TW3421x - An Introduction to Credit Risk Management

The VaR and its derivations

Coherent measures of risk and back-testing

Dr. Pasquale Cirillo

Week 3
Lesson 3

Coherent measures of risk

- A risk measure is a function that it is used to quantify risk.

- A risk measure is meant to determine the amount of an asset (or set of assets) to be kept in reserve. The aim of a reserve is to guarantee the presence of capital that can be used as a (partial) cover if the risky event manifests itself, generating a loss.

- From a mathematical point of view, a measure of risk is a function

$$\phi : \mathcal{L} \to \mathbb{R} \cup \{+\infty\}$$

where $\mathcal{L}$ is the linear space of losses.

Coherent measures of risk

- A measure of risk is said **coherent** if it is monotone, sub-additive, positive homogeneous and translation invariant:

$$Z_1, Z_2 \in \mathcal{L},\ Z_1 \le Z_2,\ \phi(Z_1) \le \phi(Z_2)$$ Monotonicity

$$Z_1, Z_2 \in \mathcal{L},\ \phi(Z_1 + Z_2) \le \phi(Z_1) + \phi(Z_2)$$ Sub-additivity

$$a \ge 0,\ Z \in \mathcal{L},\ \phi(aZ) = a\phi(Z)$$ Pos. Homogeneity

$$b \in \mathbb{R}, Z \in \mathcal{L}, \phi(Z + b) = \phi(Z)\ \ (\ldots - b)$$ Trans. Invariance

- Quite often is good to require a risk measure to be normalized as well:

$$\phi(0) = 0$$

- Coherent risk measures are of great importance in risk management.

- What are the financial implications of the properties of a coherent risk measure?

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Pos. Homogeneity	→	Proportionality of risk
Trans. Invariance	→	Neutrality/safety of liquidity

Is the VaR coherent?

- We have two independent portfolios of bonds. They both have a probability of 0.02 of a loss of £10 million and a probability of 0.98 of a loss of £1 million over a 1-year time window.
  The $VaR_{0.975}$ is £1 million for each investment.

- Let's combine the two portfolios and have a look at their joint VaR.

Is the ES coherent?

- The expected shortfall of the each portfolio at confidence level 0.975 is £8.2 million.
- If we combine the two investments in a single portfolio, the joint ES is £11.4 million.
- Notice that **11.14<8.2+8.2**

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The ES is ALWAYS coherent!

And in general?

- When you consider actual data, the VaR is very often not sub-additive (the ES yes), and this may be a problem, as we will see.
- However, there are theoretical "situations" in which the VaR is a coherent measure, and these "situations" are the building blocks of many models of CR.
- The most important case are elliptical distributions, such as the Gaussian.

Back-testing

- Back-testing is a validation procedure often used in risk management.
- The idea is simply to check the performances of the chosen risk measure on historical data.
- Consider a 99% C-VaR. What we want to verify is how that value would have performed if used in the past.

- In other terms, we count the number of days in the past in which the actual loss was higher than our 99% VaR.

- If these days, called **exceptions**, are less than 1% then our back-testing is successful.
  If exceptions are say 5%, then we are probably underestimating the actual VaR.

- Naturally it may also happen that we overestimate VaR!

Back-testing

- Standard back-testing is based on Bernoulli trials generating binomial random variables.
- If our (C)VaR$_\alpha$ is accurate, the probability **p** of observing an exception is **1-α**.
- Now, suppose we look at a total of **n** days and that we observe **m<n** exceptions.
- What we want to compare is the ratio **m/n** and the probability **p**.

- According to the binomial distribution, *which tells us the probability of observing m "successes" in n "trials"*, we have that the probability that we observe more than **m** exceptions is

$$\sum_{k=m}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

- Generally a 5% significance level for the test is chosen. This means that, if the probability of observing exceptions is less than 5%, we reject the null hypothesis that the exception's probability is **p**.
  Otherwise we cannot reject the null hypothesis.

Exercise

- We want to back-test our VaR using 900 days of data.
- α=0.99 and we observe 12 exceptions.
  According to our VaR we expect only
  9 exceptions (1% of 900).
- Should we reject our VaR?
- What happens with 20 exceptions?

Another exercise

- Let's consider the same data, but we now observe only 2 exceptions.
- These are less than what we expected!
- Is our VaR overestimated?
- What about 4 exceptions?

```
> pbinom(2,900,0.01,TRUE)
[1] 0.006058456
> # In this case our VaR is too large.
> pbinom(4,900,0.01,TRUE)
[1] 0.05412027
> # In this case our VaR is appropriate.
```

Thank You