\Big) \sim C_0^{(W)}(Q,\Lambda_f) + \frac{O_1^{(W)}(\Lambda_f)}{Q}\,, \tag{5.50}$$

where $O_0^{(W)}(\Lambda_f) = C_1^{(W)}(Q,\Lambda_f) = 1$. This separation has no renormalon, but since it uses a hard cut-off, the calculations are very difficult and the symmetries will not be preserved order by order (as was mentioned earlier in these notes).

In a general $R$-scheme (working with $\bar{C}_1 = 1$ for simplicity), one needs to start with the $\overline{MS}$ scheme and then change the scheme by moving pieces around inside elements of the OPE (rearrangement of physics):

$$\bar{O}_1(\mu) = O_1(R,\mu) - R\sum_{n=1}^\infty d_n\big(\frac{\mu}{R}\big)\Big(\frac{\alpha_s(\mu)}{4\pi}\Big)^n\bar{O}_0 \;\;,\;\; \bar{C}_0(Q,\mu) = C_0(Q,R,\mu) + \frac{R}{Q}\sum_{n=1}^\infty d_n\big(\frac{\mu}{R}\big)\Big(\frac{\alpha_s(\mu)}{4\pi}\Big)^2 \tag{5.51}$$

Then $\sigma \sim C_0(Q,R,\mu)\bar{O}_0 + \frac{O_1(R,\mu)}{Q}$, and by choosing the coefficients $d$ the $u=1$ renormalon in the $\overline{MS}$ scheme can be removed (in detail, the power-law dependence on $R$ eliminates the sensitivity of $\overline{C}_n$ to small momenta). The choice of $d$ is indeed similar to the choice of the hard cut-off, hence actually this scheme change is perturbatively going toward the Wilsonian picture (in a Lorentz and gauge symmetries preserving way), starting from $\overline{MS}$.

<u>Example 1:</u> The MSR scheme for OPE reuses the coefficients of the $\overline{MS}$ scheme $b_n\big(\frac{\mu}{Q}\big)$ (with renormalons inside) at a different scale $R$ as $d_n\big(\frac{\mu}{R}\big) = b_n\big(\frac{\mu}{R}\big)$. With $\frac{R}{Q}\overline{C}_0(R,\mu)$ acting as an IR cut-off to ensure $C_0(Q,R,\mu)$ corresponds to short distance physics, one gets a renormalon-free expression:

$$C_0(Q,R,\mu) = \sum_{n=1}^\infty\Big(b_n\big(\frac{\mu}{Q}\big) - \frac{R}{Q}b_n\big(\frac{\mu}{R}\big)\Big)\Big(\frac{\alpha_s(\mu)}{4\pi}\Big)^n = \overline{C}_0(Q,\mu) - \frac{R}{Q}\overline{C}_0(R,\mu) \tag{5.52}$$

Indeed, since the renormalon is independent of $R$ and $Q$, they cancel out in the subtraction $b_n\big(\frac{\mu}{Q}\big) - \frac{R}{Q}b_n\big(\frac{\mu}{R}\big)$. Setting $\mu = R$ and pretending that $\overline{C}_0$ has no anomalous dimension, one can easily read-off the running $\frac{\partial C_0(Q,R,R)}{\partial lnR} = -\frac{R}{Q}\gamma\big(\alpha_s(R)\big)$. The flow can be solved to be:

$$\begin{aligned} C_0(Q,R_1,R_1) &= C_0(Q,R_0,R_0) + \frac{\Lambda_{QCD}}{Q}\sum_j s_j(-1)^j e^{i\pi\widehat{b}_1}\Big(\Gamma(-\widehat{b}_1-j,t_0) - \Gamma(-\widehat{b}_1-j,t_1)\Big) \\ &= C_0(Q,R_0,R_0)\mathcal{U}(Q,R_0,R_1) \end{aligned} \tag{5.53}$$

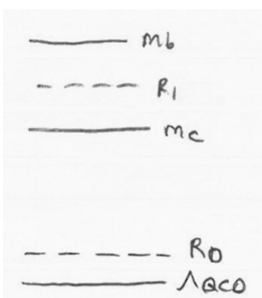
Example 2:. The OPE renormalon techniques can be used in HQET. Consider the ratio between the mass-squared difference of a symmetry multiplet $r = \frac{m_{b^*}^2 - m_B^2}{m_{D^*}^2 - m_D^2}$ perturbatively in the $\overline{MS}$ scheme:

$$r = \frac{\bar{C}_F(m_b, \mu)}{\bar{C}_F(m_c, \mu)} + \frac{\bar{\Sigma}_p(\mu)}{\mu_G^2(\mu)}\Big(\frac{1}{m_b} - \frac{1}{m_c}\Big) + ... \quad , \quad \frac{\Sigma_p(\mu) \sim \Lambda_{QCD}^3}{\mu_G^2(\mu) \sim \Lambda_{QCD}^2} \sim \Lambda_{QCD} \tag{5.54}$$

The theoretically calculated result $r = 1 - 0.113\Big|_{\alpha_s} - 0.078\Big|_{\alpha_s^2} - 0.0755\Big|_{\alpha_s^3}$ has contributions at different orders of similar sizes and doesn't seem to converge well. In terms of log order, one can rewrite this as $r = 0.8617\Big|_{LL} - 0.0696\Big|_{NLL} - 0.0908\Big|_{NNLL}$, which also doesn't help (the experimental data is close to the answer at LL order, however, and the next log order contributions are moving away). Indeed, there's a renormalon present in the calculation (can be shown to be the $u = \frac{1}{2}$ renormalon through the bubble sum), and to cure this one can use the MSR scheme by redefining $C_F(m_Q, R, R)$. This gives:

$$r = \frac{C_F(m_b, R_0, R_0)}{C_F(m_c, R_0, R_0)} + \frac{\Sigma_p(R_0, R_0)}{\mu_G^2(R_0)}\Big(\frac{1}{m_b} - \frac{1}{m_c}\Big) \quad , \quad \Sigma_p(R_0, R_0) = \bar{\Sigma}_p - R_0 \mu_G^2 \sum ...\alpha_s \tag{5.55}$$

The scale $\mu = R_0$ is chosen a little above $\Lambda_{QCD}$ so that $\Sigma_p(R_0, R_0)$ is still on the order of $\Lambda_{QCD}^3$, therefore doesn't mess up with the power counting in the the $\overline{MS}$ results. Using $R$-RGE to sum up logs between $R_0 \to m_Q$ gives




$$r = \frac{C_F(m_b, R_1, R_1)\mathcal{U}(m_b, R_1, R_0)}{C_F(m_c, R_1, R_1)\mathcal{U}(m_c, R_1, R_0)} + \frac{\Sigma_p(R_0, R_0)}{\mu_G^2(R_0)}\Big(\frac{1}{m_b} - \frac{1}{m_c}\Big)$$

The first term in this expression is given order by order numerically as $1 \to 0.88 \to 0.862 \to 0.860$, which converges pretty decently. The second terms (which are smaller than in the $\overline{MS}$ scheme) can be seen as a small uncertainty coming from varying $R_0$ and $R_1$, in the sense that $r^{NNLL} = 0.860 \pm 0.065\Big|_{\Sigma_p} \pm 0.0008\Big|_{pert}$. Note that, since the $R_0$ dependence cancels between the leading power term and the $\frac{1}{Q}$ term, it gives us a method for estimating the size of the power corrections (pretty much like what $\mu$ does for perturbative corrections in the $\overline{MS}$ scheme).

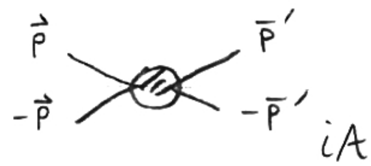
# 6 EFT with a Fine Tuning

This section is devoted to investigating an EFT where a naively irrelevant operator must be promoted to being relevant. We will see an operator, although it can be thought of as irrelevant from dimensional analysis, can have an anomalous dimension large enough that it actually becomes relevant.

## 6.1 Two Nucleon Nonrelativistic EFT

The Two Nucleon Nonrelativistic EFT (NNEFT) is a bottom up EFT that describes the SM in the limit of small momenta $p \ll m_\pi$ so that all exchanged particles, including the pions, can be integrated out. Nonlocal pion exchange becomes a local process, similar to how the massive weak gauge bosons are integrated out from QCD to give $H_{ew}$.

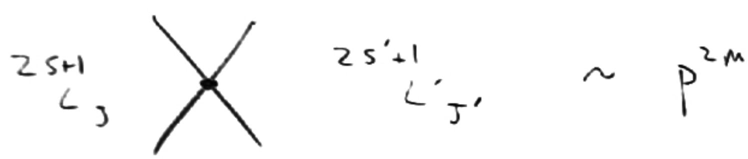
Let's start from a nonrelativistic elastic scattering in the center-of-mass frame (the particles have the same mass $M$, so energy conservation gives $|\mathbf{p}| = |\mathbf{p'}| = P$), which can be dealt with by using standard Quantum Mechanics. Indeed, the scattering can be described by a phase shift $S = e^{2i\delta} = 1 + \frac{iPM}{2\pi}\mathcal{A}$ (the familiar partial wave expansion), where $\mathcal{A} = \frac{4\pi}{M}\frac{1}{P\cot\delta - iP}$ is the scattering transition amplitude. For any short range potential the effective range expansion is $P^{2l+1}\cot\delta_l = -\frac{1}{a} + \frac{1}{2}r_o P^2 + \mathcal{O}(P^4)$, where $l$ is the particular partial wave of interest (e.g. $l = 0$ for s-wave and $l = 1$ for p-wave). This expansion is nontrivial in QM, since one needs to consider a general potential, but we can show it quite easily from EFT.




Let us prove the effective range expansion in NNEFT, which has the Lagrangian:

$$\mathcal{L} = N^\dagger\Big(i\partial_t + \frac{\nabla^2}{2m} + ...\Big)N - \sum_S \sum_{m=0}^{\infty} C_{2m}^{(S)} O_{2m}^{(S)} + ... , \tag{6.1}$$

where $N$ is the nucleon field with spin $\frac{1}{2}$ and isospin $\frac{1}{2}$ and $\sum_S \sum_{m=0}^{\infty} ...$ represents a contact interaction $O_{2m}^{(s)}$ of 4 nucleon fields $(N^\dagger N)^2$ with $2m$ derivatives (here $S$ is the spectroscopic channel ${}^{2S+1}L_J$). We are looking at the four nucleon interaction in this theory, which is diagrammatically,




It can be seen from the operators listed in the Lagrangian, that for any given in-channel and out-channel, the contribution $\sim P^{2m}$. Note that nucleons are fermions, therefore the wavefunction must be anti-symmetric. This gives us a relationship between the isospin and the angular momentum that tells us $(-1)^{s+l}$ is even for the isotriplet $I = 1$ and $(-1)^{s+l}$ is odd for the isosinglet $I = 0$. Angular momentum conservation forces $J = J'$, and for $s = 0$ we get $l = l'$, while for $s = 1$ we get $|l - l'| = 0, 2$. Going into more detail we explicitly write some of the operators in the Lagrangian:

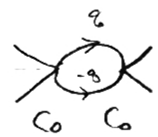
$$\sum_{m=0}^{\infty} C_{2m}^{(S)} O_{2m}^{(S)} = C_0^{(S)} (N^T \mathbb{P}_i^{(S)} N)^\dagger (N^T \mathbb{P}_i^{(S)} \overleftrightarrow{\nabla}^2 N) - \frac{C_2^{(S)}}{8}\Big((N^T \mathbb{P}_i^{(S)} N)^\dagger (N^T \mathbb{P}_i^{(S)} N) + h.c.\Big) + ... , \tag{6.2}$$

with $\overleftrightarrow{\nabla}^2 = \overleftarrow{\nabla}^2 + \overrightarrow{\nabla}^2 - 2\overleftarrow{\nabla}\overrightarrow{\nabla}$, and the matrices in spin-isospin space $\mathbb{P}^{{}^1S_0} = \frac{1}{\sqrt{8}}(i\sigma^2)(i\tau_2\tau_i)$, $\mathbb{P}^{{}^3S_1} = \frac{1}{\sqrt{8}}(i\sigma_2\sigma_i)(i\tau_2)$. The Feynman rules can be easily read-off (in the center-of-mass frame), giving the complete tree-level amplitude as follows:

Feyn. Rules  $= -iC_0$  $= -iC_2 p^2$ etc

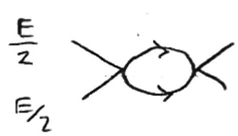
 $\equiv \sum_m$  $= -i \sum_m C_{2m} p^{2m}$ complete tree level amplitude

To study quantum effects one needs to consider loops. For simplicity, consider the following loop with total energy going in $E = 0$. One can see that by keeping only the $\partial_t$ terms (like in HQET) in the kinetic pieces the integral is ill-defined and has a pinch singularity:




$$= (-iC_0)^2 \int đ^d q \frac{i}{q^0} \frac{i}{-q^0}$$

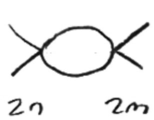
Indeed, the problem arises because the kinetic energy is a relevant operator in Quantum Mechanics (whenever one writes down the Schrödinger equation, one needs to keep it), therefore the right power counting should give $E \sim \frac{P^2}{2M}$ at leading order (which means that the $\partial_t$ terms and $\frac{\nabla^2}{2M}$ terms are about the same size, $i\partial_t \sim \frac{\nabla^2}{2M}$). It is generically true for 2 heavy particles to have this power counting for the kinetic terms, which is different than what we saw in the case of HQET. Adding the missing pieces and using dimensional regularization we get:




$$\begin{aligned} &= (-iC_0)^2 \int đ^d q \frac{i}{\left(\frac{E}{2} + q^0 - \frac{\mathbf{q}^2}{2m}\right)} \frac{i}{\left(\frac{E}{2} - q^0 - \frac{\mathbf{q}^2}{2m}\right)} \\ &= iC_0^2 \int đ^d q \frac{M}{\mathbf{q}^2 - ME} = -iC_0^2 \left(\frac{-iMP}{4\pi}\right) \end{aligned}$$

The above result has the nucleon mass (which is large) appearing in the numerator, which is usually a bad sign. Let's count the powers of $M$ while holding the spatial momentum $P$ fixed: $\nabla \sim M^0$ ($|\mathbf{x}| \sim M^0$), $\partial_t \sim \frac{1}{M}$ ($t \sim M$), $\int d^4x N^\dagger \left(i\partial_t - \frac{\nabla^2}{2M}\right) N \sim M^0 \to N \sim M^0$ and $\int d^4x C_{2m} O_{2m} \sim M^0 \to C_{2m} \sim \frac{1}{M}$ as $O_{2m} \sim M^0$. Hence, there's no issue with the counting of $M$, with the 1-loop and tree-level contributions being about the same size $\sim \frac{1}{M}$. From the dimension counting $[C_{2m}] = -2 - 2m$, therefore the coefficient is of the form $C_{2m} \sim \frac{1}{M\Lambda^{2m+1}}$ ($P \ll \Lambda$, where $\Lambda$ is set from all fields one integrates out).

We can also calculate the loop with two generic vertices:




$$= (-iC_0)^2 \int đ^d \mathbf{q}^{2n} \frac{M}{\mathbf{q}^2 - ME} \mathbf{q}^{2m} = (ME)^{n+m} \int \frac{đ^d q}{\mathbf{q}^2 - ME}$$

It is convenient that this theory is entirely made of bubbles of the same diagram type as above (indeed, one doesn't need to have anti-particles involved since it is a nonrelativistic theory). Summing all the contributions at all loop orders, the bubble chain gives a geometrical series:


$$= i\mathcal{A}_k = -i\Big(\sum_m C_{2m} P^{2M}\Big)^k \left(\frac{-iMP}{4\pi}\right)^{k-1}$$

which is easily summed as

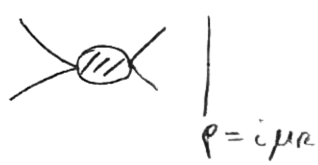
$$\mathcal{A} = \sum_k \mathcal{A}_k = -\frac{\sum_m C_{2m} P^{2m}}{1 + \frac{iMP}{4\pi}\left(\sum_m C_{2m} P^{2m}\right)} = \frac{4\pi}{M} \frac{1}{\left(-\frac{4\pi}{M}\frac{1}{\sum_m C_{2m} P^{2m}}\right) - iP} \tag{6.7}$$

The phase-shift can be found to be $\delta = \operatorname{arccot}\left(-\frac{1}{\sum_m \widehat{C}_{2m} P^{2m+1}}\right) = \delta_0$ where we've defined the shorthand $\widehat{C}_{2m} = \frac{mC_{2m}}{4\pi}$. This result is for the s-wave part, which can be Taylor-expanded as $P \cot\delta = -\frac{1}{\widehat{C}_0} + \frac{\widehat{C}_2}{\widehat{C}_0^2} P^2 + \mathcal{O}(P^4)$. The same can be done for higher partial waves, e.g. for the p-wave $l = 1$ (no $\widehat{C}_0$ contribution), $P^3 \cot\delta_1 = \frac{-P^2}{\widehat{C}_2 P^2 + \widehat{C}_4 P^4 + \dots} = -\frac{1}{\widehat{C}_2} + \frac{\widehat{C}_4}{\widehat{C}_2^2} P^2 + \dots$. At this point we have proven the effective range expansion for nonrelativistic QM. Note that this would have been much more difficult without our EFT approach.

