ABOUT THE COHERENCE OF VARIANCE AND STANDARD DEVIATION AS MEASURES OF RISK

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In the following, X and Y are two random variables. With $\text{var}(X)$ we indicate the variance of X, and with $\text{sd}(X)$ its standard deviation.
We know that $\text{sd}(X) = \sqrt{\text{var}(X)}$. And we also know that neither the variance nor the standard deviation can be negative!

Let's start by considering the **variance**.
In order to show that $\text{var}(X)$ is not coherent, we show that $\text{var}(X)$ is neither positive homogenous nor sub-additive.

- **Positive homogeneity**.
  If we multiply X by a scalar (i.e. a number) a, the properties of variance tell us that $\text{var}(aX) = a^2\text{var}(X)$, because the variance is not linear. But $a^2\text{var}(X) \neq a\text{var}(X)$, hence the variance is not positive homogenous.
- **Sub-additivity**.
  Always from the properties of variance (you can check on Wikipedia), we know that $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X,Y)$, where $\text{cov}(X,Y)$ is the so-called covariance. Using correlation $\rho(X,Y) = \frac{\text{cov}(X,Y)}{\text{sd}(X)\text{sd}(Y)}$, this can be re-written as

  $$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2\rho(X,Y)\text{sd}(X)\text{sd}(Y)$$

  Unless $\rho(X,Y)$ is zero (X and Y are not linearly dependent) or negative (they are negatively correlated), we have that the right-hand side of the previous equation is always bigger than the simple sum of the variances, therefore $\text{var}(X+Y) \geq \text{var}(X) + \text{var}(Y)$.
  As a consequence, in general, variance is not sub-additive.

Even if the other two properties (monotonicity and translation invariance) are respected[1], the variance is NOT coherent.

We now consider the **standard deviation**, which we know is defined as $\text{sd}(X) = \sqrt{\text{var}(X)}$ for a random variable X.
The standard deviation is always coherent. Notice that standard deviation, in finance, is often called volatility.

[1] The proofs are exactly as those we consider here below for the standard deviation.

- **Monotonicity**.
  If X is considered riskier than Y, in terms of standard deviations (if the standard deviation is used as a measure of risk), we have that $\text{sd}(X) \geq \text{sd}(Y)$. In other words, X is more volatile than Y.
- **Translation Invariance**.
  If we have a random variable X and we add a scalar/constant c to it, the properties of standard deviation tell us that sd does not change, i.e. $\text{sd}(X + c) = \text{sd}(X)$. In fact, a scalar does not add randomness.
- **Positive homogeneity**.
  If we multiply X by a scalar (i.e. a number) a, the properties of variance tell us that $\text{var}(aX) = a^2\text{var}(X)$. But we know that $\text{sd}(X) = \sqrt{\text{var}(X)}$, hence
  $$\text{sd}(aX) = \sqrt{\text{var}(aX)} = \sqrt{a^2\text{var}(X)} = a\text{sd}(X)$$
- **Sub-additivity**.
  We can prove sub-additivity by using the equation we have just seen for the variance, i.e.
  $$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2\rho(X, Y)\text{sd}(X)\text{sd}(Y),$$
  and then by applying a simple trick.
  In fact, we all know that $\rho(X, Y) \in [-1, 1]$, that is correlation cannot be smaller than -1 (perfect negative correlation) or larger than 1 (perfect positive correlation). This means that $\text{var}(X + Y)$ reaches its maximum when $\rho(X, Y) = 1$. Therefore
  $$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2\rho(X, Y)\text{sd}(X)\text{sd}(Y) \leq \text{var}(X) + \text{var}(Y) + 2\text{sd}(X)\text{sd}(Y),$$
  which we obtain by substituting 1 to $\rho(X, Y)$. Now, let us re-write the previous equation in terms of standard deviations:
  $$(\text{sd}(X + Y))^2 \leq (\text{sd}(X))^2 + (\text{sd}(Y))^2 + 2\text{sd}(X)\text{sd}(Y) = (\text{sd}(X) + \text{sd}(Y))^2.$$
  Now, let's take the square root on both sides of the equation,
  $$\sqrt{(\text{sd}(X + Y))^2} \leq \sqrt{(\text{sd}(X) + \text{sd}(Y))^2},$$
  so that we get
  $$\text{sd}(X + Y) \leq \text{sd}(X) + \text{sd}(Y),$$
  showing that standard deviation is sub-additive.

Hence standard deviation is always coherent!