Chapter 1

Introduction

Nature at the sub-atomic scale behaves totally differently from anything that our experience with the physical world prepares us for. **Quantum mechanics** is the name of the branch of physics that governs the world of elementary particles such as electrons and photons, and it is paradoxical, unintuitive, and radically strange. Below is a sampling of a few such odd features:

- Complete knowledge of a system's state is forbidden - A measurement reveals only a small amount of information about the quantum state of the system.
- The act of measuring a particle fundamentally disturbs its state.
- Quantum entities do not have trajectories. All that we can say for an elementary particle is that it started at A and was measured later at B. We cannot say anything about the trajectory or path it took from A to B.
- Quantum Mechanics is inherently probabilistic. If we prepare two elementary particles in identical states and measure them, the results may be different for each particle.
- Quantum entities behave in some ways like particles and in others like waves. But they really behave in their own unique way, neither particles nor waves.

These features are truly strange, and difficult to accept. To quote the great physicist Niels Bohr, "Anyone who is not shocked by quantum theory has not understood it." We start by describing a simple experiment that highlights many differences between quantum mechanics and our classical intuition. It is the famous double slit experiment. The intuition gained in the process will help us as we define qubits, and more generally in the study of quantum computing.

1.1 The Double Slit Experiment

What is the nature of light? You may have learned light is electromagnetic *waves* propagating through space. Also, you may have learned that light is made of a rain of individual *particles* called photons. But these two notions seem contradictory, how can it be both?

The debate over the nature of light goes deep into the history of science. The eminent physicist Isaac Newton believed that light was a rain of particles, called corpuscles. At the beginning of the nineteenth century, Thomas Young demonstrated with his famous double-slit interference experiment that light propagates as waves. With Maxwell's formulation of electromagnetism at the end of the nineteenth century, it was generally accepted that light is propagated as electromagnetic waves, and the debate seemed to be over. However, in 1905, Einstein was able to explain the photoelectric effect, by using the idea of light quanta, or particles which we now call photons.

Similar confusion reigned over the nature of electrons, which behaved like particles, but then it was discovered in electron diffraction experiments, performed in 1927, that they exhibit wave behavior. So do electrons behave like particles or waves? And what about photons? This great challenge was resolved with the discovery of the equations of quantum mechanics. But the theory is not intuitive, and its description of matter is very different from our common experience.

To understand what seems to be a paradox, we look to Young's double-slit experiment. Here's the set up: a source of light is shone at a screen with two very thin, identical slits cut into it. Some light passes through the two slits and lands upon a subsequent screen. Take a look at Figure 1.1 for a diagram of the experiment setup.

First, think about what would happen to a stream of bullets going through this double slit experiment. The source, which we think of as a machine gun, is unsteady and sprays the bullets in the general direction of the two slits. Some bullets pass through one slit, some pass through the other slit, and others don't make it through the slits. The bullets that do go through the slits then land on the observing screen behind them. Now suppose we closed slit 2. Then the bullets can only go through slit 1 and land in a small spread behind slit 1. If we graphed the number of times a bullet that went through slit 1 landed at

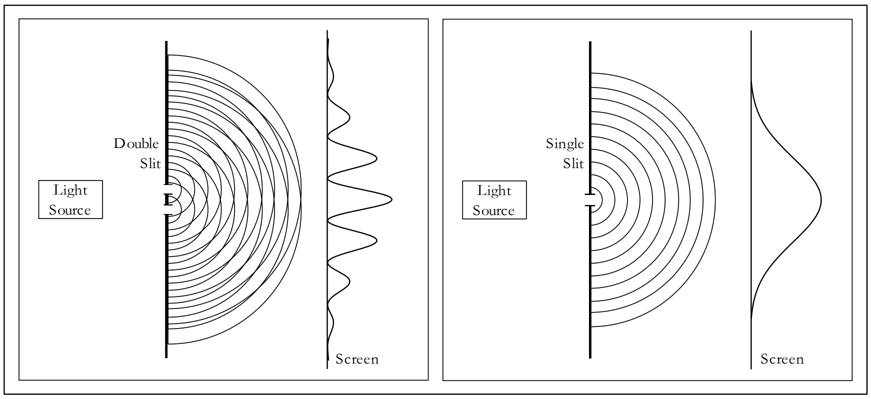

Figure 1.1: Double- and single-slit diffraction. Notice that in the double-slit experiment the two paths interfere with one another. This experiment gives evidence that light propagates as a wave.