The matching can be done easily giving $C_0 = \frac{4\pi}{M} a$ and $C_2 = \frac{4\pi}{M}\frac{a^2 r_0}{2}$, where $a$ and $r_0$ would be determined experimentally. Higher coefficients $C_{2m}$ can also be found in a similar fashion. For power counting, if $a, r_0 \sim \frac{1}{\Lambda}$ ($\Lambda \sim m_\pi$), one can reproduce $C_{2m} \sim \frac{1}{M\Lambda^{2m+1}}$. Unfortunately the value of the scattering length $a$ in nature is large (hence $C_0$ becomes huge), and $a$ seems to have a fine-tuning from the dimensional counting in the EFT point of view (note that for other scales, e.g. $r_0 \sim \frac{1}{m_\pi}$, they are of the expected size):

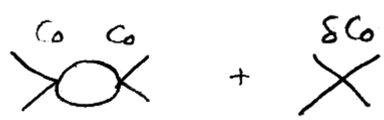
$$a^{(^1S_0)} = -23.714 \pm 0.013 (fm) \gg \frac{1}{m_\pi} \quad , \quad a^{(^3S_1)} = 5.425 \pm 0.001 (fm) \tag{6.8}$$

One needs to change the power counting a little bit, modifying $aP \ll 1$ to $aP \sim 1$ or even $aP \gg 1$. This means $C_0$ must be treated as relevant although from dimensional analysis $C_0 \sim \frac{1}{M\Lambda}$ is irrelevant. Since the problem originated from using dimensional regularization and the $\overline{MS}$ scheme, let us take a step back and use another scheme – the off-shell momentum subtraction (OS) scheme defined diagrammatically as:




$$= -i \sum_m C_{2m}(\mu_R) P^{2m}$$

where the scale $\mu_R$ keeps track of the power divergence that both dimensional regularization and the $\overline{MS}$ scheme couldn't see. The loop result is now changed with a finite correction:



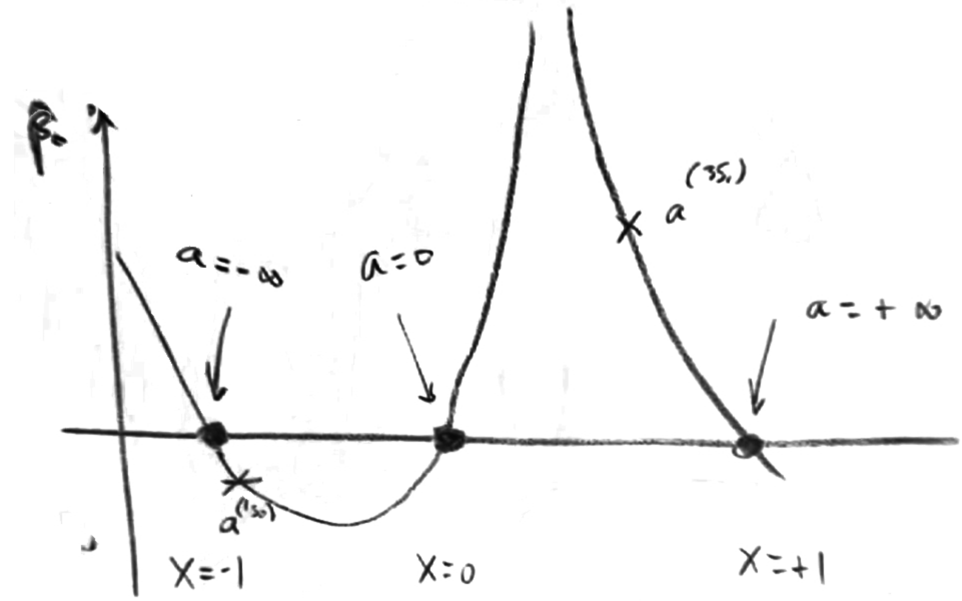

$$= \frac{iM}{4\pi} C_0(\mu_R)^2 \Big(iP + \mu_R\Big) \quad , \quad C_0^{bare} = C_0(\mu_R) + \delta C_0(\mu_R)$$

The renormalization group equation for $C_0(\mu_R)$ is straightforward,

$$\mu_R \partial_{\mu_R} C_0(\mu_R) = -\mu_R \partial_{\mu_R} \delta C_0(\mu_R) = \frac{M}{4\pi} C_0(\mu_R)^2 \quad , \quad C_0(0) = \frac{4\pi}{M} a = C_0^{\overline{MS}} \tag{6.11}$$

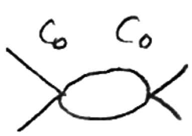
and yields the solution $C_0(\mu_R) = -\frac{4\pi}{M}\frac{1}{\mu_R - \frac{1}{a}}$. If $\mu_R \sim P \gg \frac{1}{a}$ then the correct power counting must be $C_0(\mu_R) \sim \frac{1}{M\mu_R}$, which means one has to swap the integrated-out energy scale with the physical scale of interest $\frac{1}{\Lambda} \to \frac{1}{\mu_R}$ (with this change, $C_0$ now becomes relevant as desired). One can summarize this by saying that the renormalization scheme can be chosen to make the right power counting easier.

Note that by counting $P$ one can find that $\partial_t$, $\frac{\nabla^2}{2M}$ and $C_0$ are all relevant. Another way to see that is to look at the renormalization group flow from the $\beta$-function, with $\beta_0 \sim \frac{a\mu_R}{(1-a\mu_R)^2}$ and the $a$-axis being mapped to a more compact version called the $x$-axis via $a\mu_R = \tan\left(\frac{\pi x}{2}\right)$:




There are 3 points where the $\beta$-function vanishes in the $a$-axis for a fixed value of $\mu_R$: $a = 0$ (noninteracting, since the only relevant terms are the kinetic pieces while all interactions are irrelevant) and $a = \pm\infty$ (interacting, since there are relevant interactions). Classically $a$ measures the interaction size, therefore the fixed points are either so big or so small that it's basically the same on all scales (conformal symmetry at these points). When one does perturbation theory, it's best to expand about the fixed points of the theory, and the problem pops out from naive dimensional analysis coming from perturbing around the wrong fixed point $a = 0$. There is an interesting feature where $a \to \frac{1}{\mu_R}$ and $\beta_0 \to \infty$ blows up, this corresponds to a deuteron bound state – from the $a = 0$ side one never sees a deuteron in perturbative calculations, while from the $a = +\infty$ side this is a true pole in the scattering amplitude as a deuteron must exist in the theory. Also, $a = \pm\infty$ are conformal fixed points with an enhanced $SU(4)$ symmetry (combined spin-isospin symmetry).

There is another scheme called the power divergence subtraction (PDS) scheme, which doesn't only subtract poles at $d = 4$ like in the $\overline{MS}$ scheme ($\sim \ln \Lambda$) but also poles in $d = 3$ ($\sim \Lambda$). The calculation is the following:




$$= iC_0^2\left(\frac{\mu}{2}\right)^{4-d}\int \frac{\bar{d}^dqM}{\mathbf{q}^2 - ME} = \frac{iC_0^2M}{(4\pi)^{\frac{d-1}{2}}}\Gamma\left(\frac{3-d}{2}\right)(-ME)^{\frac{d-3}{2}}\left(\frac{\mu}{2}\right)^{4-d}$$

At $d = 4$ the answer is $i\left(\frac{iMP}{4\pi}\right)$ and at $d = 3$ there's a pole $iC_0^2\frac{M}{4\pi}\frac{\mu}{3-d}$, which results in a counter-term to cancel this (the renormalized answer is then basically the same as OS scheme):


$$= i\frac{M}{4\pi}C_0(\mu)^2\left(iP + \mu\right)$$

This scheme (dimensional regularization style) indeed tracks the power divergent corrections, just like OS does. Now, with the knowledge of $C_0(\mu)$, one can find the running behavior of higher coefficients.

Example 1: Let's deal with $C_2(\mu)$:


have $p^2$

$$\mu\partial_\mu C_2(\mu) = \frac{M\mu}{4\pi} 2C_0(\mu)C_2(\mu) \quad , \quad C_2(0) = C_2^{\overline{MS}} = \frac{4\pi}{M} a^2 r_0 \tag{6.14}$$

This yields the solution $C_2(\mu) = \frac{4\pi}{M}\left(\frac{1}{\mu - \frac{1}{a}}\right)^2 \frac{r_0}{2}$. The RGE enhances $C_2$ from $\frac{1}{\Lambda^3}$ to $\frac{1}{\mu^2\Lambda}$.

Example 2: In general the RGE tells the enhancement due to fine-tuning from $a \to \infty$ of all operators in the theory (since fine-tuning messes up the naive power counting, one needs to fix it everywhere):

$$\mu\partial_\mu C_{2k}(\mu) = \frac{M\mu}{4\pi} \sum_{j=0}^{k} C_{2j} C_{2(k-j)} \quad , \quad C_{2j}C_{2(k-j)} \sim P^{2k} \tag{6.15}$$

The naive power counting gives $\widehat{C}_{2n} \sim \frac{1}{\Lambda^{2n+1}}$, while the improved one gives $\widehat{C}_{2n} \sim \frac{1}{\mu^{n+1}\Lambda^n} + \dots$.

For the few first coefficients:

| Naive $p \ll 1/a$ | Improved $pa \gg 1$ | |
|---|---|---|
| $\hat{C}_0 \sim 1/\Lambda$ | $\hat{C}_0 \sim 1/\mu$ | |
| $\hat{C}_2 \sim 1/\Lambda^3$ | $\hat{C}_2 \sim \frac{1}{\mu^2\Lambda}$ | irrelevant still |
| $\hat{C}_4 \sim 1/\Lambda^5$ | $\hat{C}_4 \sim \frac{1}{\mu^3\Lambda^2} + \frac{1}{\mu^2\Lambda^3}$ | |
| | no new constant / new constant | |

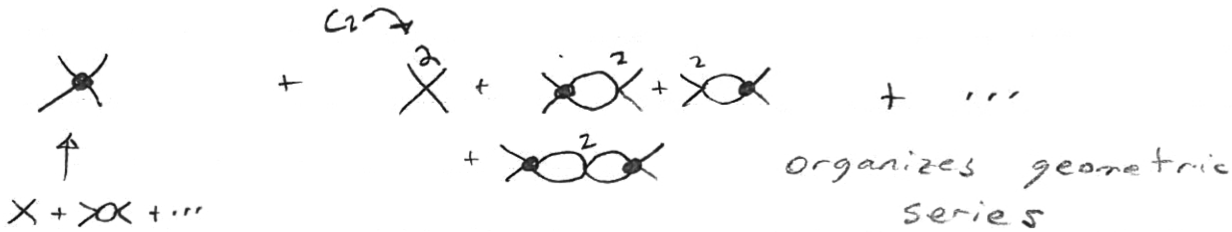
The scattering amplitude is now changed to:

$$\frac{M\mathcal{A}}{4\pi} = -\frac{\sum_m C_{2m}(\mu)P^{2m}}{1 + (\mu + iP)\Big(\sum_m C_{2m}(\mu)P^{2m}\Big)} \tag{6.16}$$

$$= -\frac{\widehat{C}_0(\mu)}{1 + \widehat{C}_0(\mu)\Big(\mu + iP\Big)} - \frac{\widehat{C}_2(\mu)P^2}{\Big(1 + \widehat{C}_0(\mu)\big(\mu + iP\big)\Big)^2} + \left(\frac{\Big(\widehat{C}_2(\mu)P^2\Big)^2(\mu + iP)}{(\dots)^3} - \frac{\widehat{C}_4(\mu)P^4}{(\dots)^2}\right) + \dots \tag{6.17}$$

$$= -\frac{1}{\frac{1}{a} + iP} - \frac{\frac{r_0}{2}P^2}{\left(\frac{1}{a} + iP\right)^2} - \frac{\left(\frac{r_0}{2}\right)^2 P^4}{\left(\frac{1}{a} + iP\right)^3} - \frac{\frac{r_1}{2\Lambda^2}P^4}{\left(\frac{1}{a} + iP\right)^2} + \dots \tag{6.18}$$

The answer is $\mu$-independent order by order (indeed, the scale $\mu$ only helps to do the power counting right, and in the end of the day, the physical prediction is free of $\mu$). In the diagram representation:




## 6.2 Symmetries of NNEFT

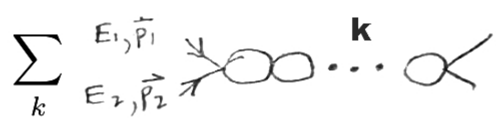
### 6.2.1 Conformal Invariance for Nonrelativistic Field Theory

The Schrödinger group – an extension of the Galilean group, instead of the Poincare group – contains:

- Translation transformation: 4 generators
- Rotation transformation: 3 generators
- Galilean Boosts transformation: 3 generators, $\mathbf{x}' = \mathbf{x} + ct$ and $t' = t$
- Scale transformation: 1 generator, $\mathbf{x}' = e^s\mathbf{x}$ and $t' = e^{2s}t$
- Conformal transformation: 1 generator, $\mathbf{x}' = \frac{\mathbf{x}}{1+ct}$ and $\frac{1}{t'} = \frac{1}{t} + c$

The NNEFT has this conformal symmetry at $a \to \infty$ ($C_0(\mu) \to -\frac{4\pi}{M\mu}$, a fixed point of the $\beta$-function), as the Lagrangian is invariant under $\mu \to e^{-s}\mu$. The Green's function is also invariant under this symmetry of the free Schrödinger equation (the operator $i\partial_t + \frac{\nabla^2}{2M}$).

Example: At leading order, adding up bubbles in a general frame gives:




$$\Rightarrow \mathcal{A}^{LO} = \frac{8\pi}{M}\frac{1}{\sqrt{-4M(E_1+E_2) + (\mathbf{p}_1 + \mathbf{p}_2)^2}}$$

This expression is both scale and conformal invariant, which leads to the cross section $\sigma = \frac{4\pi}{P^2}$.

### 6.2.2 SU(4) spin-isospin symmetry – Wigner's SU(4)

The infinitesimal form for Wigner's $SU(4)$ transformation is $\delta N = i\alpha_{\mu\nu}\sigma^\mu\tau^\nu N$, with $\sigma$ for spin and $\tau$ for isospin. To visualize the symmetry, let's rewrite the Lagrangian in a slightly different basis:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}C_0^S(N^\dagger N)^2 - \frac{1}{2}C_0^T(N^\dagger\overrightarrow{\sigma}N)^2 && (6.20)\\ &= -\frac{1}{4}(C_0^{(^1S_0)} + 3C_0^{(^3S_1)})(N^\dagger N)^2 - \frac{1}{8}(C_0^{(^1S_0)} - C_0^{(^3S_1)})(N^\dagger\overrightarrow{\sigma}N)^2 && (6.21) \end{aligned}$$

The first term has an explicit $SU(4)$ symmetry and when $a \to \infty$, $C_0^{(^1S_0)} = C_0^{(^3S_1)}$, gives rise to $SU(4)$ symmetry for the second term. In order to see this, we go to $a^{(^1S_0)}, a^{(^3S_1)} \to \infty$:

$$C_0^S(\mu) = -\frac{4\pi}{M\mu} \quad , \quad -\frac{M}{4\pi}C_0^T(\mu) = \frac{\frac{1}{a^{(^3S_1)}} - \frac{1}{a^{(^1S_0)}}}{\left(\mu - \frac{1}{a^{(^3S_1)}}\right)\left(\mu - \frac{1}{a^{(^1S_0)}}\right)} \tag{6.22}$$

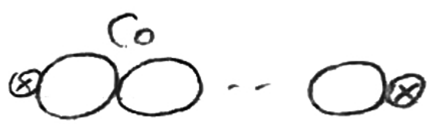
Note that nature is far from the $a^{(^1S_0)} = a^{(^3S_1)}$ limit, but $SU(4)$ can still be realized when both of them become large.

## 6.3 Deuteron

The deuteron $d = np$ is a bound state of a proton and a neutron, has isospin $I = 0$ and spin $s = 1$ ($^3S_1$ state). To look for a bound state in field theory, we approach from the lattice QFT point of view. So, write down some interpolating fields that overlap with that state by choosing operators with correct quantum numbers, such as $d_i = N^T\mathbb{P}^{(^3S_1)}N$ for the specific isospin and spin equal to deuteron's. Then we look for a pole and see whether the deuteron is in the theory or not:

$$G(\mathcal{E})\delta_{ij} = \int d^4x e^{ipx} \langle 0| T\Big(d_i^\dagger(x)d_j(0)\Big)|0\rangle \overset{?}{\sim} \frac{iZ(\mathcal{E})}{\mathcal{E}+B_d}\delta_{ij} \quad , \quad \mathcal{E} = E - \frac{\mathbf{p}^2}{2M} + \dots , \tag{6.23}$$

where $\mathcal{E}$ is the 2 nucleon center-of-mass energy. In our theory this is simple to calculate using our bubble chain,



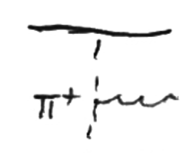

The sum of the bubble chain is nothing but a simple geometric series, which gives $G = \frac{\Sigma}{1+iC_0\Sigma}$ with $\Sigma$ encoding 2PI$^{C_0}$ diagrams. At leading order in the PDS renormalization scheme we get:

$$= \Sigma^{(1)}(\mu) = -\frac{iM}{4\pi}\Big(\mu - \sqrt{-M\mathcal{E}}\Big)$$

Define $E_B = -\mathcal{E} > 0$ and $\gamma_B = \sqrt{-M\mathcal{E}} = \sqrt{ME_B} = -iP$, therefore $G = \frac{1}{\frac{1}{a}-\gamma_B}$. There is a pole for $\gamma_B = \frac{1}{a^{(^3S_1)}} \approx 36$ MeV $> 0$, corresponding to a physical deuteron state. A similar calculation can be done at the $^1S_0$ channel, but no pole is found. The binding energy of the deuteron is $E_B = \frac{\gamma_B^2}{M} = 1.4$ MeV, which is about the same order as the experimental value $E_d = 2.2$ MeV.

