Default Probabilities

Merton’s Model - Part 2

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In the last lesson, we have seen that, according to Merton’s model,

\[ S_T = (V_T - B)^+ \]

This means that the value of the firm’s equity at time \( T \) corresponds to the payoff of a European call option on \( V_T \).
The famous Black-Scholes-Merton formula tells us that the value of the equity today is

\[ S_0 = V_0 \Phi(d_1) - B e^{-rT} \Phi(d_2) \]  

EQUATION 1

with

\[ d_1 = \frac{\log(V_0/B) + (r + \sigma^2_V/2)T}{\sigma_V \sqrt{T}} \]

\[ d_2 = d_1 - \sigma_V \sqrt{T} \]

and where \( \Phi(\cdot) \) is the cumulative distribution function of a standard Gaussian (i.e. N(0,1)).
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and where $\Phi(\cdot)$ is the cumulative distribution function of a standard Gaussian (i.e. N(0,1)).
Under Merton’s model, a company defaults if, at maturity $T$, the value of its assets falls below the liabilities’ threshold $B$, that is to say when

$$V_T \leq B$$

With some maths, it can be show that the probability that $V_T \leq B$, i.e $P(V_T \leq B)$ is equal to

$$P(V_T \leq B) = \Phi \left( \frac{\log(B/V_0) - (r - \sigma_V^2/2)T}{\sigma_V \sqrt{T}} \right) = \Phi(-d_2)$$
We have just seen that the probability of default in $T$ (and only in $T$, not before!) is

$$P(V_T \leq B) = \Phi \left( \frac{\log(B/V_0) - (r - \sigma^2 V/2)T}{\sigma V \sqrt{T}} \right) = \Phi(-d_2)$$

This probability depends on several different quantities, but only two of them are really problematic for us: $V_0$ and $\sigma_V$.

These two quantities cannot be observed directly. They need to be somehow inferred from data.
In particular, for the market value of the company’s assets, we have the following:

* Market value can strongly differ from the value of a company as measured by accountancy rules.

* The market value of a company corresponds to the sum of the market values of its equity and debt. But only a relatively small part of firm’s debt (i.e. issued bonds) is fully knowable.
If the company we are analyzing is publicly traded, we can observe $S_0$.

It can be shown that the value of equity today must satisfy the following equation

$$S_0 = \frac{\sigma_V}{\sigma_S} \Phi(d_1)V_0$$

This equation, together with equation 1, can be used to find $V_0$ and $\sigma_V$. 
Computing $V_0$ and $\sigma_V$

* Let us rearrange equations 1 and 2, so that

$$F(V_0, \sigma_V) = V_0 \Phi(d_1) - Be^{-rT} \Phi(d_2) - S_0 = 0$$

$$G(V_0, \sigma_V) = \frac{\sigma_V}{\sigma_S} \Phi(d_1)V_0 - S_0 = 0$$

* $V_0$ and $\sigma_V$ can be found by minimizing

$$[F(V_0, \sigma_V)]^2 + [G(V_0, \sigma_V)]^2$$

* This can be done numerically, for example with R.
Consider a company whose equity is 3 million euros. The volatility of equity is 0.80. The company’s debt is equal to 10 million euros and it must be paid in one year. The risk-free rate on the market is 5% per annum.

This means that

\[ S_0 = 3, \sigma_S = 0.8, r = 0.05, T = 1, B = 10 \]

**Question 1:** What are the values of \( V_0 \) and \( \sigma_V \)?

**Question 2:** What is the probability of default of the company in one year?
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**Question 2:** What is the probability of default of the company in one year?

We will use R to solve this exercise.
Merton’s model in a nutshell.

<table>
<thead>
<tr>
<th>Pros</th>
<th>Cons</th>
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<tbody>
<tr>
<td>Simple yet able to obtain meaningful results.</td>
<td>Default only in $T$.</td>
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<tr>
<td>Effective view of the potential conflict of interest between shareholders and debt holders in a company (see $P(V_T \leq B)$, slide 6)</td>
<td>Default = Liquidation.</td>
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<tr>
<td>“Easily” computable.</td>
<td>Assumes that the world is “Gaussian”.</td>
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Thank You