the position y on the observation screen, we would see a normal distribution centered directly behind slit 1. That is, most land directly behind the slit, but some stray off a little due to the small amount randomness inherent in the gun, and because of they ricochet off the edges of the slit. If we now close slit 1 and open slit 2, we would see a normal distribution centered directly behind slit 2.

Now let's repeat the experiment with both slits open. If we graph the number of times a bullet that went through *either* slit landed at the position y, we should see the sum of the graph we made for slit 1 and a the graph for slit 2.

Another way we can think of the graphs we made is as graphs of the probability that a bullet will land at a particular spot y on the screen. Let $P_1(y)$ denote the probability that the bullet lands at point y when only slit 1 is open, and similarly for $P_2(y)$. And let $P_{12}(y)$ denote the probability that the bullet lands at point y when both slits are open. Then $P_{12}(y) = P_1(y)+P_2(y)$.

Next, we consider the situation for waves, for example water waves. A water wave doesn't go through *either* slit 1 or slit 2, it goes through both. You should imagine the crest of 1 water wave as it approaches the slits. As it hits the slits, the wave is blocked at all places but the two slits, and waves on the other side are generated at each slit as depicted in Figure 1.1.

When the new waves generated at each slit run into each other, interference occurs. We can see this by plotting the intensity (that is, the amount of energy carried by the waves) at each point y along the viewing screen. What we see is the familiar interference pattern seen in Figure 1.1. The dark patches of the interference pattern occur where the wave from the first slit arrives perfectly out of sync with wave from the second slit, while the bright points are where the two arrive in sync. For example, the bright spot right in the middle is bright because each wave travels the exact same distance from their respective slit to the screen, so they arrive in sync. The first dark spots are where the wave from one slit traveled exactly half of a wavelength longer than the other wave, thus they arrive at opposite points in their cycle and cancel. Here, it is not the intensities coming from each slit that add, but height of the wave. This differs from the case of bullets: $I_{12}(y) \neq I_1(y) + I_2(y)$, but $h_{12}(y) = h_1(y) + h_2(y)$, and $I_{12}(y) = h(y)^2$, where $h(y)$ is the height of the wave and $I(y)$ is the intensity, or energy, of the wave.

Before we can say what light does, we need one more crucial piece of information. What happens when we turn down the intensity in both of these examples?

In the case of bullets, turning down the intensity means turning down the rate at which the bullets are fired. When we turn down the intensity, each time a bullet hits the screen it transfers the same amount of energy, but the frequency at which bullets hit the screen becomes less.

With water waves, turning down the intensity means making the wave amplitudes smaller. Each time a wave hits the screen it transfers *less* energy, but the frequency of the waves hitting the screen is unchanged.

Now, what happens when we do this experiment with light. As Young observed in 1802, light makes an interference pattern on the screen. From this observation he concluded that the nature of light is wavelike, and reasonably so! However, Young was unable at the time to turn down the intensity of light enough to see the problem with the wave explanation.

Picture now that the observation screen is made of thousands of tiny little photo-detectors that can detect the energy they absorb. For high intensities the photo-detectors individually are picking up a lot of energy, and when we plot the intensity against the position y along the screen we see the same interference pattern described earlier. Now, turn the intensity of the light very very very low. At first, the intensity scales down lower and lower everywhere, just like with a wave. But as soon as we get low enough, the energy that the photo-detectors report reaches a minimum energy, and all of the detectors are reporting the same energy, call it E_0, just at different rates. This energy corresponds to the energy carried by an individual photon, and at this stage we see what is called the quantization of light.