With the $LSZ$ reduction for the bound state, one can calculate the deuteron electromagnetic form-factor. The matrix element of the electromagnetic current, $\langle p', j|\, J^\mu_{EM}\,|p, i\rangle$, can have 3 possible form-factors: electric $F_E(q^2)$ (from charge conservation $F_E(0) = 1$), magnetic $F_M(q^2)$ ($\frac{eF_M(0)}{2m_d} = \mu_M$) and quadrupole $F_Q(q^2)$. To put electromagnetism in the theory use the covariant derivative $D_\mu N = \Big(\partial_\mu + ie\mathbb{Q}_{EM}eA_\mu\Big)$ , $\mathbb{Q}_{EM} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. There is another operator that should be written down, describing how magnetism couples to a deuteron $eL_2(\mu)(N^T\mathbb{P}_iN)^\dagger(N^T\mathbb{P}_i\vec{\sigma}\mathbf{B}N) + h.c.$ ($L_2(\mu) \sim \frac{1}{Mmu^2\Lambda}$), which diagrammatically looks like


comes from eg



The LSZ reduction for the matrix elements of interest (on-shell, $\mathcal{E}, \mathcal{E}' \to -B_d$) becomes:

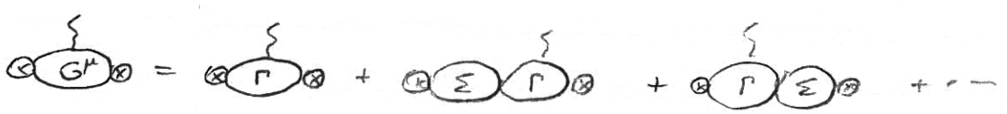
$$\langle p', j | \, J^\mu_{EM} \, | p, i \rangle = Z_B \Big( G^{-1}(\mathcal{E}) G^{-1}(\mathcal{E}') G^\mu_{ij}(\mathcal{E}, \mathcal{E}', \tau) \Big) \Big|_{\mathcal{E}, \mathcal{E}' \to -B_d} , \tag{6.25}$$

$$G^\mu_{ij} = \int d^4x d^4y e^{-ipx + ip'y} \langle 0 | \, T \Big( d_i^\dagger(x) J^\mu_{EM}(0) d_j(y) \Big) \, | 0 \rangle \tag{6.26}$$

where $Z_B$ is our bound state $Z$-factor $Z_B$, $G^{-1}(\mathcal{E}) G^{-1}(\mathcal{E}')$ give our truncation by two point functions and $G^\mu_{ij}(\mathcal{E}, \mathcal{E}', \tau)$ gives the 3-point function. In terms of diagrams (defining $\Sigma$ as the part irreducible by $C_0$), the 2-point function can be represented as:


$$= \frac{\Sigma}{1 + iC_0\Sigma}$$

Also with $\Gamma^\mu$ being irreducible by $C_0$, the 3-point function can be drawn as:




$$= \frac{\Gamma^\mu}{(1 + iC_0\Sigma)(1 + iC_0\Sigma)}$$

To find the $Z$-factor note that $G = \frac{iZ_B(\mathcal{E})}{\mathcal{E} + B_d}$ has a residue:

$$Z_B(-B_d) = -i\partial_{\mathcal{E}} G^{-1}(\mathcal{E}) \Big|_{\mathcal{E} = -B_d} = \frac{i\Sigma^2}{\partial_{\mathcal{E}}\Sigma} \Big|_{\mathcal{E} = -B_d} \tag{6.29}$$

All pieces can be put together back in the LSZ reduction matrix element of interest:

$$\langle p', j | \, J^\mu_{EM} \, | p, i \rangle = i \frac{\Gamma^\mu_{ij}(\mathcal{E}, \mathcal{E}', q)}{\partial_{\mathcal{E}}\Sigma} \Big|_{\mathcal{E}, \mathcal{E}' = -B_d} \tag{6.30}$$

At lowest order the electromagnetic current $J^0_{EM}$ for the electric case is:

$$\to \partial_{\mathcal{E}} \Sigma^{(1)} \Big|_{\mathcal{E} = -B_d} = -\frac{iM^2}{8\pi\gamma_B}$$

$$\to \Gamma^{(-1)}_{ij} = -e \frac{M^2}{2\pi q} \arctan\left( \frac{q}{4\gamma_B} \right) \delta_{ij}$$

One can go to higher orders in perturbation series (e.g. for the next leading order, one needs $\Sigma^{(2)}$ and $\Gamma^{(0)}_{ij}$; note that without $\Sigma^{(2)}$ the charge of the deuteron is messed up as $F_E(0) \neq 1$). The physics of the deuteron from this theory fits extremely well with experimental data.

We can also study phenomenology with NNEFT. Other processes of interest are neutron-proton scattering $np \to d\gamma$ (Big Bang Nucleosynthesis, which is calculated up to $N^4LO$), deuteron break-up $\gamma d \to np$, neutrino-deuteron scattering $\nu d \to ppe^-$ and $\nu d \to pn\nu$ (these are a charge current and a neutral current process studied at the Sudbury Neutrino Observatory), and more.

Another process of great interest is the nucleon-nucleon scattering that produces an axion, $NN \to NN+$ axion, which comes from the following pieces in the Lagrangian:

$$\mathcal{L}_{int} = g_0(\nabla^i X^0)\Big|_{\mathbf{X}=0} N^\dagger \sigma^i N + g_1(\nabla^i X^0)\Big|_{\mathbf{X}=0} N^\dagger \sigma^i \tau^3 N \tag{6.31}$$

It's important to decide kinematically what region of the phase space to look at, and for bounding axion physics for axions in the sun $E_{axion} \sim E_{nucleon}$ and $|\mathbf{k}_{axion}| \ll |\mathbf{p}_{nucleon}|$ (to implement this particular region use the multipole expansion by choosing the spatial part of the axion field going to 0 to make $(\nabla X^0)$ exchanging energy but not momentum with the rest of the operators at lowest order). Note that $N^\dagger \sigma^i N$ and $N^\dagger \sigma^i \tau^3 N$ are related to the conserved charge of Wigner's $SU(4)$ symmetry in the EFT $(Q_{\mu\nu} = \int d^3\mathbf{x} N^\dagger \sigma_\mu \tau_\nu N)$, therefore since the charges of a field theory are time-independent (no energy exchange) the axion has vanishing energy, which means no scattering. Indeed, $NN(^1S_0) \to NN(^1S_0)X^0$ must be gone because of angular momentum ($X^0$ in p-wave), $NN(^3S_0) \to NN(^3S_0)X^0$ vanishes for any $a$ ($Q_{i0}$ is conserved spin) and $NN(^1S_0) \to NN(^2S_0)X^0$ disappears as $a \to \infty$ (the amplitude for this process is $\mathcal{A} \sim \mathbf{k}\epsilon(^3S_1)\left(\frac{1}{a^{(^1S_0)}} - \frac{1}{a^{(^3S_1)}}\right)\frac{1}{(\frac{1}{a^{(^1S_0)}}+iP)}\frac{1}{(\frac{1}{a^{(^3S_1)}}+iP)} \to 0$ as $a^{(^1S_0)}$ and $a^{(^3S_1)}$ become huge). To summarize: at lowest order in the EFT the process is suppressed.

# A Introduction to the Standard Model

Here we give an overview of the symmetries and quantum numbers in the Standard Model.

## A.1 U(1) gauge symmetry (QED) and SU(3) gauge symmetry (QCD)

Consider electromagnetism with a single fermion $\psi(x)$ with charge $Q$ ($Q = -1$ for $e^-$). To get the Lagrangian to be invariant under the gauge transformation $\psi(x) \to e^{iQ\alpha(x)}\psi(x) = U(x)\psi(x)$ (for an infinitesimal $\alpha$, $U = 1 + iQ\alpha(x) + ...$), one needs to introduce a gauge field that transforms as $A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu\alpha$ (which can be written in terms of $U(x)$ as $QA_\mu(x) \to QA_\mu(x) - \frac{i}{e}\Big(\partial_\mu U\Big)U^{-1}$) and change the purely spatial derivatives (which mess up the $U(1)$ symmetry) to a gauge covariant version $i\partial_\mu \to iD_\mu\psi = (i\partial_\mu + eQA_\mu)\psi$, which leads to the transformation becoming nice again $iD_\mu\psi \to U(x)iD_\mu\psi$. One can form the gauge field strength $F_{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A^\mu$ ($F_{\mu\nu}$ is gauge invariant, $F_{\mu\nu} \to F_{\mu\nu}$) from $[D_\mu, D_\nu]\psi = iQeF_{\mu\nu}\psi$. A $U(1)$ gauge invariant QED Lagrangian can be written down:

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{A.1}$$

For $SU(3)$ (color) gauge symmetry, a triplet of fermion fields (3 colors) $\psi(x)$ transforms as $\psi(x) \to U(x)\psi(x)$ with $U(x) = e^{i\alpha^A(x)T^A}$ ($T^A = \frac{\lambda^A}{2}$) where $\lambda^A$ ($A = 1, \dots, 8$) are the familiar Gell-Mann matrices. Define the non-Abelian gauge covariant derivative $iD_\mu\psi = (i\partial_\mu + gT^A A^A_\mu)\psi$, and in terms of $A_\mu = A^A_\mu T^A$, the $A^A_\mu$ component transforms as $A_\mu \to U\Big(A_\mu + \frac{i}{g}\partial_\mu\Big)U^{-1}$ (this is indeed similar to QED if one writes the latter as $QA_\mu \to QA_\mu + \frac{i}{e}U(\partial_\mu U^{-1})$, using $\partial_\mu(UU^{-1}) = 0$). We can use the same commutation trick $[D^\mu, D^\nu]\psi = igF^{\mu\nu}\psi$ to get the gauge field strength $F_{\mu\nu} = F^A_{\mu\nu}T^A$, $F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + gf^{ABC}A^B_\mu A^C_\nu$ ($F_{\mu\nu}$ transforms as $F_{\mu\nu} \to UF_{\mu\nu}U^{-1}$ so $\mathrm{Tr}\,(F^{\mu\nu}F_{\mu\nu})$ is invariant). The gauge fields transform as an octet in the adjoint representation $iD_\mu A^A_\nu = (i\partial_\mu\delta^{AC} + igf^{ABC}A^B_\mu)A^C_\nu$, where the second term shows that the gauge fields are charged. The $SU(3)$-invariant Lagrangian for QCD is:

$$\mathcal{L}_{QCD} = \sum_{i=1,2,3} \bar{\psi}_i(i\not{D} - m)\psi_i - \sum_{A=1,\dots,8} \frac{1}{4}F^A_{\mu\nu}F^{\mu\nu A} \tag{A.2}$$

Note that this Lagrangian contains only renormalizable interactions: operators with dimensions $\leq 4$ (for example, the dimension-5 operator $g'\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi$ is not included). The standard reasoning is that one needs to impose a cutoff $\Lambda$ for the UV divergences, and demands that all the divergences can be absorbed into parameters of the theory (e.g. $g(\Lambda)$, $M(\Lambda)$), then takes $\Lambda\to\infty$ (in dimensional regularization this means $\epsilon\to 0$ where $d=4-2\epsilon$). The renormalization group will actually allow one to make an even a stronger statement, for a finite $\Lambda$. Actually, there is still another gauge-invariant dimension-4 operator $\theta\epsilon^{\mu\nu\lambda\tau}F^A_{\mu\nu}F^A_{\lambda\tau}$ ($\theta$ is a coupling constant) – although this term can be written as a total derivative, it is nevertheless topologically meaningful in QCD, but it's not a term of interest here, so we will ignore it.

## A.2 The Standard Model $SU(3)\times SU(2)\times U(1)$ gauge symmetry

Phenomenology is used to infer the charges (representations) of the fields, and from the gauge symmetry one can find $\mathcal{L}_{SM}$. The Lagrangian, schematically, can be split as $\mathcal{L}=\mathcal{L}_{gauge}+\mathcal{L}_{fermi}+\mathcal{L}_{Higgs}+\mathcal{L}_{\nu_R}$. We will examine each of these pieces in turn.

The pure gauge kinetic part is:

$$\mathcal{L}_{gauge}=-\frac{1}{4}B_{\mu\nu}B^{\mu\nu}-\frac{1}{4}W^a_{\mu\nu}W^{\mu\nu a}-\frac{1}{4}F^A_{\mu\nu}F^{\mu\nu A} \tag{A.3}$$

where $B_{\mu\nu}$ is field strength for the $U(1)$ gauge field, $W^a_{\mu\nu}$ ($a=1,2,3$) gives the $SU(2)$, and $F^A_{\mu\nu}$ ($A=1,\ldots,8$) is $SU(3)$. The $SU(2)$ is similar to $SU(3)$, except that the generators are $T^a=\sigma_a/2$ (where $\sigma_a$ are the Pauli matrices) and the generators satisfy $[T^a,T^b]=i\epsilon^{abc}T^c$. All the gauge bosons transform under the adjoint representations of their corresponding groups, and they form 3 distinct sectors (e.g., no $U(1)$ charge for gluons).

The fermion part is simply $\mathcal{L}_{fermi}=\bar{\psi}i\not{D}\psi$ and one has to specify what the field degrees of freedom $\psi$ and gauge covariant derivatives $D$ there are. Since $\psi$ should describe 6 flavors of quarks, one can arrange them in doublets for quarks $\begin{pmatrix}u\\d\end{pmatrix}$, $\begin{pmatrix}c\\s\end{pmatrix}$, $\begin{pmatrix}t\\b\end{pmatrix}$ and leptons $\begin{pmatrix}\nu_e\\e^-\end{pmatrix}$, $\begin{pmatrix}\nu_\mu\\\mu^-\end{pmatrix}$, $\begin{pmatrix}\nu_\tau\\\tau^-\end{pmatrix}$. These can be viewed as 3 families of fermions, in the order written above (for example, the first family includes $u$, $d$, $\nu_e$, and $e^-$). The covariant derivative has the general form $iD_\mu=i\partial_\mu+ig_1YB_\mu+ig_2T^aW^a_\mu+igT^AA^A_\mu$ ($g$, $g_1$ and $g_2$ are coupling constants, $Y$ is the $U(1)$ charge, $T^a$ is an $SU(2)$ generator in a particular representation, and $T^A$ is an $SU(3)$ generator in a particular representation). For the degrees of freedom, quarks transform as triplets (fundamental representation) under the color $SU(3)$ (all the quarks transform together, and color is flavor-blind) and leptons are singlets.

$SU(2)$ breaks parity – it acts only on the left-handed fields. The left handed field is obtained as $\psi_L=P_L\psi$ ($P_L=\frac{1-\gamma^5}{2}$ is the projection operator). Charge matter comes in $SU(2)$ doublets: $Q_L=\begin{pmatrix}u_L\\d_L\end{pmatrix}$ and $L_L=\begin{pmatrix}\nu_{eL}\\e_L\end{pmatrix}$ (the analysis is identical for the other families, so let's look at $i=1$ only). The right-handed fields, given by $\psi_R=P_R\psi$ ($P_R=\frac{1+\gamma^5}{2}$), are singlets: $u_R$, $d_R$, $e_R$, and maybe $\nu_R$. One might also include $\nu_R$ in a special term in the Lagrangian $\mathcal{L}_{\nu_R}$, since if the neutrino is massless it can be completely dropped because $\nu_R$ is uncharged (colorless, right-handed, neutral). However, it is known that the neutrino masses are non-zero ($m_\nu\neq 0$).

A mass term for the fermions is $m\bar{\psi}\psi=m\psi^\dagger\gamma^0(P_LP_L+P_RP_R)\psi=m(\bar{\psi}_R\psi_L+\bar{\psi}_L\psi_R)$ ($P_L+P_R=1$, $P_L=P_L^2$, $P_R=P_R^2$, $\gamma^0\gamma^5=-\gamma^5\gamma^0$). Under $SU(2)$, $\psi_L\to U\psi_L$ and $\psi_R\to\psi_R$, the mass term violates $SU(2)$ gauge symmetry, hence it cannot be included. Instead masses are generated (for fermions, and also for the gauge bosons) through the Higgs mechanism, which will appear in the Higgs sector, $\mathcal{L}_{Higgs}$.

