Lecture 11: Quantum Search

Needle in a haystack
Searching for a needle in a haystack
Unstructured search

“Digital haystack”

Goal: Search for the marked entry.

Classically: try random entries. O(N/2) expected time.

Quantum??
## Unstructured search

“Digital haystack”

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Quantum??
Finding a solution to an NP-complete problem can be viewed as a search problem.

$$f(x_1, \ldots, x_n) = (x_1 \lor \neg x_2 \lor x_3) \land (x_2 \lor \neg x_5 \lor x_6) \land \cdots$$

Is there a configuration of $x_1, x_2, \cdots$ that satisfy the above formula?

There are $2^n$ possible configurations.

$N = 2^n$
Unstructured search

“Digital haystack”
Unstructured search

“Digital haystack”

Quantum??

Grover’s Algorithm: Quantum algorithm for unstructured search that takes $O(\sqrt{N})$ time.

\[
N = 2^n
\]

$\text{size } n \cdot \sqrt{N} = 2^{\frac{n}{2}}$
**Unstructured search**

“Digital haystack”

**Problem.** Given $f: \{0, 1, \ldots, N-1\} \to \{0,1\}$, find $x: f(x) = 1$.

**Hardest case:** There is exactly one $x: f(x) = 1$. 

$$N = 2^n$$
Lecture 11: Quantum Search

Grover’s Algorithm
Unstructured search

“Digital haystack”

**Problem.** Given $f: \{0, 1, \ldots, N-1\} \rightarrow \{0, 1\}$, find $x: f(x) = 1$.

**Hardest case:** There is exactly one $x: f(x) = 1$. 
Phase Inversion

\[ f(x^*) = 1 \]

\[ \sum_x \alpha_x \left< x \right| \rightarrow \text{Phase inversion.} \]

\[ \sum_{x \neq x^*} \alpha_x \left< x \right| - \alpha_{x^*} \left< x^* \right| \]
Inversion About Mean

\[ \sum_x \alpha_x |x\rangle \]

\[ \mu = \frac{\sum_{x=0}^{N-1} \alpha_x}{2} \]

\[ \alpha'_x \rightarrow (2 \mu - \alpha_x) = \mu + (\mu - \alpha_x) \]

\[ \sum_x \alpha_x |x\rangle \rightarrow \sum_x (2 \mu - \alpha_x) |x\rangle \]
Grover’s algorithm

**Problem.** Given $f : \{0, \ldots, N - 1\} \to \{0, 1\}$ such that $f(x) = 1$ for exactly one $x$, find $x$. 
Grover's algorithm

**Problem.** Given $f : \{0, \ldots, N - 1\} \rightarrow \{0, 1\}$ such that $f(x) = 1$ for exactly one $x$, find $x$. 
Grover’s algorithm

**Problem.** Given $f : \{0, \ldots, N - 1\} \to \{0, 1\}$ such that $f(x) = 1$ for exactly one $x$, find $x$. 
**Grover’s algorithm**

**Problem.** Given $f : \{0, \ldots, N - 1\} \to \{0, 1\}$ such that $f(x) = 1$ for exactly one $x$, find $x$.

How many steps?
Grover’s algorithm

What is the amplitude of the rest when the needle has $\frac{1}{\sqrt{2}}$?

We will reach $\frac{1}{\sqrt{2}}$ in $O(\sqrt{N})$ steps.

At this point how much improvement are we making per step?

We will reach $\frac{1}{\sqrt{2}}$ in $O(\sqrt{N})$ steps.
Problem. Given $f : \{0, \ldots, N - 1\} \to \{0, 1\}$ such that $f(x) = 1$ for exactly one $x$, find $x$. 

\[ \sum_{x} \alpha_{x} |x\rangle \xrightarrow{\text{Phase inversion}} \sum_{x} \alpha_{x} (-1)^{f(x)} |x\rangle \]
Phase Inversion

\[
\begin{align*}
\sum_x \alpha_x |x\rangle \left[\begin{array}{c|c}
|0\rangle & |1\rangle \\
\hline
0 & 0 \\
0 & 0
\end{array}\right] U_f \left[\begin{array}{c|c}
|0\rangle & |1\rangle \\
\hline
0 & 0 \\
0 & 0
\end{array}\right] \sum_x \alpha_x (-1)^{f(x)} |x\rangle \\
\sum_x \alpha_x |x\rangle \left[\begin{array}{c|c}
|0\rangle & |1\rangle \\
\hline
0 & 0 \\
0 & 0
\end{array}\right] \left[\begin{array}{c|c}
|0\rangle & |1\rangle \\
\hline
0 & 0 \\
0 & 0
\end{array}\right] \sum_x \alpha_x (-1)^{f(x)} |x\rangle
\end{align*}
\]

\[
\frac{\sqrt{2}}{\sqrt{3}} |0\rangle - \frac{\sqrt{2}}{\sqrt{3}} |1\rangle \quad \frac{\sqrt{2}}{\sqrt{3}} |1\rangle - \frac{\sqrt{2}}{\sqrt{3}} |0\rangle = - \left[ \frac{\sqrt{2}}{\sqrt{3}} |0\rangle - \frac{\sqrt{2}}{\sqrt{3}} |1\rangle \right] \quad |\rightarrow\ |
\]

\[
\sum_x \alpha_x |x\rangle \left[\begin{array}{c}
|0\rangle \\
|1\rangle
\end{array}\right] (-1)^{f(x)} |\rightarrow\ |
\]
Inversion About Mean

\[
\sum_x \alpha_x |x\rangle
\]

\[
\mathcal{M} = \frac{\sum_{x=0}^{N-1} \alpha_x}{2}
\]

\[
\alpha_x \rightarrow (2\mathcal{M} - \alpha_x) = \mathcal{M} + (\mathcal{M} - \alpha_x)
\]

\[
\sum_x \alpha_x |x\rangle \rightarrow \sum_x (2\mathcal{M} - \alpha_x) |x\rangle
\]
Inversion About Mean

\[ g : \{0, 1\}^n \rightarrow \{0, 1\} \]
\[ g(0 \cdots 0) = 0 \]
\[ g(y) = 1 \text{ if } y \neq 0 \cdots 0. \]
Inversion about the mean is the same as doing reflection about

\[ |u\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle \]

\[ |v\rangle = \exists \alpha \cdot |u\rangle \]

1) Transform \( |u\rangle \) into \( |0, \ldots, 0\rangle \)
2) Reflection about \( |0, \ldots, 0\rangle \)
3) Transform \( |0, \ldots, 0\rangle \) into \( |u\rangle \)
Inversion about the mean is the same as doing reflection about

\[ |u\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle \]
Quantum Ckt fn Grover's Alg.

\[ N = 2^n \]

Initialization

Phase inversion

Inversion about mean

\[ H \otimes n \]

\[ U_f \]

\[ H \otimes n \]

\[ U \neq 0^n \]

\[ H \otimes n \]

\[ |0\rangle \]

\[ \cdots \]

\[ |0\rangle \]

\[ \langle 0 | \]

\[ \langle - | \]

\[ O(\sqrt{N}) \text{ iterations}. \]