Photo-detectors that are in the bright spots of the interference pattern report the energy E_0 very frequently, while darker areas report the energy E_0 at lower rates. Totally dark points still report nothing. This behavior is the behavior of *bullets*, not waves! We now see that photons behave unlike either bullets or waves, but like something entirely different.

Turn down the intensity so low that only one photo-detector reports something each second. In other words, the source only sends one photon at a time. Each time a detector receives a photon, we record where on the array it landed and plot it on a graph. The distribution we draw will reflect the *probability* that a single photon will land at a particular point.

Logically we think that the photon will either go through one slit or the other. Then, like the bullets, the probability that the photon lands at a point should be y is $P_{12}(y) = P_1(y) + P_2(y)$ and the distribution we expect to see is the two peaked distribution of the bullets. *But this not what we see at all.*

What we actually see is the same interference pattern from before! But how can this be? For there to be an interference pattern, light coming from one slit must interfere with light from the other slit; but there is only one photon going through at a time! The modern explanation is that the photon actually goes through *both* slits at the same time, and interferes with *itself*. The mathematics is analogous to that in the case of water waves. We say that the probability $P(y)$ that a photon is detected at y is proportional to the square of some quantity $a(y)$, which we call a probability amplitude. Now probability amplitudes for different alternatives add up. So $a_{12}(y) = a_1(y) + a_2(y)$. But $P_{12}(y) = |a_{12}(y)|^2 \neq |a_1(y)|^2 + |a_2(y)|^2 = P_1(y) + P_2(y)$.

Logically, we can ask which slit the photon went through, and try to measure it. Thus, we might construct a double slit experiment where we put a photodetector at each slit, so that each time a photon comes through the experiment we see which slit it went through and where it hits on the screen. But when such an experiment is preformed, the interference pattern gets completely washed out! The very fact that we know which slit the photon goes through makes the interference pattern go away. This is the first example we see of how measuring a quantum system alters the system.

Here the photon looks both like a particle, a discreet package, and a wave that can have interference. It seems that the photon acts like both a wave and a particle, but at the same time it doesn't exactly behave like either. This is what is commonly known as the wave-particle duality, usually thought of as a paradox. The resolution is that the quantum mechanical behavior of matter is unique, something entirely new.

What may be more mind blowing still is that if we conduct the exact same experiment with *electrons* instead of light, we get the exact same results! Although it is common to imagine electrons as tiny little charged spheres, they are actually quantum entities, neither wave nor particle but understood by their wavefunction.

The truth is that there is no paradox, just an absence of intuition for quantum entities. Why should they be intuitive? Things on our scale do not behave like wavefunctions, and unless we conduct wild experiments like this we do not see the effects of quantum mechanics. The following sections describe in more detail some of the basic truths of quantum mechanics, so that we can build an intuition for a new behavior of matter.

1.2 The Axioms of Quantum Mechanics

"I think I can safely say that nobody understands quantum mechanics."

-Richard Feynman

Paradoxically, the fundamental principles of quantum mechanics can be stated very concisely and simply. The challenge lies in understanding and applying these principles, which is the goal of the rest of the book. Here are two basic

- The superposition principle explains how a particle can be superimposed between two states at the same time.
- The measurement principle tells us how measuring a particle changes its state, and how much information we can access from a particle.
- The unitary evolution axoim governs how the state of the quantum system evolves in time.

In keeping with our philosophy, we will introduce the basic axioms gradually, starting with simple finite systems, and simplified basis state measurements, and building our way up to the more general formulations. This should allow the reader a chance to develop some intuition about these topics.

1.3 The Superposition Principle

Consider a system with k distinguishable (classical) states. For example, the electron in a hydrogen atom is only allowed to be in one of a discrete set of energy levels, starting with the ground state, the first excited state, the second excited state, and so on. If we assume a suitable upper bound on the total energy, then the electron is restricted to being in one of k different energy levels — the ground state or one of $k-1$ excited states. As a classical system, we might use the state of this system to store a number between 0 and $k-1$. The superposition principle says that if a quantum system can be in one of two states then it can also be placed in a linear superposition of these states with complex coefficients.