Note that the $U(1)$ factor is not electromagnetism, as it can only have a single charge under the $U(1)$ (otherwise it would mess up the $SU(2)$ gauge symmetry). Electromagnetism is actually hiding inside $SU(2) \times U(1)$. The difference in the electromagnetic charge between the upper and the lower components of the doublets is always 1 (e.g. $\frac{2}{3} - (-\frac{1}{3}) = 1$, $0 - (-1) = 1$), so let's take the electromagnetic charge to be $Q = T^3 + Y$ ($Y$ is the charge corresponding to the $U(1)$ – hypercharge, and $T^3$ is the $a = 3$ generator of $SU(2)$ – it's $\frac{1}{2}$ for the upper component of the left-handed doublet, $-\frac{1}{2}$ for the lower component, and 0 for the right-handed singlets). Since the neutrino is electrically neutral ($Q = 0$) while the electron has charge $Q = -1$, the $U(1)$ charge of this doublet should be $-1/2$. Using similar arguments one finds $Y(Q_L) = \frac{1}{6}$, $Y(u_R) = \frac{2}{3}$, $Y(d_R) = -\frac{1}{3}$, and $Y(e_R) = -1$ (although these charges seem somewhat arbitrary, they're constrained by anomalies – the possible mismatch between classical and quantum symmetries). From this analysis we can write down the quantum numbers for the matter content of the standard model (including some content that we will get to soon):

| Field | $SU(3)$ | $SU(2)$ | $U(1)$ | Lorentz |
|---|---|---|---|---|
| $Q_L^i = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix}$ | 3 | 2 | $1/6$ | $(\frac{1}{2}, 0)$ |
| $u_R^i$ | 3 | 1 | $2/3$ | $(0, \frac{1}{2})$ |
| $d_R^i$ | 3 | 1 | $-1/3$ | $(0, \frac{1}{2})$ |
| $L_L^i = \begin{pmatrix} \nu_L^i \\ e_L^i \end{pmatrix}$ | 1 | 2 | $-1/2$ | $(\frac{1}{2}, 0)$ |
| $e_R^i$ | 1 | 1 | $-1$ | $(0, \frac{1}{2})$ |
| $\nu_R^i$ | 1 | 1 | 0 sterile | $(0, \frac{1}{2})$ |
| $A_\mu^A$ | 8 | 1 | 0 | $(\frac{1}{2}, \frac{1}{2})$ |
| $W_\mu^a$ | 1 | 3 | 0 | $(\frac{1}{2}, \frac{1}{2})$ |
| $B_\mu$ | 1 | 1 | 0 | $(\frac{1}{2}, \frac{1}{2})$ |
| $H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ | 1 | 2 | $Y_H = ?$ | $(0, 0)$ |
| | leave $SU(3)$ unbroken | break $SU(2)$ | | |

Example: With these ingredients one can write the full $iD_\mu$ term for each of the fermions, for example:

$$iD_\mu Q_L = i\partial_\mu Q_L + g_1 \frac{1}{6} B_\mu Q_L + g_2 W_\mu^a \frac{\sigma^a}{2} Q_L + g A_\mu^A \frac{\lambda^A}{2} Q_L \ , \tag{A.4}$$

$$iD_\mu e_R = i\partial_\mu e_R - g_1 B_\mu e_R + 0 + 0 \ , \ \ldots \tag{A.5}$$

A family index $i = 1, 2, 3$ should be included for the sake of completeness (e.g. $u_L^1 = u_L$, $u_L^2 = c_L$, $u_L^3 = t_L$, etc).

The Higgs field $H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$, which is a doublet of complex scalars, will break $SU(2)_L \times U(1)_Y$ into $U(1)_Q$ and generate masses for the particles. To find $Y_H$ consider the following terms, which are allowed by gauge symmetry for the Higgs Lagrangian:

$$\mathcal{L}_{Higgs} = (D_\mu H)^\dagger (D^\mu H) + \mu^2 (H^\dagger H) - \lambda (H^\dagger H)^4 + \mathcal{L}_{Yukawa} \tag{A.6}$$

$$= (D_\mu H)^\dagger (D^\mu H) - \lambda \left( H^\dagger H - \frac{\mu^2}{2\lambda} \right)^2 + \mathcal{L}_{Yukawa} \tag{A.7}$$

We can see that the Higgs potential is minimized for a non-zero vev $\langle 0 | H | 0 \rangle \sim \sqrt{\frac{\mu^2}{2\lambda}} \neq 0$. Recall that $Q = T^3 + Y$, so $QH = \begin{pmatrix} (1/2 + Y_H) h_1 \\ (-1/2 + Y_H) h_2 \end{pmatrix}$. When the Higgs gets a vev, if one doesn't want to break electromagnetism, one picks $Y_H = 1/2$ and gives the vev only to the neutral component $\langle 0|h^0|0\rangle = \frac{v}{\sqrt{2}}$ in $H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$ (also, with $Y_H = -1/2$ an equivalent result is obtained). The most general Yukawa couplings consistent with the gauge symmetries are $\mathcal{L}_{Yukawa} = -g_e^{ij}\bar{e}_R^i H^t L_L^j - g_d^{ij}\bar{d}_R^i H^\dagger Q_L^j + g_u^{ij}\bar{u}_R^i H^T \epsilon Q_L^j + h.c.$ ($i, j = 1, 2, 3$ are family indices and $\epsilon = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Each term in this Lagrangian is a Standard Model singlet (even without the summation over family indices).

Example: Consider, for example, the first term: it is $SU(3)$-neutral, $H^t L_L$ is an $SU(2)$ singlet, and the sum of the hypercharges $Y_H = +1 - 1/2 - 1/2 = 0$. The fact that $H^T \epsilon Q_L$ in the last term forms an $SU(2)$ singlet follows from the fact that 2 is a pseudoreal representation of $SU(2)$, so that $\bar{2} = 2$. Let $U = e^{i\alpha^a\sigma^a}$. Then $\epsilon U \epsilon = -U^*$, as can be checked using $\epsilon^2 = -1$ in the Taylor expansion $U = 1 + i\alpha^a\sigma^a + \ldots$, $U^* = 1 - i\alpha^a\sigma^{*a} + \ldots$. It follows that $U^T\epsilon U = \epsilon$. Then under $SU(2)$ transformation $H^T\epsilon Q_L \to H^T U^T \epsilon U Q_L = H^T \epsilon Q_L$

Let's see how the vev of the Higgs gives masses to particles. Start with $H = \begin{pmatrix} 0 \\ h^0 \end{pmatrix}$ and $h^0 = \frac{v}{\sqrt{2}}$:

$$D_\mu H \frac{g_1}{2} B_\mu \begin{pmatrix} 0 \\ h^0 \end{pmatrix} + g_2 W_\mu^a \frac{\sigma^a}{2} \begin{pmatrix} 0 \\ h^0 \end{pmatrix} = \frac{h^0}{2} \begin{pmatrix} g_2(W_\mu^1 - iW_\mu^2) \\ g_1 B_\mu - g_2 W_\mu^3 \end{pmatrix} \tag{A.8}$$

$(D_\mu H)^\dagger (D^\mu H)$ includes the term $\frac{g_2^2 v^2}{8} W^{1\mu} W_\mu^1$, which is a mass term for $W_\mu^1$. Fermions get their mass from the Yukawa couplings (e.g. the term $g_e \bar{e}_R H^\dagger L_L + h.c.$ includes $g_e h_0 \bar{e}_R e_L + h.c.$, and after adding the hermitian conjugate and substituting the vev of $h^0$ one arrives at $\frac{g_e v}{\sqrt{2}}\bar{e}e = \frac{g_e v}{\sqrt{2}}(\bar{e}_R e_L + \bar{e}_L e_R)$ – a mass term for the electron).

Note that the construction of $\mathcal{L}_{SM}$ is based on gauge symmetry. One could instead have listed the bosons and fermions and constructed all $d \le 4$ operators first (e.g. $\partial_\mu\phi\partial^\mu\phi$, $\phi\partial^\mu\phi A_\mu$, $\phi\bar{\psi}\psi$, $\bar{\psi}\psi$, $\phi^2 A^\mu A_\mu$, $A^\mu A_\mu$, $\bar{\psi}\not{A}\psi$, $\bar{\psi}i\not{\partial}\psi$), then imposing gauge invariance would relate coefficients of these operators and set some of them to 0.

## A.3 Symmetries of the Standard Model

Let's look into the symmetries of the Standard Model in more detail:

1. Discrete Symmetries

   The discrete symmetries are parity ($P$: $(x^0, \mathbf{x}) \to (x^0, -\mathbf{x})$, $x \to x_P$), time-reversal ($T$: $(x^0, \mathbf{x}) \to (-x^0, \mathbf{x})$, $x \to x_T$) and charge conjugation ($C$: particles $\to$ anti-particles).

   Example: A fermion field transforms as $P\psi(x)P^{-1} = \gamma^0\psi(x_P)$, $C\psi(x)C = e(\bar{\psi})^T$ ($e$ is representation-dependent, e.g. in Peskin-Schröder notation $e = -i\gamma^2\gamma^0$ and in Bjorken-Drell notation $e = -i\gamma^2$). The term $\int d^4x \bar{\psi}(i\not{D} - m)\psi$ is invariant under $P$, $C$, and $T$. Under $P$ one has $m\bar{\psi}\psi \to m\bar{\psi}\psi(x_P)$ ($d^4x = d^4x_P$), under $C$ one has $\bar{\psi}\gamma^\mu\psi \to -\bar{\psi}\gamma^\mu\psi$ and $A_\mu \to -A_\mu$. So, we can see that QED is invariant under these symmetries, and QCD is also invariant, up to the term $\theta F^{\mu\nu}F^{\lambda\tau}\epsilon_{\mu\nu\lambda\tau}$ which breaks $C$ and $CP$. Experimentally, $\theta \le 10^{-10}$, so the violations, if they exist, are small. Weak interactions violate $P$, $C$, and $CP$. The $W$ boson couples to the current $\bar{\psi}_1\gamma^\mu P_L\psi_2$. Under $P$ one has $\bar{\psi}_1\gamma^\mu P_L\psi_2 \to \bar{\psi}_1\gamma^\mu P_R\psi_2$, under $C$ one has $\bar{\psi}_1\gamma^\mu P_L\psi_2 \to \bar{\psi}_2\gamma^\mu P_R\psi_1$, then under $CP$ one gets $\bar{\psi}_1\gamma^\mu P_L\psi_2 \to \bar{\psi}_2\gamma^\mu P_L\psi_1$ and if one adds the complex conjugate terms, the current is invariant under $CP$. However, if the coupling constant isn't real, $\lambda\bar{\psi}_1\gamma^\mu P_L\psi_2 + \lambda^*\bar{\psi}_2\gamma^\mu P_L\psi_1$ isn't invariant anymore (there is one such case like this in the Standard Model).

2. Classify global symmetries

   A global symmetry has the form $\psi \to e^{i\alpha^A T^A}\psi$ ($\alpha^A$ are spacetime-independent). There are symmetries that are exact symmetries of $\mathcal{L}_{SM}$, and there are also approximate symmetries: broken by small terms in $\mathcal{L}_{SM}$. For example, the $SU(2)$ isospin symmetry acting on the doublet $\begin{pmatrix} u \\ d \end{pmatrix}$ is broken by $\frac{m_u - m_d}{\Lambda}$ and by $\alpha_{EM} = \frac{1}{137}$, but is still a good symmetry of bound hadrons in QCD. The scale $\Lambda$ describes the strength of the first term in $\bar{\psi}(i\not{D} - m_u)\psi$. Since the size of a hadron is $\sim 1$ fm, the momentum is $p \sim \left(1 \text{ fm}\right)^{-1} \sim 200$ MeV $\sim \Lambda \gg m_u \sim 4$ MeV. In addition, there are symmetries that are spontaneously broken by the vacuum expectation value (hidden symmetries). In these cases, the Lagrangian $\mathcal{L}_{SM}$ is symmetric, but the ground state is not. Still, such symmetries have implications for the dynamics (Goldstone bosons and their interactions). Finally, there are symmetries that are anomalous: classical symmetry is not a symmetry of the quantum theory (breaking could be large or small).

3. Conserved charge

   Let's look at the symmetries classically. Suppose the Lagrangian is invariant under the transformation of the fields $\phi^i \to U^{ij}\epsilon\phi^j$ (infinitesimally $\phi^i \to (\delta^{ij} + i\epsilon T^{ij})\phi^j$), define $\pi_i^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}$, then $\partial_\mu \pi_i \mu = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} = \frac{\partial \mathcal{L}}{\partial \phi_i}$ where in the last step we used the equation of motion following from demanding $\delta \int d^4x \mathcal{L} = 0$. The next step is to define the current $J^\mu = \pi_i^\mu (iT^{ij})\phi_j$, hence this current is conserved:

   $$\partial_\mu J^\mu = \partial_\mu \pi^\mu \cdot iT \cdot \phi + \pi^\mu \cdot iT \cdot \partial_\mu \phi = \frac{\partial \mathcal{L}}{\partial \phi} \cdot iT \cdot \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot iT \cdot \partial_\mu \phi = \delta \mathcal{L} = 0 \tag{A.9}$$

   The conserved charge is simply $Q = \int d^3\mathbf{x} J^0 = \int d^3\mathbf{x} \pi_i^0 (iT^{ij})\phi_j$

   In the Hamiltonian formulation, the momentum conjugate to the field $\phi^j$ is $\pi^j = \pi^j_{\mu=0}$ and it satisfies the canonical commutation relations $[\phi^i(\mathbf{x},t), \pi^j(\mathbf{y},t)] = i\delta^{ij}\delta^3(\mathbf{x}-\mathbf{y})$, which give:

   $$[Q(t), \phi^j(\mathbf{y},t)] = \int d^3x [\pi^i(\mathbf{x},t), \phi^j(\mathbf{y},t)](iT^{ik}\phi_k(\mathbf{x},t)) = T^{jk}\phi_k(\mathbf{y},t) \ , \ [Q(t), \pi^j(\mathbf{y},t)] = T^{jk}\pi_k(\mathbf{y},t) \tag{A.10}$$

   This implies that for any operator $\mathcal{O}(t)$ built out of $\phi$ and $\pi$, one has the infinitesimal transformation generator popping out from the conserved charge operator-wise $[Q(t), \mathcal{O}(t)] = -i\delta\mathcal{O}$. In particular, for the Hamiltonian $H$ one gets $[Q(t), H] = -i\delta H = 0$ if the Hamiltonian is invariant under the symmetry. This result implies that $Q$ is conserved $\frac{d}{dt}Q = 0$.

4. Baryon Number and Lepton Number

   The baryon number, a $U(1)$ global symmetry, corresponds to the transformation $\psi_i \to e^{-i\theta}\psi_i$ ($i$ refers to each of the quarks, and in this case $T_{ij} = (-1)_{ij}$). The Lagrangian that's invariant under this symmetry is:

   $$\mathcal{L} = \bar{\psi} i\not{D}\psi - g_d \bar{d} H^\dagger Q_L + g_u \bar{u}_R H^T \epsilon Q_L \tag{A.11}$$

   It can be read-off that $\pi^i_\mu = \bar{\psi}^i i\gamma_\mu$ and $J_\mu = \sum_i \bar{\psi}^i \gamma_\mu \psi^i$. Baryon number is an accidental symmetry of the Standard Model. It is anomalous, but the effect of the anomaly is very small.

   Similar to the baryon number, the lepton number corresponds to the transformation $\psi^i \to e^{-i\theta}\psi^i$ where $i$ refers to each of the leptons. In fact, neglecting $\mathcal{L}_{\nu_R}$ one can diagonalize $\mathcal{L}_{SM}$ in family space by unitary redefinitions of $e_R$ and $L_L$ such that $g_e$ in $g_e^{ij}\bar{e}^i_R H^\dagger L^j_L$ is diagonal. The lepton number inside each family is then conserved separately.

Example: For example, for the first family, the symmetry transformation can be written as $\psi^i \to e^{-i\theta_e}\psi^i$, where $i$ includes only the electron and electron neutrino.

5. Quark Numbers

   The number of a given quark flavor, such as strangeness and charm, is approximately conserved in the standard model. The symmetry is broken due to non-diagonal $g_u^{ij}$ and $g_d^{ij}$. For example, weak interactions have a vertex with $s$, $u$, and $W$, so strangeness is violated. However, strangeness is a good symmetry of $\mathcal{L}_{QCD}$ and $\mathcal{L}_{QED}$. Strangeness is useful also in weak interactions: for example one can consider the matrix element $\langle \pi^+ e^- \bar{\nu}_e | H_{weak} | \bar{K}^0 \rangle$, where $\pi^+$ is a bound state of $u$ and $\bar{d}$, and $\bar{K}^0$ is a bound state of $s$ and $\bar{d}$, so the strangeness changes by 1. The Wigner-Eckart theorem can be used to obtain useful information.

6. Axial $U(1)$

   Consider the transformation $\psi^i \to e^{-i\theta\gamma^5}\psi^i$ ($i = u, d$ in the limit $m_u, m_d \ll \Lambda$). The corresponding current is $J_{\mu(5)} = \sum_{i=u,d} \bar{\psi}^i \gamma_\mu \gamma_5 \psi^i$. Another example is $\begin{pmatrix} u \\ d \end{pmatrix} \to \begin{pmatrix} e^{-i\theta\gamma^5} & 0 \\ 0 & e^{i\theta\gamma^5} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$ with $J_{\mu(5)} = \bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma^\mu\gamma^5 d$. These $U(1)$ are anomalous, and the effects of the anomalies are strong. These are not symmetries in the quantum theory.

7. Flavor $SU(3)$

   Flavor $SU(3)$ is a generalization of isospin $SU(2)$ to include 3 quarks: $u$, $d$, and $s$.

8. Heavy quark symmetries

   There exist approximate symmetries involving the heavy quarks $c$ and $b$, which become exact in the limit $m_c, m_b \gg \Lambda$. There is $U(2)$ flavor symmetry and $SU(2)$ spin symmetry that are combined into $U(4)$ symmetry that acts on the vector $(c^\uparrow, c^\downarrow, b^\uparrow, b^\downarrow)^T$.

9. Chiral symmetry

   The QCD Lagrangian has an $SU(2)_L \times SU(2)_R$ symmetry (in the limit $m_u, m_d \ll \Lambda$), corresponding to the two independent $SU(2)$ transformations $\psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \to \exp(-i\sigma \cdot \theta_L)\psi_L$ and $\psi_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix} \to \exp(-i\sigma \cdot \theta_R)\psi_R$. The corresponding currents are $J_L^{\mu a} = \bar{\psi}_L \gamma^\mu \sigma^a \psi_L$ and $J_R^{\mu a} = \bar{\psi}_R \gamma_\mu \sigma^a \psi_R$. This symmetry is spontaneously broken into $SU(2)_{isospin}$. Similarly, in the limit $m_u, m_d, m_s \ll \Lambda$, the QCD Lagrangian has an $SU(3)_L \times SU(3)_R$ symmetry that is spontaneously broken into $SU(3)_{flavor}$.

# B Renormalization Techniques

The goals for studying renormalization, in general:

1. Improve our understanding of renormalizable QFT (the operator $O$ has mass dimension $[O] \le 4$ in 4 spacetime dimensions) by showing that it is often the low energy limit of QFT with no restriction on operator dimensions (non-renormalizable).

2. Understand why quantum fluctuations at short distances (large momenta $\Lambda$) only affect the value of a few parameters $g(\Lambda)$, $m(\Lambda)$, etc. in renormalizable QFT (or at low energy).

3. Explore and exploit the scheme dependence of coupling constants (renormalization schemes).

4. Derive "renormalization group equations" (RGEs) which allow us to avoid a breakdown of perturbation theory due to large logs, e.g. $\alpha \ln(q^2/m_e^2)$ for $q^2 \gg m_e^2$ in QED, by using a smart choice of coupling $\alpha(q^2)$. If $q_1^2 \gg q_2^2 \gg m_e^2$ we'll see that the RGE connects $\alpha(q_1^2)$, $\alpha(q_2^2)$, and $\alpha(m_e^2)$.

Points 1 and 2 are usually thought of as Wilsonian RG. Point 4 is Gell-Mann–Low RG.

Consider for example non-renormalizable massless QED:

$$\mathcal{L}_{QED}^{\Lambda_0} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}i\not{D}\psi + \frac{g_5}{\Lambda_0}\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi + \frac{g_6}{\Lambda_0^2}(\bar{\psi}\psi)^2 + ... \tag{B.1}$$

The term with $g_5$ is a dimension-5 operator, the term with $g_6$ is a dimension-6 operator, and we have omitted an infinite number of other terms consistent with gauge symmetry. We can see that $g_5$ and $g_6$ are dimensionless, and $\Lambda_0$ is a dimension-1 constant, the mass scale of irrelevant operators that defines what low energy means. By dimensional analysis, the electron-positron scattering cross section is given by $\sigma(e^+e^- \to e^+e^-) \sim \frac{\alpha^2}{E^2} + \frac{\alpha g_5^2}{\Lambda_0^2} + ...$, where the first term comes from squaring the amplitude of a diagram with two $e$ vertices, and the second from squaring the amplitude of a diagram with one $e$ vertex and one $g_5$ vertex (the cross-term vanishes by chirality). For $E \ll \Lambda_0$, the first term dominates and the contributions from $g_5$, $g_6$,... are irrelevant. Therefore the operators with dim $> 4$ are called irrelevant operators. Point 1 says $\mathcal{L}_{QED}^{\Lambda_0}\Big|_{E \ll \Lambda_0} = \mathcal{L}_{QED} + \mathcal{O}\left(\frac{E}{\Lambda_0}\right)$.

## B.1 Wilsonian point of view

In the Wilsonian picture, a QFT should be regarded as an effective field theory valid in a certain range of energies with a finite physical UV cutoff $\Lambda_0$ (imposing $p_E^2 \lesssim \Lambda_0^2$). For $\mathcal{L}_{SM}^{\Lambda_0}$ this cutoff might be the scale of quantum gravity or a heavy particle we haven't seen.

One should write down all interactions consistent with symmetries of the theory. For $E \ll \Lambda_0$, $\mathcal{L}_{SM}^{\Lambda_0}$ will look like $\mathcal{L}_{SM}$. For example, our $\frac{g_5}{\Lambda_0}$ term contributes $\frac{4g_5}{\Lambda_0}$ to the electron magnetic moment. The value calculated from the Standard Model agrees with experiment to $10^{-10}\frac{e}{2m_e}$, so $\frac{\Lambda_0}{g_5} \gtrsim 8 \times 10^{10} m_e = 4 \times 10^7 (GeV)$ . Note that, in the Standard Model, $\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi$ is not consistent with $SU(2)$ gauge symmetry.

## B.2 Loops, Regularization, and Renormalization

Regularization is a cutoff on UV loop momenta (dimreg, cutoff $\Lambda$, Pauli-Villars, etc). Renormalization is picking up a scheme to give definite meaning to parameters in $\mathcal{L}$. Consider the $\phi^4$ theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 = \mathcal{L}[\phi_0, m_0, \lambda_0] = \mathcal{L}[\phi, m, \lambda] + \frac{1}{2}\delta Z(\partial_\mu\phi)^2 - \frac{1}{2}\delta m\phi^2 - \frac{\delta\lambda}{4!}\phi^4 \tag{B.2}$$

In the last equality, the Lagrangian is expressed in terms of the renormalized field $\phi$, and defined $\phi_0 = Z^{1/2}\phi$, $\delta Z = Z - 1$, $\delta m^2 = m_0^2 Z - m^2$, $\delta\lambda = \lambda_0 Z^2 - \lambda$. The counter terms "$\delta$..." remove UV divergences, but can also remove finite terms, and one needs a way of specifying those, and this is the renormalization scheme.

### B.2.1 On-shell renormalization scheme

In the on-shell scheme one requires that the 1PI 2-point function $\Pi(p^2)$ satisfies $\Pi(m^2) = 0$ and $\Pi'(m^2) = 0$. The first condition fixes $\delta m$, and the second fixes $\delta Z$. To fix $\delta\lambda$, the 1PI 4-point function is chosen to be equal to $-i\lambda$ at $s = t = u = \frac{4}{3}m^2$, where $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$ are the Mandelstam variables. For regularizing loop diagrams either dimensional regularization (dimreg) or a cutoff can be used, and the obtained results are the same for observables $\sigma(E, m^2, \lambda)$.

Example: The 1-loop contribution to the 4-particle scattering amplitude (regularized by a cutoff $\Lambda$ in Euclidean space) is given by the sum of the 3 contributing diagrams:

$$\mathcal{A} = \frac{i\lambda^2}{32\pi^2} \int_0^1 dx \Bigg( \ln\Big(\frac{\Lambda^2}{m^2 - x(1-x)s}\Big) + (t) + (u) - 3 \Bigg) \tag{B.3}$$

If regularized by dimensional regularization with $d = 4 - 2\epsilon$ ($B$ is a known number), then:

$$\mathcal{A} = \frac{i\lambda^2}{32\pi^2} \int_0^1 dx \Bigg( \frac{1}{\epsilon} + \ln\Big(\frac{\Lambda^2}{m^2 - x(1-x)s}\Big) + (t) + (u) + B \Bigg) \tag{B.4}$$

Here $(t)$ and $(u)$ denote the same term with $s$ replaced by $t$ or $u$. Since by our definition of $\lambda$ the 4-point function at $s = t = u = \frac{4}{3}m^2$ is exactly given by the tree level result, the contribution from the loops should be canceled by the counter-term. Substituting $s = t = u = \frac{4}{3}m^2$ into the loop result gives $\delta\lambda = \frac{\lambda^2}{32\pi^2}\Big(3\ln\big(\frac{\Lambda^2}{m^2}\big) + A - 3\Big)$ in the case of the cutoff, and $\delta\lambda = \frac{\lambda^2}{32\pi^2}\Bigg(\frac{3}{\epsilon} + 3\ln\big(\frac{1}{m^2}\big) + A + B\Bigg)$ in the case of dimreg (where $A$ is a known number). The sum of the loops and the counter-terms for general momentum gives $\mathcal{A}^{ren} = \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\Big(\ln\big(\frac{m^2}{m^2 - x(1-x)s}\big) + (t) + (u) - A\Big)$ independent of the method of regularization.

The fact that the argument of the log in $\ln\Big(\frac{1}{m^2}\Big)$ isn't dimensionless might seem to be a problem. The reason is that in dimensional regularization the dimension of fields and couplings changes. From the kinetic term for the scalar field $d^dx(\partial_\mu\phi\partial^\mu\phi)$ it can be seen that $[\phi] = 1 - \epsilon$, and from the $\phi^4$ term $d^dx\lambda\phi^4$ it can be seen that $[\lambda] = 2\epsilon$. It is then convenient to write $\lambda = \mu^{2\epsilon}\lambda(\mu)$ where $\lambda(\mu)$ is dimensionless. In the above analysis $\mu$ plays no role in $\mathcal{A}^{ren}$ because it is a part of the regulator, but the expression for $\delta\lambda$ is given by $\delta\lambda = \frac{\lambda(\mu)}{32\pi^2}\Big(\frac{1}{\epsilon} + \ln\big(\frac{\mu^2}{m^2}\big) + \dots\Big)$. Now suppose $s, t, u \gg m^2$, then $\mathcal{A}^{ren} \sim \lambda^2\ln(m^2/s)$. The large log could potentially spoil the $\lambda$-expansion, and in this limit it's natural to consider taking $m \to 0$, but both $\mathcal{A}^{ren}$ and $\lambda = \lambda_0 + \frac{\lambda_0^2}{32\pi^2}\Big(3\ln\big(\frac{\Lambda^2}{m^2}\big) + \dots\Big)$ blow up. Writing for any observable of dimension $D$, one gets $\Gamma(s, x, \lambda, m^2) = s^{D/2}\Gamma(x, \lambda, m^2/s)$, and the dimensionless function $\Gamma$ on the right-hand side doesn't have a good $m^2 \to 0$ limit ($x$ stands for the ratio $t/s$ and $u/s$).

### B.2.2 Off-shell renormalization scheme ($\mu_R$)

To solve the problem of the on-shell scheme, another scheme might be needed, where one might require the 1PI 4-point function to be equal $-i\lambda(\mu_R)$ at $s = t = u = -\mu_R^2$ where $\mu_R$ is an arbitrary renormalization scale (different values of $\mu_R$ are different schemes). The tree level contribution is now $-i\lambda = -i\lambda(\mu_R)$, and the 1-loop calculation gives:

$$\delta\lambda = \frac{\lambda^2(\mu_R)}{32\pi^2}\int_0^1 dx\Bigg(3\ln\Big(\frac{\Lambda^2}{m^2 + x(1-x)\mu_R^2}\Big) + \dots\Bigg) \tag{B.5}$$

This expression has a good $m \to 0$ limit. The total amplitude in this limit is given by:

$$\mathcal{A}^{ren} = \frac{i\lambda^2(\mu_R)}{32\pi^2}\int_0^1 dx\Bigg(\ln\Big(\frac{\mu_R^2}{-x(1-x)s}\Big) + \dots\Bigg) \tag{B.6}$$

$$\Rightarrow \Gamma(s, x, \lambda(\mu_R), m^2, \mu_R^2) = s^{D/2}\Gamma(x, \lambda(\mu_R), m^2/s, \mu_R^2/s) \tag{B.7}$$

These also have a good $m \to 0$ limit. Moreover, for $-\mu_R^2 \simeq s$, $\mathcal{A}^{ren}$ and $\Gamma$ have no large logs, just $\ln\left(\frac{\mu_R^2}{-s}\right)$, so perturbation theory is fine.

### B.2.3 Relating schemes

Different renormalization schemes are related. The bare couplings are scheme-independent, so $\lambda_0 = \lambda + \delta\lambda = \lambda(\mu_R) + \delta\lambda(\mu_R)$ with $\lambda$ and $\delta\lambda$ refering to the on-shell scheme. One has:

$$\lambda(\mu_R) = \lambda + \frac{\lambda^2}{32\pi^2}\left(3\ln\left(\frac{m^2 + x(1-x)\mu_R^2}{m^2}\right) + ...\right) \tag{B.8}$$

If $\mu_R \gg m$ this series may not converge. Consider $\lambda(\mu'_R) = G\Big(\lambda(\mu_R), z = \frac{\mu'_R}{\mu_R}, \frac{m}{\mu_R}\Big)$, where $G$ is some arbitrary function. Take $\mu'_R\partial_{\mu'_R}$ of this equation and set $\mu'_R = \mu_R$. Then one arrives at the Callan-Symanzik equation $\mu'_R\partial_{\mu'_R}\lambda(\mu_R) = \beta\Big(\lambda(\mu_R), \frac{m}{\mu_R}\Big)$where the $\beta$-function is $\beta = \partial_z G\Big(\lambda(\mu_R), z, \frac{m}{\mu_R}\Big)\Big|_{z=1}$. For our $\phi^4$ theory, the $\beta$-function can be computed for $\mu'_R \simeq \mu_R$ where there are no large logs, and then we can integrate the Callan-Symanzik equation to relate the coupling at different scales. The answer is $\lambda(\mu'_R) = \lambda(\mu_R) + \frac{3\lambda^2(\mu_R)}{32\pi^2}\ln\left(\frac{\mu_R'^2}{\mu_R^2}\right) + \mathcal{O}\Big(\lambda(\mu_R)^3\Big)$, and the $\beta$-function is $\beta = \frac{3\lambda^2(\mu_R)}{16\pi^2} + ...$

For $m = 0$, $\beta(\gamma(\mu_R)) = \frac{3\lambda^2(\mu_R)}{16\pi^2}$. Integrating this solution we obtain the result $\lambda(\mu_R) = \frac{\lambda}{1 - \frac{3\lambda}{16\pi^2}\ln(\mu_R/m)} = \lambda + \lambda\sum_{k=1}^{\infty} a_k\left(\lambda\ln\left(\frac{\mu_R}{m}\right)\right)^k$ ($\lambda$ is the coupling constant in the on-shell renormalization scheme). Note that $\lambda(\mu_R) \simeq \lambda + \mathcal{O}(\lambda^2)$ for $\lambda\ln\left(\frac{\mu_R}{m}\right) \ll 1$. The lesson is that the most appropriate coupling depends on the probed scale.

One can extend the off-shell treatment to the other coupling constant. Requiring the 2-point 1PI diagram $\Pi(p^2)$ to satisfy $\Pi(-\mu_R^2) = 0$ gives $m(\mu_R)$, and the condition $\Pi'(-\mu_R^2) = 0$ gives $Z(\mu_R)$, where $Z$ is the field renormalization $\phi = Z^{-\frac{1}{2}}(\mu_R)\phi_0$. Defining $\frac{Z^{\frac{1}{2}}(\mu'_R)}{Z^{\frac{1}{2}}(\mu_R)} = G^\phi\Big(\lambda(\mu_R), \frac{\mu'_R}{\mu_R}, \frac{m}{\mu_R}\Big)$, taking $\mu'_R\partial_{\mu'_R}$, setting $\mu'_R = \mu_R$ one gets $\mu_R\frac{d}{d\mu_R}Z^{\frac{1}{2}}(\mu_R) = \gamma^\phi\Big(\lambda(\mu_R), \frac{m}{\mu_R}\Big)Z^{\frac{1}{2}}(\mu_R)$ or $\mu_R\frac{d}{d\mu_R}\ln Z^{\frac{1}{2}}(\mu_R) = \gamma^\phi\Big(\lambda(\mu_R), \frac{m}{\mu_R}\Big)$, where $\gamma^\phi = \partial_z G^\phi\Big(\lambda(\mu_R), z, \frac{m}{\mu_R}\Big)\Big|_{z=1}$ is the anomalous dimension. Local products of operators, e.g. $O_0 = (\phi_0^2)(x)$ are renormalized as $O = Z^O O_0$ and this means $\mu_R\partial_{\mu_R}\ln Z^O(\mu_R) = \gamma^O\Big(\lambda(\mu_R), \frac{m}{\mu_R}\Big)$. In the $\phi^4$ theory the first-order (in $\lambda$) contribution to $Z$ vanishes (i.e. $Z = 1 + \mathcal{O}(\lambda^2)$) and $\gamma^\phi = 0 + \mathcal{O}(\lambda^2)$, so let's look into QED for more interesting physics.