Let us introduce some notation. We denote the ground state of our k-state system by $|0\rangle$, and the succesive excited states by $|1\rangle, \ldots, |k-1\rangle$. These are the k possible classical states of the electron. The superposition principle tells us that, in general, the quantum state of the electron is $\alpha_0 |0\rangle + \alpha_1 |1\rangle + \cdots + \alpha_{k-1} |k-1\rangle$, where $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ are complex numbers normalized so that $\sum_j |\alpha_j|^2 = 1$. α_j is called the *amplitude of the state* $|j\rangle$. For instance, if $k=3$, the state of the electron could be

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{2} |1\rangle + \frac{1}{2} |2\rangle$$

or

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{2} |1\rangle + \frac{i}{2} |2\rangle$$

or

$$|\psi\rangle = \frac{1+i}{3} |0\rangle - \frac{1-i}{3} |1\rangle + \frac{1+2i}{3} |2\rangle .$$

The superposition principle is one of the most mysterious aspects about quantum physics — it flies in the face of our intuitions about the physical world. One way to think about a superposition is that the electron does not make up its mind about whether it is in the ground state or each of the $k-1$ excited states, and the amplitude α_0 is a measure of its inclination towards the ground state. Of course we cannot think of α_0 as the probability that an electron is in the ground state — remember that α_0 can be negative or imaginary. The measurement priniciple, which we will see shortly, will make this interpretation of α_0 more precise.

1.4 The Geometry of Hilbert Space

We saw above that the quantum state of the k-state system is described by a sequence of k complex numbers $\alpha_0, \ldots, \alpha_{k-1} \in \mathbb{C}$, normalized so that $\sum_j |\alpha_j|^2 = 1$. So it is natural to write the state of the system as a k dimensional vector:

$$|\psi\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{k-1} \end{pmatrix}$$

The normalization on the complex amplitudes means that the state of the system is a unit vector in a k dimensional complex vector space — called a Hilbert space.

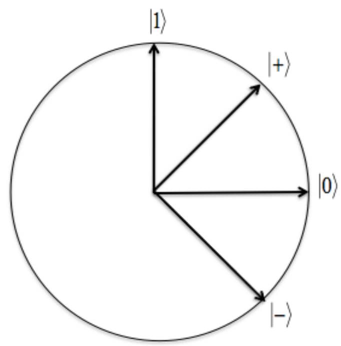

Figure 1.2: Representation of qubit states as vectors in a Hilbert space.

But hold on! Earlier we wrote the quantum state in a very different (and simpler) way as: $\alpha_0 |0\rangle + \alpha_1 |1\rangle + \cdots + \alpha_{k-1} |k-1\rangle$. Actually this notation, called Dirac's ket notation, is just another way of writing a vector. Thus

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad |k-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

So we have an underlying geometry to the possible states of a quantum system: the k distinguishable (classical) states $|0\rangle, \ldots, |k-1\rangle$ are represented by mutually orthogonal unit vectors in a k-dimensional complex vector space. i.e. they form an orthonormal basis for that space (called the standard basis). Moreover, given any two states, $\alpha_0 |0\rangle + \alpha_1 |1\rangle + \cdots + \alpha_{k-1} |k-1\rangle$, and $\beta |0\rangle + \beta |1\rangle + \cdots + \beta |k-1\rangle$, we can compute the inner product of these two vectors, which is $\sum_{j=0}^{k-1} \alpha_j^* \beta_j$. The absolute value of the inner product is the cosine of the angle between these two vectors in Hilbert space. You should verify that the inner product of any two basis vectors in the standard basis is 0, showing that they are orthogonal.

The advantage of the ket notation is that the it labels the basis vectors explicitly. This is very convenient because the notation expresses both that the state of the quantum system is a vector, while at the same time explicitly writing out the physical quantity of interest (energy level, position, spin, polarization, etc).

1.5 Bra-ket Notation

In this section we detail the notation that we will use to describe a quantum state, $|\psi\rangle$. This notation is due to Dirac and, while it takes some time to get used to, is incredibly convenient.

Inner Products

We saw earlier that all of our quantum states live inside a Hilbert space. A Hilbert space is a special kind of vector space that, in addition to all the usual rules with vector spaces, is also endowed with an inner product. And an inner product is a way of taking two states (vectors in the Hilbert space) and getting a number out. For instance, define

$$|\psi\rangle = \sum_k a_k |k\rangle ,$$

where the kets $|k\rangle$ form a basis, so are orthogonal. If we instead write this state as a column vector,

$$|\psi\rangle = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix}$$

Then the inner product of $|\psi\rangle$ with itself is

$$\langle\psi,\psi\rangle = \begin{pmatrix} a_0^* & a_1^* & \cdots & a_{N_1}^* \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix} = \sum_{k=0}^{N-1} a_k^* a_k = \sum_{k=0}^{N-1} |a_k|^2$$

The complex conjugation step is important so that when we take the inner product of a vector with itself we get a real number which we can associate with a length. Dirac noticed that there could be an easier way to write this by defining an object, called a "bra," that is the conjugate-transpose of a ket,

$$\langle\psi| = |\psi\rangle^\dagger = \sum_k a_k^* \langle k| \, .$$

This object acts on a ket to give a number, as long as we remember the rule,

$$\langle j| \, |k\rangle \equiv \langle j|k\rangle = \delta_{jk}$$

Now we can write the inner product of $|\psi\rangle$ with itself as

$$\begin{aligned}
\langle\psi|\psi\rangle &= \left(\sum_j a_j^* \langle j|\right) \left(\sum_k a_k |k\rangle\right) \\
&= \sum_{j,k} a_j^* a_k \langle j|k\rangle \\
&= \sum_{j,k} a_j^* a_k \delta_{jk} \\
&= \sum_k |a_k|^2
\end{aligned}$$

Now we can use the same tools to write the inner product of any two states, $|\psi\rangle$ and $|\phi\rangle$, where

$$|\phi\rangle = \sum_k b_k |k\rangle \, .$$

Their inner product is,

$$\langle\psi|\phi\rangle = \sum_{j,k} a_j^* b_k \langle j|k\rangle = \sum_k a_k^* b_k$$

Notice that there is no reason for the inner product of two states to be real (unless they are the same state), and that

$$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^* \in \mathbb{C}$$

In this way, a bra vector may be considered as a "functional." We feed it a ket, and it spits out a complex number.

The Dual Space

We mentioned above that a bra vector is a *functional* on the Hilbert space. In fact, the set of all bra vectors forms what is known as the *dual space*. This space is the set of *all* linear functionals that can act on the Hilbert space.

1.6 The Measurement Principle

This linear superposition $|\psi\rangle = \sum_{j=0}^{k-1} \alpha_j |j\rangle$ is part of the private world of the electron. Access to the information describing this state is severely limited — in particular, we cannot actually measure the complex amplitudes α_j. This is not just a practical limitation; it is enshrined in the measurement postulate of quantum physics.

A measurement on this k state system yields one of at most k possible outcomes: i.e. an integer between 0 and $k-1$. Measuring $|\psi\rangle$ in the standard basis yields j with probability $|\alpha_j|^2$.

One important aspect of the measurement process is that it alters the state of the quantum system: the effect of the measurement is that the new state is exactly the outcome of the measurement. I.e., if the outcome of the measurement is j, then following the measurement, the qubit is in state $|j\rangle$. This implies that you cannot collect any additional information about the amplitudes α_j by repeating the measurement.

Intuitively, a measurement provides the only way of reaching into the Hilbert space to probe the quantum state vector. In general this is done by selecting an orthonormal basis $|e_0\rangle, \ldots, |e_{k-1}\rangle$. The outcome of the measurement is $|e_j\rangle$ with probability equal to the square of the length of the projection of the state vector ψ on $|e_j\rangle$. A consequence of performing the measurement is that the new state vector is $|e_j\rangle$. Thus measurement may be regarded as a probabilistic rule for projecting the state vector onto one of the vectors of the orthonormal measurement basis.