## B.3 Renormalization of QED

The QED Lagrangian in Feynman gauge $\xi = 1$ can be written as:

$$\mathcal{L} = \bar{\psi}(iD - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial\cdot A)^2 + c.t. \tag{B.9}$$

The relations between the bare and the renormalized quantities are parameterized as $\psi_0 = Z_\psi^{\frac{1}{2}}\psi$, $A_0^\mu = Z_A^{\frac{1}{2}}A^\mu$, $m_0 = m - \delta m$, $e_0 = Z_e e$ and $\xi_0 = Z_A\xi$. $Z_\psi$ comes from a diagram in which the electron emits a photon and then absorbs it and $Z_A$ comes from a diagram in which the photon turns into an electron-positron pair, which then annihilate to create a photon. There is a relation $\bar{\psi}_0 e_0 A_0 \psi_0 = Z_\psi Z_A^{\frac{1}{2}} Z_e \bar{\psi} e A \psi \equiv Z_1 \bar{\psi} e A \psi$, with 4 diagrams being relevant to it, giving modifications to the electron-electron-photon vertex. In one of them, an additional photon is traveling between the incoming electron and the outgoing electron, in two others either the incoming or the outgoing electron is decorated with a $Z_\psi$ contribution and in the last one the photon is decorated with a $Z_A$ contribution. By the Ward identity $Z_1 = Z_\psi$ it can be shown that $Z_e = Z_A^{-\frac{1}{2}}$.

### B.3.1 QED: On-shell scheme

In the on-shell scheme one has $\alpha \simeq \frac{1}{137}$. In this scheme one requires that the 1PI electron-electron-photon vertex is equal to $-ie\gamma^\mu$ for $q^\mu = 0$ (where $q^\mu$ is the momentum of the photon) or $\Pi(q^2 = 0) = 0$. The photon propagator is given by $\frac{-iZ_A^{-1}}{q^2(1-\Pi_0(q^2))}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\frac{-i}{q^4}q^\mu q^\nu$. In the on-shell scheme this is picked to be $\frac{-i}{q^2(1-\Pi(q^2))}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\frac{-i}{q^4}q^\mu q^\nu$ ($Z_A$ is absent, and $\Pi_0$ is replaced by $\Pi$).

$$\Pi(q^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left(\frac{m^2 - q^2 x(1-x)}{m^2}\right) + \mathcal{O}(e^4) \tag{B.10}$$

$$Z_A = \Big(1 - \Pi_0(0)\Big)^{-1} = 1 - \frac{e^2}{12\pi^2}\ln\left(\frac{\Lambda^2}{m^2}\right) + \ldots \tag{B.11}$$

These results diverge in the massless $m = 0$ limit.

### B.3.2 QED: Off-shell momentum subtraction scheme

In the off-shell momentum subtraction scheme the photon propagator at $q^2 = -\mu_R^2$ is equal to $\frac{-i}{q^2}g^{\mu\nu} + q^\mu q^\nu \ldots$. This implies that $Z_A^{-1}(\mu_R) = 1 - \Pi_0(-\mu_R^2) = Z_A^{OS^{-1}}(1 - \Pi^{OS}(-\mu_R^2))$ and $G^{(A)} = \left(\frac{1-\Pi(-\mu_R'^2)}{1-\Pi(-\mu_R^2)}\right)^{-\frac{1}{2}}$:

$$G^{(A)}\Big(e(\mu_R), \frac{\mu_R'}{\mu_R}, \frac{m}{\mu_R}\Big) = \Big(\frac{Z_A(\mu_R')}{Z_A(\mu_R)}\Big)^{\frac{1}{2}} = 1 + \frac{e(\mu_R)^2}{4\pi^2}\int_0^1 dx\, x(1-x)\ln\Big(\frac{m^2 + \mu_R'^2 x(1-x)}{m^2 + \mu_R^2 x(1-x)}\Big) \tag{B.12}$$

$$\gamma^{(A)}\Big(e(\mu_R), \frac{m}{\mu_R}\Big) = \frac{e(\mu_R)^2}{2\pi^2}\int_0^1 dx\, \frac{x^2(1-x)^2\mu_R^2}{m^2 + \mu_R^2 x(1-x)} \quad , \quad e(\mu_R) = Z_A(\mu_R)^{\frac{1}{2}} e_0 \tag{B.13}$$

Note that $e_0 A_0 = (e_0 Z_A(\mu_R)^{\frac{1}{2}})(A_0 Z_A(\mu_R)^{-\frac{1}{2}} = eA$ should be $\mu_R$-independent. So:

$$\beta = \mu_R \partial_{\mu_R} e(\mu_R) = \frac{1}{Z_A}\mu_R \partial_{\mu_R} Z_A^{\frac{1}{2}} e_0 = e(\mu_R)\gamma^{(A)} = \frac{e(\mu_R)^3}{2\pi^2}\int_0^1 dx \frac{x^2(1-x)^2\mu_R^2}{m^2 + \mu_R^2 x(1-x)} \tag{B.14}$$

In the limit $\mu_R \gg m$ one gets $\beta \simeq \frac{e(\mu_R)^3}{12\pi^2}$ and this is known as the QED $\beta$-function. In the limit $\mu_R \ll m$, $\beta \simeq \frac{e(\mu_R)^3}{60\pi^2}\frac{\mu_R^2}{m^2}$. In general, the function passes smoothly through $\mu_R = m$, and can be directly matched to $e(\mu_R = 0) = e$. For $m > \infty$ (the non-relativistic limit) we get $\beta = 0$ and $e(\mu_R) = e$, the charge we use in quantum mechanics. Integrating the expression for $\mu_R \gg m$ leads to $\alpha(\mu_R) = \frac{\alpha}{1 - \frac{2\alpha}{3\pi}\ln(\mu_R/m)}$, which is similar to what is found for the $\phi^4$ theory. For a general $\mu_R/m$, the expression has the form $\alpha(\mu_R) = \frac{\alpha}{1-\frac{2\alpha}{3\pi}f(\mu_R/m)}$ (the function $f(\mu_R/m)$ can be computed from the $\beta$ function, and it satisfies $f(0) = 0$, $f(1) \neq 0$). For $\frac{\mu_R}{m} \lesssim 1$, the function $\alpha(\mu_R)$ is approximately flat with the value $\alpha(0) \simeq \frac{1}{137}$, then grows slowly with $\frac{mu_R}{m}$ (e.g. reaching $\alpha(10) \simeq \frac{1}{136.5}$ for $\frac{mu_R}{m} = 10$). At the mass of the $W$ boson ($\sim 80$ GeV) $\alpha(\mu_R = m_W) \simeq \frac{1}{128}$. People take these changes into account for precision weak physics.

### B.3.3 QED: $MS$ scheme and $\bar{M}S$ scheme

The minimal subtraction scheme is an efficient method to get $\beta$ and $\gamma$ in dimensional regularization ($d \to d - 2\epsilon$) where $\mu$ is a sliding scale and the counter-terms are simple. From the term in the action $\int d^d x \bar{\psi}_0 \not{A}_0 \psi_0 e_0$ one sees that the dimension of $e_0$ in dimensional regularization is $[e_0] = \epsilon$, therefore $e_0 = Z_e \mu^\epsilon e(\mu)$ ($e(\mu)$ is dimensionless, and all objects in the equation depend on $d = 4 - 2\epsilon$). A Laurent series expansion in $\epsilon$ gives:

$$\mu^{-\epsilon} e_0(d) = e(\mu, d)\Big(1 + \sum_{k=1}^{\infty} \frac{a_k(e(\mu, d))}{\epsilon^k}\Big) = e(\mu, d) Z_e \,. \tag{B.15}$$

Define $e(\mu, d)$ to be analytic for all $d$, ergo $Z_e$ only has pole terms. By acting with $\mu\partial_\mu$ and defining $\mu\partial_\mu e(\mu, d) = \beta(e(\mu, d), d)$ one gets:

$$-\epsilon\Big(e + e\sum_{k=1}^{\infty} \frac{a_k(e)}{\epsilon^k}\Big) = \beta(e, d)\Big(1 + \sum_{k=1}^{\infty} \frac{a_k(e)}{\epsilon^k}\Big) + e\sum_{k=1}^{\infty} \frac{\dot{a}_k(e)\beta(e, d)}{\epsilon^k} \tag{B.16}$$

where $\dot{a}_k = \partial_e a_k$),$e$ stands for $e(\mu_R, d)$. We also know that $\beta$ is analytic in $d$ since $e(\mu_R, d)$ is, and it must be linear in $\epsilon$ and cannot be quadratic or higher order. This means $\beta(e, d) = -\epsilon e + \beta(e)$, and to 0th order:

$$\beta(e) = -e a_1(e) + e a_1(e) + e^2 \partial_e a_1(e) = e^2 \partial_e a_1(e) \to \mu\partial_\mu e = -\epsilon e + e^2 \partial_e a_1(e) \tag{B.17}$$

In the limit $d \to 4$ ($\epsilon \to 0$), $\mu\partial_\mu e(\mu) = e^2 \partial_e a_1(e)$ or in terms of $\alpha(\mu) = \frac{e(\mu)^2}{4\pi}$ then $\mu\partial_\mu \alpha(\mu) = 4\alpha^2(\mu)\partial_\alpha a_1(\alpha)$. At any order in perturbation theory, the $\beta$-function is determined by simple poles in $Z_e$. This makes it clear that one only needs the divergent part of the graphs to compute $\beta$, e.g. $Z_A = 1 - \frac{e^2}{12\pi^2\epsilon} + ...$, $Z_e = 1 + \frac{e^2}{24\pi^2\epsilon} + ...$, $a_1 = \frac{e^2}{24\pi^2} + \mathcal{O}(e^4)$ and $\beta = \frac{e^3(\mu)}{12\pi^2}$, which is the same as the massless $\beta$-function obtained in the $\mu_R$ scheme. In fact the massless $\beta(e)$ is scheme-independent up to the first 2 orders.

The dimensional regularization scales $\mu$ in $MS$ scheme and $\overline{MS}$ scheme are different by a constant factor $\mu_{MS}^2 = \mu_{\overline{MS}}^2 e^{\gamma_E} (4\pi)^{-1}$ ($\gamma_E \simeq 0.5772$ in the Euler constant). The $\overline{MS}$ scheme is used to simplify the finite piece ($\epsilon^0$ terms). A typical contribution from a loop looks like:

$$\frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \frac{\mu_{MS}^{2\epsilon}}{s^\epsilon} = \frac{1}{16\pi^2}\left(\frac{1}{\epsilon} + \ln\Big(\frac{\mu_{\overline{MS}}^2}{s}\Big) + ...\right) \tag{B.18}$$

At $\mathcal{O}\left(\frac{1}{\epsilon^k}\right)$, $0 = \beta(e) a_k - e^2 \dot{a}_{k+1} + e\beta(e, d)\dot{a}_k$ so $e^2 \partial_e a_{k+1} = \beta(e)\partial_e(e a_k(e))$ and one finds a recursion relation for the higher order terms in $Z_e$ – the coefficients of the higher poles are determined by $a_1(e)$.

<u>Example:</u> For $k = 1$

$$e^2 \partial_e a_2 = \Big(\frac{e^3}{12\pi^2} + ...\Big)\partial_e\Big(\frac{e^3}{24\pi^2} + ...\Big) = \frac{e^5}{96\pi^4} + ... \tag{B.19}$$

So $a_2 = \frac{e^4}{384\pi^4}$ is a 2-loop $\frac{1}{\epsilon^2}$ pole in $Z_e$. The anomalous dimension of the operator $O$ is given by $\gamma^O = \mu\partial_\mu \ln Z^O(\mu) = -e\partial_e a_1^O$, with the renormalized factor $Z^O = 1 + \sum_{k=1}^{\infty} \frac{a_k^O}{\epsilon^k}$.

There can be multiple couplings $g_1, g_2, ..., g_l, ... = \mathbf{g}$ with different dimensions:

$$g_l^{\text{bare}} \mu^{-\Delta_l(d)} = g_l(\mu, d)\Big(1 + \sum_k \frac{a_k^l(\mathbf{g})}{\epsilon^k}\Big) \quad , \quad \Delta_l(d) = \Delta l + \epsilon\rho_l \quad , \tag{B.20}$$

$$\mu\partial_\mu g_l(\mu, d) = -\epsilon\rho_l g_l - \Delta_l g_l + g_l \sum_m \frac{d a_1^l}{d g_m} \rho_m g_m \tag{B.21}$$

Note that $g_l(\mu, d)$ is dimensionless (or dimensionful $\widehat{g}_l(\mu, d) = \mu^{\Delta_l} g_l(\mu, d)$ and $[\widehat{g}_l]\Big|_{d=4} = [g_l^{bare}]\Big|_{d=4}$):

$$\mu\partial_\mu \widehat{g}_l(\mu, d) = -\epsilon\rho_l \widehat{g}_l + \widehat{g}_l \sum_m \frac{da_1^l(\widehat{\mathbf{g}})}{d\widehat{g}_m}\rho_m \widehat{g}_m \tag{B.22}$$

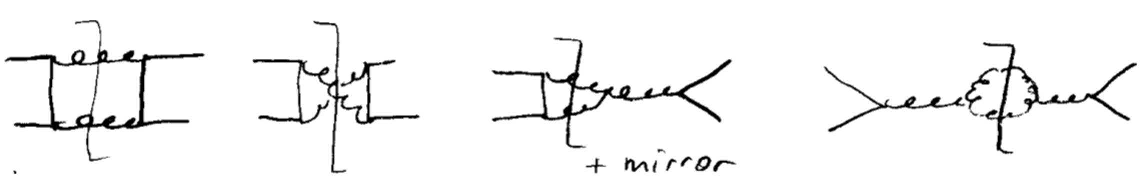
In dimensional regularization, powers of a cutoff never appear ($\Lambda$, $\Lambda^2$, ...). Poles at $d = 4$ correspond to log divergences $\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{-s}\right) + \ldots$. If a product of couplings $\widehat{g}_a$, $\widehat{g}_b$, $\widehat{g}_c$, ... appears in $a_1^l(\widehat{\mathbf{g}})$ then dimensional analysis implies $0 = \Delta_a + \Delta_b + \Delta_c + \ldots$. In a renormalizable theory $\Delta_m \geq 0$, so $\gamma$ and $\beta$ functions only depend on $\Delta_m = 0$ (relevant) couplings. In a nonrenormalizable theory with $\Delta_m \leq 0$ for all operators (marginal and irrelevant), the nonrenormalizable interactions do not affect the renormalizable ones with $\Delta_\ell = 0$.

## B.4 QCD: renormalization, $\beta$-function, asymptotic freedom

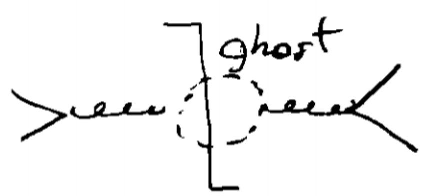
In the Faddeev-Popov gauge fixing one has:

$$\mathcal{L}_{QCD} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}(i\not{D} - m)\psi - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar{c}^A(-\partial^\mu D^{AC}_\mu)c^C \tag{B.23}$$

Here the $c$ are ghosts, which are negative degrees of freedom that cancel unphysical degrees of freedom, e.g. the timelike and the longitudinal polarization states of the gauge bosons. For example, in the process $q\bar{q} \to gg$ one can calculate the squared amplitude $|\mathcal{M}|^2$ in two ways. One way is to consider a tree diagram in which the two gluons are attached to a quark line, and square its absolute value, while explicitly requiring the polarizations of the gluons to be transverse. Another way is to do the multiplication $|\mathcal{M}|^2$ by considering all diagrams with incoming quark and antiquark and outgoing quarks and antiquark, that are possible to cut into two pieces by cutting through two gluon lines. Without ghosts, there are 5 diagrams like that, which look as follows. All of them have a quark line for the incoming quark-antiquark pair, and another quark line for the outgoing quark-antiquark pair. The lines are connected as follows:



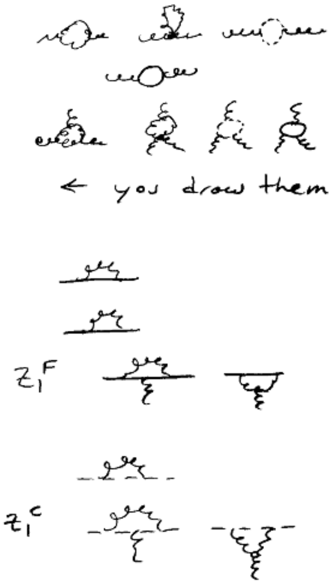

In order to get the right answer, one also must include diagrams with ghosts, that in this case is the following diagram:




Define the renormalization factors $A_0 = Z_3^{\frac{1}{2}} A$, $\psi_0 = Z_2^{\frac{1}{2}}\psi$, $c_0 = Z_2^{c\,\frac{1}{2}} c$, $g_0 = Z_g g$, $m_0 = (m + \delta m)Z_2^{-1}$ and $\xi_0 = Z_3\xi$. $Z_3$ in the expressions is used for both $A$ and $\xi$, which is allowed based on Ward identities. Let's write the Lagrangian term by term, state the renormalization prefactor of each term, and draw the contributing diagrams:

Renormalization

graphs




| | | |
|---|---|---|
| $\mathcal{L} = -\frac{1}{4}(\partial_\mu A_{0\nu} - \partial_\nu A_{0\mu})^2$ | $Z_A$ | |
| $- g_0 f^{ABC} (\partial_\mu A^A_{0\nu}) A_0^{B\mu} A_0^{C\nu}$ | $Z_g Z_A^{3/2} = Z_1$ | |
| $- \frac{1}{4} g_0^2 (f^{EAB} A^A_{0\mu} A^B_{0\nu})(f^{ECD} A_0^{C\mu} A_0^{D\nu})$ | $Z_g^2 Z_A^2 = Z_1'$ | ← you draw them |
| $+ \bar{\psi}_0 i \not{\partial} \psi_0$ | $Z_\psi$ | |
| $- m_0 \bar{\psi}_0 \psi_0$ | $\delta m \bar{\psi}\psi$ | |
| $+ g_0 \bar{\psi}_0 \not{A}_0 \psi_0$ | $Z_g Z_\psi Z_A^{1/2} = Z_1^F$ | |
| $- \bar{c}_0^A \partial^2 c_0^A$ | $Z_c$ | |
| $- g_0 \bar{c}_0^A f^{ABC} \partial^\mu A^B_{0\mu} c_0^C$ | $Z_g Z_c Z_A^{1/2} = Z_1^c$ | |
| $- \frac{1}{2\xi_0} (\partial^\mu A^A_{0\mu})^2$ | 1 ↑ longitudinal part of propagator not altered. | |

Note that the same $g_0$ appears in all interactions. This allows one to write the relations:

$$Z_g = \frac{g_0}{g} = Z_1 Z_3^{-\frac{3}{2}} = Z_1'^{\frac{1}{2}} Z_3^{-1} = Z_1^F Z_3^{-\frac{1}{2}} Z_2^{-1} = Z_1^c Z_2^{c-1} Z_3^{-\frac{1}{2}} \tag{B.24}$$

Note that unlike QED, $Z_1^F \neq Z_2$. This gives 4 different ways to get $Z_g$ and hence the $\beta$-function. These relations can also be written as $\frac{Z_1}{Z_3} = \frac{Z_1'}{Z_1} = \frac{Z_1^F}{Z_2} = \frac{Z_1^c}{Z_2^c}$, which are called Slavnov-Taylor identities (this is the QCD analog of the QED Ward identities $Z_1 = Z_2$). It's also possible to derive Slavnov-Taylor identities based on gauge symmetry, which would give relations between the couplings. A manageable derivation of Slavnov-Taylor identities requires using the BRST symmetry of the gauge-fixed action. The analog in the $U(1)$ case is that the renormalized coupling $e = \frac{Z_2}{Z_1} Z_3^{\frac{1}{2}} e_0 = Z_3^{\frac{1}{2}} e_0$ does not depend on the species of fermions.