Some of you might be puzzled about how a measurement is carried out physically? We will get to that soon when we give more explicit examples of quantum systems.

1.7 Qubits

Qubits (pronounced "cue-bit") or quantum bits are basic building blocks that encompass all fundamental quantum phenomena. They provide a mathematically simple framework in which to introduce the basic concepts of quantum physics. Qubits are 2-state quantum systems. For example, if we set $k = 2$, the electron in the Hydrogen atom can be in the ground state or the first excited state, or any superposition of the two. We shall see more examples of qubits soon.

The state of a qubit can be written as a unit (column) vector $\binom{\alpha}{\beta} \in \mathbb{C}^2$. In Dirac notation, this may be written as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad \text{with} \quad \alpha, \beta \in \mathbb{C} \quad \text{and} \quad |\alpha|^2 + |\beta|^2 = 1.$$

This linear superposition $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is part of the private world of the electron. For us to know the electron's state, we must make a measurement. Making a measurement gives us a single classical bit of information — 0 or 1. The simplest measurement is in the standard basis, and measuring $|\psi\rangle$ in this $\{|0\rangle, |1\rangle\}$ basis yields 0 with probability $|\alpha|^2$, and 1 with probability $|\beta|^2$.

One important aspect of the measurement process is that it alters the state of the qubit: the effect of the measurement is that the new state is exactly the outcome of the measurement. *I.e.*, if the outcome of the measurement of $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ yields 0, then following the measurement, the qubit is in state $|0\rangle$. This implies that you cannot collect any additional information about α, β by repeating the measurement.

More generally, we may choose any orthogonal basis $\{|v\rangle, |w\rangle\}$ and measure the qubit in that basis. To do this, we rewrite our state in that basis: $|\psi\rangle = \alpha'|v\rangle + \beta'|w\rangle$. The outcome is v with probability $|\alpha'|^2$, and $|w\rangle$ with probability $|\beta'|^2$. If the outcome of the measurement on $|\psi\rangle$ yields $|v\rangle$, then as before, the the qubit is then in state $|v\rangle$.

Examples of Qubits

Atomic Orbitals

The electrons within an atom exist in quantized energy levels. Qualitatively these electronic orbits (or "orbitals" as we like to call them) can be thought of as resonating standing waves, in close analogy to the vibrating waves one observes on a tightly held piece of string. Two such individual levels can be isolated to configure the basis states for a qubit.

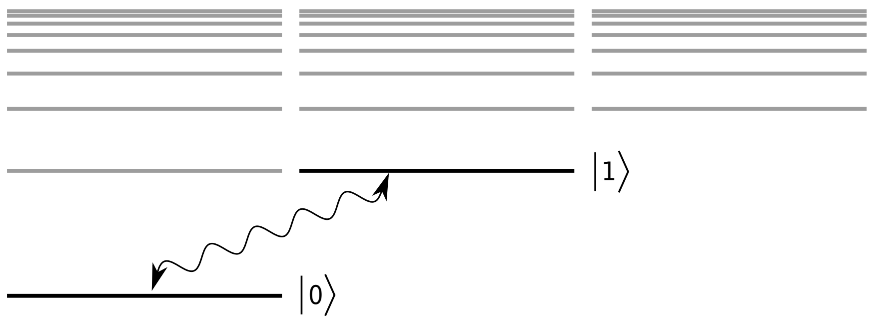

Figure 1.3: Energy level diagram of an atom. Ground state and first excited state correspond to qubit levels, $|0\rangle$ and $|1\rangle$, respectively.

Photon Polarization

Classically, a photon may be described as a traveling electromagnetic wave. This description can be fleshed out using Maxwell's equations, but for our purposes we will focus simply on the fact that an electromagnetic wave has a *polarization* which describes the orientation of the electric field oscillations (see Fig. 1.4). So, for a given direction of photon motion, the photon's polarization axis might lie anywhere in a 2-d plane perpendicular to that motion. It is thus natural to pick an orthonormal 2-d basis (such as $\vec{x}$ and $\vec{y}$, or "vertical" and "horizontal") to describe the polarization state (i.e. polarization direction) of a photon. In a quantum mechanical description, this 2-d nature of the photon polarization is represented by a qubit, where the amplitude of the overall polarization state in each basis vector is just the projection of the polarization in that direction.