In QCD things are more complicated because $Z_1 \neq Z_2$, and $Z_3$ is gauge dependent. However, you can show that the first two terms in $\beta(g) = b_0 g^3 + b_1 g^5 + ...$ are gauge-independent, and that in $\overline{MS}$ the full $\beta(g)$ is gauge-independent $\beta = \mu \partial_\mu g(\mu) = g \partial_g a_1(g)$ ($Z_g = 1 + \sum_k \frac{a_k}{\epsilon^k}$). One needs to compute $Z_1^F$, $Z_3$, and $Z_2$ in order to get $a_1$. A simpler method is the background field method.

### B.4.1 Background field method

The idea of the background field method is to find a gauge where $gA^\mu$ is not renormalized, so $Z_g = Z_3^{-\frac{1}{2}}$. The gauge fixing is needed for obtaining a well-behaved propagator, but one can still keep gauge invariance on the external lines. Let $A^\mu \to A^\mu + Q^\mu$ where $A^\mu$ is a fixed background, and $Q^\mu$ is a quantum field. The QCD action $S[A+Q]$ (without the gauge fixing terms) is invariant under the gauge transformation

$A_\mu + Q_\mu \to A_\mu + Q_\mu + \frac{1}{g}\partial_\mu\alpha - i[A_\mu + Q_\mu, \alpha]$. There are 2 gauge symmetries, the first is the gauge transformation of the quantum field $A_\mu \to A_\mu$ and $Q_\mu \to Q_\mu + \frac{1}{g}[D_\mu^{A+Q}, \alpha]$ where $iD^{A+Q} = i\partial + g(A+Q)$, the second is the gauge transformation of the background $A_\mu \to A_\mu + \frac{1}{g}[D_\mu^A, \alpha]$ and $Q_\mu \to Q_\mu - i[Q_\mu, \alpha]$ (can also be written as $A \to U\Big(A + \frac{i}{g}\partial\Big)U^{-1}$ and $Q \to UQU^{-1}$). Here $Q$ behaves like an adjoint matter field. Let's do gauge fixing for the transformations of the first type, but leave gauge symmetry of the second type.

The generating function for Green's functions for $A = 0$ is:

$$Z[J] = \int \mathcal{D}Q \det\Big(\frac{\delta G^a}{\delta\alpha^b}\Big) \exp\bigg(i\int d^4x\Big(\mathcal{L}(Q) - \frac{1}{2\xi}(G^a)^2 + J_\mu^a Q^{\mu a}\Big)\bigg) \tag{B.25}$$

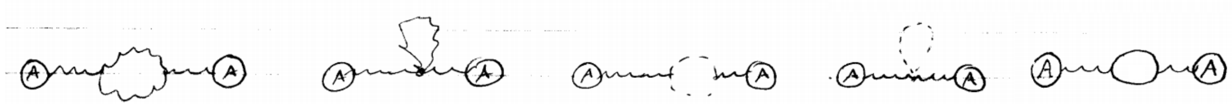
The generating function of the connected Green's functions is $W[J] = -i\ln Z[J]$ and for the 1PI Green's functions $\Gamma[\bar{Q}] = W[J] - \int d^4x J_\mu^a \bar{Q}^{\mu a}$ where $\bar{Q} = \frac{\delta W}{\delta J_\mu^a}$. Consider:

$$Z[J, A] = \int dQ \exp\Big(iS[Q+A] + iJ_\mu^a Q^{\mu a} + i(S_{gf}) + i(S_{ghost})\Big) \tag{B.26}$$

Let the gauge-fixing term be $\mathcal{L}_{gf} = -\frac{1}{2\xi}(G^a)^2$ with $G^a = (D_\mu Q^\mu)^a = \partial_\mu Q^{\mu a} + gf^{abc}A_\mu^b Q^{\mu c}$. This fixes the gauge in the first type of transformation discussed above, but is invariant under the second ($c \to UcU^{-1}$ is also invariant under this second type of gauge transformation):

$$\frac{\delta Q_\mu}{\delta\alpha} = \frac{1}{g}D_\mu^{A+Q} \quad , \quad \frac{\delta G}{\delta\alpha} = \frac{1}{g}D_\mu^A D^{\mu A+Q} \quad , \quad \mathcal{L}_{ghost} = \bar{c}(-D_\mu^A D^{\mu A+Q})c \tag{B.27}$$

Thus $S[Q+A] + S_{gf} + S_{ghost}$ is invariant. So $Z[g, A] = Z[UJU^{-1}, UAU^{-1} + U\frac{i}{g}\partial U^{-1}]$ (make a change of variable $Q \to UQU^{-1}$ in the path integral). Similarly one can talk about $Z[J, A] = e^{iW[J,A]}$ and $\Gamma[\tilde{Q}, A] = W[J, A] - \int d^4x J_\mu^a \tilde{Q}^{\mu a}$ ($\tilde{Q} = \frac{\delta W}{\delta J}$ and $\Gamma[\tilde{Q}, A] = \Gamma\Big(U\tilde{Q}U^{-1}, UAU^{-1} + U\frac{i}{g}\partial U^{-1}\Big)$). We can then use the gauge-invariant $\Gamma[0, A]$ to compute the $\beta$-function. As $\tilde{Q} = 0$ we have only external $A$ fields and we are integrating $DQ$ so we have only $Q$-type gluons on internal lines. It can be shown that the background field action with $\tilde{Q} = 0$ is the standard effective action in a strange gauge, with the gauge fixing term $\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu Q^{\mu a} - \partial_\mu A^{a\mu} + gf^{abc}A_\mu^b Q^{c\mu})^2$. The divergences must preserve gauge the invariance of $\Gamma[0, A]$, so $(F_{\mu\nu}^a)^2$ must be multiplicatively renormalized $F_{\mu\nu_0}^a = Z_2^{\frac{1}{2}}\Big(\partial_\mu A_\nu - \partial_\mu A_\nu + gZ_g Z_3^{\frac{1}{2}} f^{abc} A_\mu^b A_\nu^c\Big)$ with $Z_g Z_3^{\frac{1}{2}} = 1$. There are 5 diagrams that contribute to $Z_3$ – they have two external $A$-gluons:




Diagrams 2 and 4 are proportional to $\int \frac{d^dk}{k^2}$ so they have no $\frac{1}{\epsilon}$ pole, thus they don't contribute. There are the following Feynman rules: an $AQQ$ vertex with an $A$-field with outgoing momentum $p$ and indices $a$, $\mu$, a $Q$-field with outgoing momentum $q$ and indices $b$, $\nu$, and another $Q$-field with outgoing momentum $r$ and indices $c$, $\lambda$ is (along with $\xi = 1$ in the background field Feynman gauge):

$$gf^{abc}\bigg(g_{\mu\lambda}\Big(p - r - \frac{1}{\xi}q\Big)_\nu + g_{\nu\lambda}(r-q)_\mu + g_{\mu\nu}\Big(q - p + \frac{1}{\xi}r\Big)_\lambda\bigg) \tag{B.28}$$

There is another vertex that includes an outgoing ghost with momentum $p$ and index $a$, an incoming ghost with momentum $q$ and index $b$, and an $A$ gluon with indices $c$, $\mu$, with value $gf^{abc}(p+q)_\mu$). So, diagram 1 (gluons in the loop) gives $\frac{ig^2 C_A}{16\pi^2}\delta^{ab}(g_{\mu\nu}k^2 - k_\mu k_\nu)\frac{1}{3\epsilon}$ with $c_A = 3$ for $SU(3)$. Diagram 3 (ghost in the loop) gives 10 times the same result. So $Z_A = 1 + \frac{11c_A}{3}\frac{g^2}{16\pi^2\epsilon}$, $Z_g = 1 - \frac{11c_A}{3}\frac{g^2/2}{16\pi^2\epsilon}$, and one arrives at the $\beta$-function:

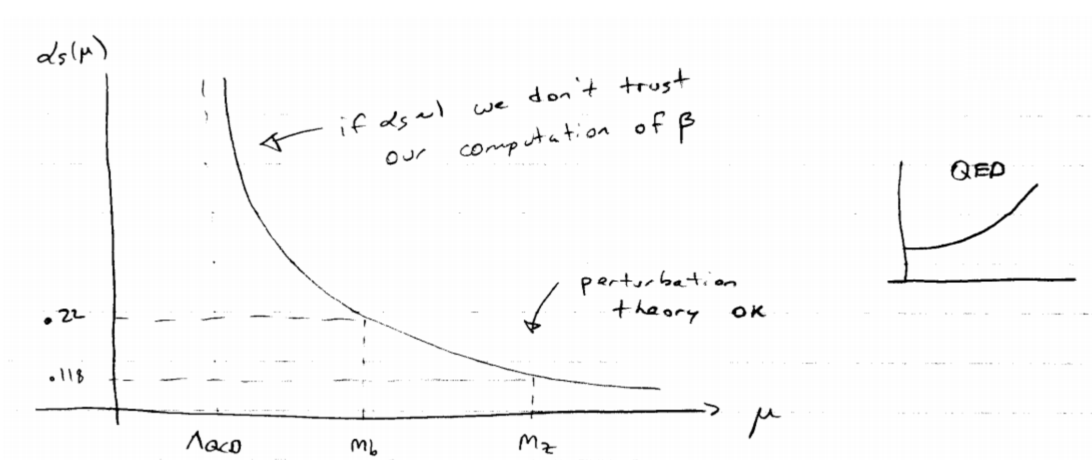
$$\beta_{QCD} = g^2\partial_g a_1 = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}c_A - \frac{4}{3}n_f T_F\right) = -\frac{g^3}{16\pi^2}\beta_0 \quad , \quad \beta_0 = \frac{11}{3}c_A - \frac{4}{3}n_f T_F \tag{B.29}$$

The term proportional to $n_f T_F$ comes from $n_f$ flavors of fermions contributing through diagram 5, where the calculation is like in QED except $\mathrm{Tr}\,(T^A T^B) = T_F\delta^{AB}$.

For QED, $c_A = 0$ and $T_F = 1$, and assuming a single fermion ($n_f = 1$) one has $\beta_0^{QED} = -\frac{4}{3}$ so $\beta_{QED} = \frac{g^3}{12\pi^2}$ as expected from previous calculation. For QCD with $n_f < 17$, $\beta_0 > 0$ which gives a negative $\beta_{QCD}$. The interaction strength behaves as:

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{\alpha_s(\mu_0)}{2\pi}\beta_0 \ln\left(\frac{\mu}{\mu_0}\right)} = \frac{2\pi}{\beta_0 \ln\left(\frac{\mu}{\Lambda_{QCD}}\right)} \tag{B.30}$$

At large $\mu$ the interaction is relatively weak: $\alpha_s(m_Z) = 0.118$, $\alpha_s(m_b) = 0.22$, so perturbation theory is good, but as one goes to low energies, the expression for $\alpha_s$ grows more and more, diverging at $\mu = \Lambda_{QCD}$. Hence starting from $\alpha_s \sim 1$ perturbative calculations cannot be trusted. As $\mu \to \infty$, $\alpha_s \to 0$ and the quarks become free at large energies (asymptotic freedom). The dimensionless parameter $\alpha_s$ is traded for the dimensionful parameter $\Lambda_{QCD}$. This is called dimensional transmutation. If the fields become massless in $\mathcal{L}_{\mathrm{QCD}}$ then it's scale invariant, but this symmetry is still broken at the scale $\Lambda_{\mathrm{QCD}}$ ($\Lambda_{\mathrm{QCD}} \sim 250$ MeV experimentally). At small $\mu$ (e.g. long distances as in the lab) quarks are confined into color singlet hadrons: baryons $qqq$ (since $3\times3\times3$ allows a singlet) and mesons $q\bar{q}$ (since $3\times\bar{3}$ allows a singlet). If we take $m_{u,d} \to 0$, then the only dimensionful parameter is $\Lambda_{\mathrm{QCD}}$, ergo $m_{proton} \sim \Lambda_{\mathrm{QCD}}$ and $(\mathrm{radius_{proton}})^{-1} \sim \Lambda_{\mathrm{QCD}}$. A strong dependence on $\mu$ means that the renormalization group calculation is crucial to the interaction strength, e.g. $\alpha_s$ is twice as big for $b$-physics as for $Z$-physics.




### B.4.2 Heuristic explanation of asymptotic freedom

Consider vacuum fluctuations in QED, with a photon turning into a fermion-antifermion pair which then annihilates back into a photon. They give rise to $e^2(\mu)$ as a dielectric medium. Hence $e^2(k) = \frac{e_0^2}{1+b_0 e_0^2 \ln\left(\frac{\Lambda}{k^2}\right)} =$

$\frac{e_0^2}{\epsilon(k)}$ where $\Lambda$ is a cut-off scale and $e_0$ is the bare charge. In QED $b_0 = -\frac{\beta_0}{16\pi^2} > 0$ so $\epsilon > 1$ – fluctuations in the vacuum are screening the charge and $e^2(k)$ is smaller at long distances $k^{-1}$. An intuitive picture is that the bare charge $e_0$ (say negative) is surrounded by fermion-antifermion pairs, and each pair points with its positive particle towards the bare charge. In QCD $b_0 < 0$ so $\epsilon < 1$ and gluon fluctuations are antiscreening the charge. To understand why, note that $\mu\epsilon = 1$ in the vacuum, so vacuum screens color magnetic charge.

Let's look at 2 interesting possible effects of magnetic field: paramagnetism – the magnetic field $\mathbf{B}$ causes intrinsic magnetic moments to line up with $\mathbf{B}$ giving $\mu > 1$ and $\mathbf{m} = \frac{\mu-1}{4\pi\mu}\mathbf{B}$, diamagnetism – current loop develops a magnetic moment to oppose the applied $\mathbf{B}$ field, giving $\mu < 1$. Magnetic susceptibility is defined as $\chi(k) = \frac{\mu-1}{\mu} = -b_0 e_0^2 \ln\left(\frac{\Lambda^2}{k^2}\right)$. The energy density in magnetic field is given by $U = -\frac{1}{2}\chi B^2 = b_0 e_0^2 \ln\left(\frac{\Lambda}{k}\right) B^2$, so the steps are: compute the energy density for free bosons or fermions with an arbitrary spin in a magnetic field, find the term proportional to $\ln(\Lambda)B^2$ and then read off $b_0$.

The diamagnetic term can be obtained as follows. In a magnetic field, the continuous free particle spectrum turns into discrete energy levels – Landau levels (one can think of particles as executing a quantized circular motion). Consider a massless particle of charge $e$ in a magnetic field $\mathbf{B} = B\widehat{z}$, then the corresponding vector potential is $\mathbf{A} = Bx\widehat{y}$ and the Hamiltonian is given by ($p_z$ and $p_x$ commute with the Hamiltonian, while the term involving $p_y$ is like a shifted oscillator; the degeneracy per unit area is $g_n = \frac{eB}{2\pi}$):

$$H^2 = (\mathbf{p} - e\mathbf{A})^2 = p_z^2 + p_x^2 + (p_y - eBx)^2 \quad , \quad E^2 = p_z^2 + (2n+1)eB \tag{B.31}$$

The vacuum energy per unit volume is given by $U = \sum_{n=0}^{\infty} \int \frac{dp_z}{2\pi} \left[p_z^2 + (2n+1)eB\right]^{1/2} \frac{eB}{2\pi}$. As $B \to 0$ one regains the continuous spectrum. One needs to take the limit carefully to get the $B^2$ term. In general:

$$\begin{aligned}
\int_{-\frac{1}{2}\epsilon}^{(N+\frac{1}{2})t} dx F(x) &= \sum_{n=0}^{\infty} \int_{(n-\frac{1}{2})t}^{(n+\frac{1}{2})t} dx F(x) && \text{(B.32)}\\
&= \sum_{n=0}^{\infty} \int_{(n-\frac{1}{2})t}^{(n+\frac{1}{2})t} dx \Big(F(nt) + (x-nt)F'(nt) + \frac{1}{2}(x-nt)^2 F''(nt) + \dots\Big) && \text{(B.33)}\\
&= \sum_{n=0}^{\infty} \Big(tF(nt) + \frac{t^3}{24}F''(nt) + \dots\Big) && \text{(B.34)}
\end{aligned}$$

Invert this relation to get:

$$\lim_{N\to\infty} \sum_{n=0}^{N} tF(nt) = \int_0^{\infty} dx F(x) - \frac{t^2}{24}\int_0^{\infty} dx F''(x) + \dots \tag{B.35}$$

Note that $t = eB$, $(n + \frac{1}{2})eB > x$, $F(x) = (p_z^2 + 2x)^{1/2}$, $F''(x) = -(p_z^2 + 2x)^{-3/2}$ and make the change of variable $x = 2\mathbf{q}_\perp^2$ so $dx = \frac{dq_\perp^2}{2} = \frac{d^2 q_\perp}{2\pi}$, then the energy density becomes (taking the continuous limit $\sum_n \int \frac{dp_z}{2\pi}\frac{eB}{2\pi} \to \int \frac{d^3p}{(2\pi)^3}$):

$$U = \int \frac{d^3p}{(2\pi)^3}(p_z^2 + q^2)^{1/2} + \frac{(eB)^2}{24}\int^{\Lambda} \frac{d^3p}{(2\pi)^3}\frac{1}{p^{3/2}} + \dots = U(B=0) + \frac{e^2B^2}{48\pi^2}\ln\Lambda \tag{B.36}$$

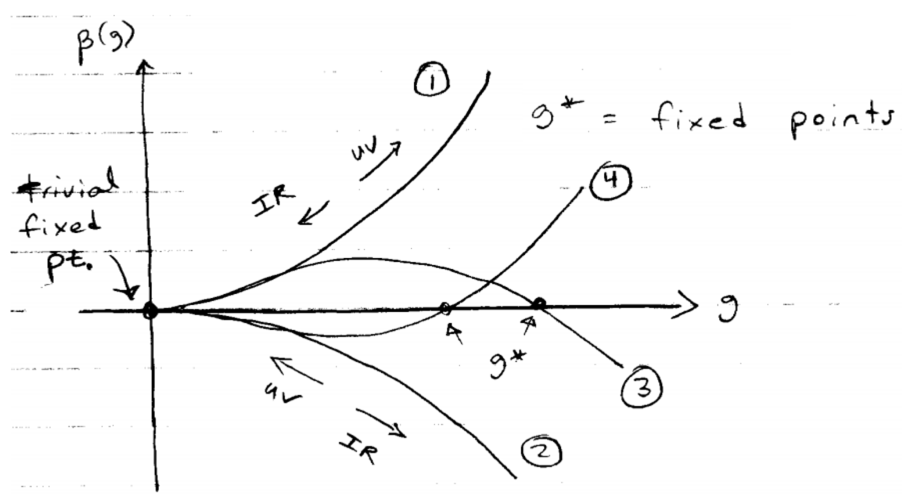
The sign of the vacuum energy for fermions is opposite to that of bosons (one can think of the fermions as holes in the negative energy states). Then the diamagnetic contribution to $b_0$ is $b_0^{\text{diamag}} = \pm\frac{1}{48\pi^2}$ where the upper sign (+) is for bosons and the lower sign for fermions, and it is spin-independent.

To calculate the paramagnetic contribution one includes the interaction of the spin with the magnetic field $H^2 = (\mathbf{p} - e\mathbf{A})^2 - 2eBS_z$ where the $g$-factor is taken to be $g = 2$. This changes the energy levels as $E^2 = E^2_{\text{Landau}} - 2eBS_z$ hence $E = E_{\text{Landau}} - \frac{eBS_z}{E_{\text{Landau}}} - \frac{e^2S_z^2B^2}{2E^3_{\text{Landau}}} + ...$, so the paramagnetic contribution to the energy is $b_0^{\text{paramag}} = \mp\frac{S_z^2}{4\pi^2}$ where the upper sign $(-)$ is for bosons, and the lower sign for fermions.

The total $b_0$ (per spin state) is $b_0 = \mp\frac{1}{16\pi^2}(4S_z^2 - \frac{1}{3})$. In QED, for fermions with $S_z = \pm 1/2$ and charge $Q$, one gets (for both helicities together) $b_0 = 2\frac{Q^2}{16\pi^2}\left(1 - \frac{1}{3}\right) = \frac{1}{12\pi^2}$ exactly as obtained in the field theory calculation. For vectors (e.g., the gluons in QCD) with $S_z = \pm 1$ and charges $Q$, $b_0 = -\frac{Q^2}{16\pi^2}\frac{22}{3}$. For comparison, the field theory calculation gave $b_0^{QCD} = -\frac{1}{16\pi^2}\left(11 - \frac{2}{3}n_f\right)$. There is a way of treating the $Q$'s correctly so that the results agree. In a sense asymptotic freedom is a consequence of the large magnetic moments of spin-1 charged particles making the vacuum paramagnetic. Magnetic moments of fermions make vacuum diamagnetic because their zero-point fluctuations have negative energy.

## B.5 Asymptotic behavior and fixed points

Recall that the coupling $g$ changes with the scale $\mu$ as $\mu\partial_\mu g(\mu) = \beta(g)$, so $\int_{g(\mu_0)}^{g(\mu)} \frac{dg}{\beta(g)} = \ln\left(\frac{\mu}{\mu_0}\right)$. As $\mu \to 0$ (IR flow) or $\mu \to \infty$ (UV flow), $\ln\left(\frac{\mu}{\mu_0}\right)$ diverges. This can happen either because $g$ goes to a value $g^*$ in which $\beta(g^*) = 0$, or $g$ goes towards $\infty$. There are several possibilities for how $\beta(g)$ can behave as a function of $g$. In all cases $\beta(0) = 0$:




1. $\beta(g)$ grows with $g$ (i.e., always positive).
2. $\beta(g)$ decreases with $g$ (i.e., always negative).
3. $\beta(g)$ first grows, but then decreases, crosses zero, and continues towards negative values.
4. $\beta(g)$ first decreases, but then grows, crosses zero, and continues towards positive values.

In case 1, $\beta(g)$ looks like QED and $\phi^4$ theory at small coupling. $\int^\infty \frac{dg}{\beta(g)} < \infty$ and $g$ diverges at a finite scale $\mu = M$ given by $M = \mu_0 \exp\int_{g(\mu_0)}^\infty \frac{dg}{\beta(g)}$. This leads to unphysical effects. However, these theories are fine as low energy effective theories with $p \ll M$ (and some new operators, new degrees of freedom become relevant at $p \sim M$). For example, in QED, $M = e^{647} m_e$ which is enormous (weak interactions enter much earlier).

Case 2 is like QCD. Since $\beta < 0$ at small $g$, large energy behavior is under control. The flow goes toward a trivial fixed point ($g = 0$) as $\mu \to \infty$. $\int_{g(\mu_0)}^{g} \frac{dg'}{\beta(g')} \to \infty$ for $g \to 0$.

Cases 3 and 4 have fixed points at an intermediate energy. In case 3, $g(\mu \to \infty) = g^*$ is UV stable fixed point (because in the limit $\mu \to \infty$ we flow towards $g^*$ from either side). Similarly, in case 4 $g(\mu \to 0) = g^*$ is IR stable fixed point. The existence of these fixed points and the slope of $\beta$ at the fixed points are scheme independent. The anomalous dimensions at the fixed points ($\gamma(g^*)$) are also scheme-independent.

Let's come back to the statement that (with some qualifications) non-renormalizable theories with $[O] > 4$ flow to renormalizable theories at low energy. Recall the following example: at tree level $\frac{g_5}{\Lambda_0} \bar{\psi} \sigma^{\mu\nu} F_{\mu\nu} \psi$ gave $\sigma \sim \frac{\alpha^2}{E^2} + \frac{\alpha g_5^2}{\Lambda_0^2} + ...$ in massless QED. In dimensional regularization ($\overline{MS}$ scheme) there are no powers of a cut-off in loop computations and the naive dimensional analysis saying one can drop $g_5$ for $E \ll \Lambda_0$ carries through. Unfortunately this doesn't really explain what's going on. To do so one needs to consider the Wilsonian RG with a hard cut-off, and show that RG flow is related to removing high energy modes, and that operators with mass dimension larger than 4 are suppressed at low energy. Take a scalar field theory with a physical Euclidean cut-off $\Lambda_0$, then write down the action (include a general term $g_i^0 O_i[\phi]$ and we'll use it to describe low-energy physics, below energy $E$, and $E \ll \Lambda_0$):

$$S_0(\Lambda_0) = \int d^4x \Big( \frac{1}{2}(\partial^\mu \phi)^2 + g_2^0 \phi^2 + g_4^0 \phi^4 + g_6^0 \phi^6 + g_{6'}^0 \phi^2 (\partial^\mu \phi)^2 + g_8 \phi^8 + ... \Big) \tag{B.37}$$

The dependence on $\Lambda_0$ comes due to the fact that $g_i^0$ depend on $\Lambda_0$, hence:

$$Z[J, \Lambda_0] = \int_{|p|<\Lambda_0} \mathcal{D}\phi_0 \exp\Big( - S[\phi_0, \Lambda_0] - \int J \phi_0 \Big) \quad , \quad \int_{|p|<\Lambda_0} \mathcal{D}\phi_0 = \int \prod_{|p|<\Lambda_0} d\phi_0(p) \tag{B.38}$$

The current $J(p)\theta(E^2 - p^2)$ is the current for small momentum. Now introduce another cutoff $\Lambda_1 < \Lambda_0$. Let $\phi_0(p) = \phi_1(p) + \chi(p) = \phi_1(p)\theta(\Lambda_1 - |p|) + \chi(p)\theta(\Lambda_1 < |p| < \Lambda_0)$ and denote $\Lambda_1 = b\Lambda_0$ ($b < 1$), then the $\chi$ propagator is proportional to $\frac{\theta(\Lambda_1 < |k| < \Lambda_0}{k^2}$. Integrate out $\chi$ (use $J(-p)\phi_0(p) = J(-p)\phi_1(p)$):

$$Z[J, \Lambda_0] = \int_{|p|<\Lambda_1} \mathcal{D}\phi_1 \exp\left[ -S_1[\phi_1, \Lambda_1] + \int J \phi_1 \right] \tag{B.39}$$

$S_1$ can be written as:

$$S_1[\phi_1, \Lambda_1] = \int d^4x \Big( \frac{(\partial \phi_1)^2}{2} + \sum_i g_i^{(1)}(\Lambda_1, \Lambda_0, \mathbf{g}^{(0)}) O_i[\phi_1] \Big) \tag{B.40}$$

It's easy to imagine using perturbation theory, and working to all orders. Expand:

$$\phi_0^4 = (\phi_1 + \chi)^4 = \phi_1^4 + 4\phi_1^3 \chi + 6\phi_1^2 \chi^2 + 4\phi_1 \chi^3 + \chi^4 \tag{B.41}$$

The $\phi_1^2 \chi^2$ vertex gives a diagram in which a $\chi$ loop is attached to an incoming-outgoing $\phi_1$ line. This diagram is proportional to $g_4^{(0)} \phi_1^2 \int_{\Lambda_1}^{\Lambda_0} \frac{d^4k}{k^2} = \phi_1^2 g_4^{(0)} f(\Lambda_0, \Lambda_1)$, where $f(\Lambda_0, \Lambda_1)$ is some function. This contributes to the $g_2^{(1)}$ term in $S_1$. Another vertex, that could come from a $g_6^{(0)}$ term in the original action, is one at which four external $\phi_1$ lines meet at a vertex, and a $\chi$ loop is attached to the same vertex. This would contribute to $g_4^{(1)}$. Another is a tree diagram in which three incoming $\phi_1$ lines turn into $\chi$, which then turns into three outgoing $\phi_1$ lines (two $g_4^{(0)}$ vertices). This will contribute to $g_6^{(1)}$ (so $g_6^{(1)}$ is generated through loop corrections even if initially $g_6^{(0)} = 0$).

One may only make a small change in the cutoff ($b$ close to 1) in order to keep the action local, but it can always be repeated many times, until the energy scale gets down to a cut-off of $E$. Make the couplings dimensionless as $\lambda_i(\Lambda) = \Lambda^{-\Delta_i} g_i(\Lambda)$ then the above process gives $\lambda_i(\Lambda') = G_i\Big(\lambda_i(\Lambda), \frac{\Lambda'}{\Lambda}\Big)$. Take $\Lambda'\partial_{\Lambda'}$ and set $\Lambda' = \Lambda$, then $\Lambda'\partial_{\Lambda'}\lambda_i(\Lambda) = \beta_i\Big(\lambda(\Lambda)\Big)$ with $\beta_i = \partial_z G_i\Big(\lambda(\Lambda), z\Big)\Big|_{z=1}$. This is Wilsonian RGE (compare to the earlier $\beta$-function calculation). The space of local interactions can be thought as an $\infty$-dimensional surface parameterized by the couplings. One would like to show that for $\Lambda \ll \Lambda_0$ it flows to a stable subspace parameterized only by renormalizable couplings (and independent of $\Lambda_0$ and the initial conditions).

Example: Consider a theory with two couplings $\lambda_4 = g_4$ and $\lambda_6 = \Lambda^2 g_6$, with $\Lambda\partial_\Lambda\lambda_4 = \beta_4(\lambda_4, \lambda_6)$ and $\Lambda\partial_\Lambda\lambda_6 = 2\lambda_6 + \beta_6(\lambda_4, \lambda_6)$. Consider a solution $\bar{\lambda}_i$ of these equations, and take a small perturbation $\lambda_i = \bar{\lambda}_i + \epsilon_i$, then the equations for $\epsilon_i$ (up to first order) are $\Lambda\partial_\Lambda\epsilon_4 = \partial_{\lambda_4}\bar{\beta}_4\epsilon_4 + \partial_{\lambda_6}\bar{\beta}_6\epsilon_6$ and $\Lambda\partial_\Lambda\epsilon_6 = 2\epsilon_6 + \partial_{\lambda_6}\bar{\beta}_6\epsilon_6$ ($\bar{\beta}$ means that the $\beta$-function should be evaluated at $\bar{\lambda}_i$). The goal is to show that as the cutoff is lowered, the perturbed and the unperturbed solutions become close in $\lambda_6$. It's possible that the curves in the $\lambda_4$-$\lambda_6$ plane will get close to each other but the close points from the two curves will correspond to different values of $\Lambda$. To take this into account, define $\xi_6 = \epsilon_6 - \partial_\Lambda\bar{\lambda}_6(\partial_\Lambda\bar{\lambda}_4)^{-1}\epsilon_4$ and let's hope that $\xi_6 \to 0$ after lowering the cutoff. From $\Lambda\partial_\Lambda\xi_6 = (2 + \partial_{\lambda_6}\bar{\beta}_6 + \partial_{\lambda_4}\bar{\beta}_4 - \Lambda\partial_\Lambda \ln\bar{\beta}_4)\xi_6$, the solution can be read off:

$$\xi_6(\Lambda) = \xi_6(\Lambda_0)\frac{\Lambda^2}{\Lambda_0^2}\frac{\bar{\beta}_4(\Lambda_0)}{\bar{\beta}_4(\Lambda)}\exp\left(\int_{\Lambda_0}^{\Lambda}\frac{d\Lambda'}{\Lambda'}\Big(\partial_{\lambda_6}\bar{\beta}_6 + \partial_{\lambda_4}\bar{\beta}_4\Big)(\Lambda')\right) \tag{B.42}$$

If the couplings are small enough that the integrand and $\frac{\bar{\beta}_4(\Lambda_0)}{\bar{\beta}_4(\Lambda)}$ remain small, then $\xi_6(\Lambda) \to 0$ for $\Lambda \ll \Lambda_0$. This can be converted to a trajectory in the $\lambda_4$-$\lambda_6$ plane where the value of $\lambda^4(\Lambda)$ determines $\lambda_6$, independent of $\Lambda_0$ and initial conditions. So the action depends only on the renormalizable couplings. The advantages of the Wilsonian RG are that there are no subdivergences or overlapping divergences or IR divergences, and that the exact correspondence with modes is clear. The disadvantages are that we always have non-renormalizable operators, and that the cutoff destroys symmetries like manifest gauge invariance, chiral symmetry etc.

Note that it was shown that the Wilsonian RGE can be used to set the nonrenormalizable couplings to 0 (effectively, at high energy scale), in the sense that in the low energy limit the physics is still well-approximated as the running value of nonrenormalizable couplings is insensitive to physics at the UV region.