The polarization of a photon can be measured by using a polaroid film or a calcite crystal. A suitably oriented polaroid sheet transmits x-polarized photons and absorbs y-polarized photons. Thus a photon that is in a superposition $|\phi\rangle = \alpha\,|\mathsf{x}\rangle + \beta\,|\mathsf{y}\rangle$ is transmitted with probability $|\alpha|^2$. If the photon now encounters another polariod sheet with the same orientation, then it is transmitted with probability 1. On the other hand, if the second polaroid sheet has its axes crossed at right angles to the first one, then if the photon is transmitted by the first polaroid, then it is definitely absorbed by the second sheet. This pair of polarized sheets at right angles thus blocks all the light. A somewhat counter-intuitive result is now obtained by interposing a third polariod sheet at a 45 degree angle between the first two. Now a photon that is transmitted by the first sheet makes it through the next two with probability

1/4.

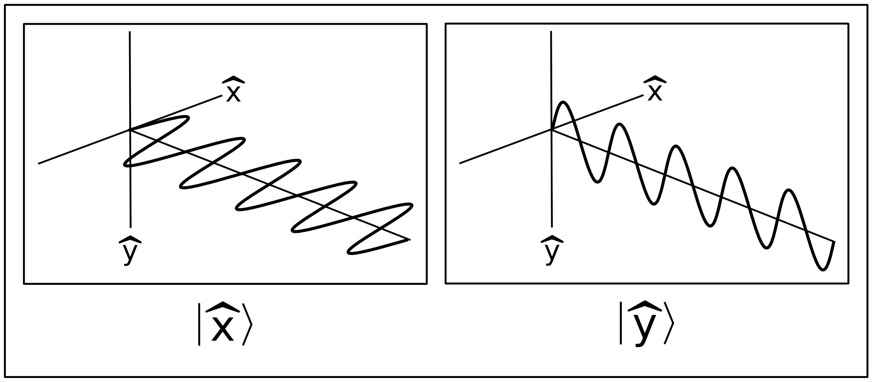

Figure 1.4: Using the polarization state of light as the qubit. Horizontal polarization corresponds to qubit state, $|\hat{x}\rangle$, while vertical polarization corresponds to qubit state, $|\hat{y}\rangle$.

To see this first observe that any photon transmitted through the first filter is in the state, $|0\rangle$. The probability this photon is transmitted through the second filter is 1/2 since it is exactly the probability that a qubit in the state $|0\rangle$ ends up in the state $|+\rangle$ when measured in the $|+\rangle, |-\rangle$ basis. We can repeat this reasoning for the third filter, except now we have a qubit in state $|+\rangle$ being measured in the $|0\rangle, |1\rangle$-basis — the chance that the outcome is $|0\rangle$ is once again 1/2.

Spins

Like photon polarization, the spin of a (spin-1/2) particle is a two-state system, and can be described by a qubit. Very roughly speaking, the spin is a quantum description of the magnetic moment of an electron which behaves like a spinning charge. The two allowed states can roughly be thought of as clockwise rotations ("spin-up") and counter clockwise rotations ("spin-down"). We will say much more about the spin of an elementary particle later in the course.

Measurement Example I: Phase Estimation

Now that we have discussed qubits in some detail, we can are prepared to look more closely at the measurement principle. Consider the quantum state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle .$$

If we were to measure this qubit in the standard basis, the outcome would be 0 with probability 1/2 and 1 with probability 1/2. This measurement tells us only about the norms of the state amplitudes. Is there any measurement that yields information about the phase, θ?

To see if we can gather any phase information, let us consider a measurement in a basis other than the standard basis, namely

$$|+\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad \text{and} \qquad |-\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

What does $|\phi\rangle$ look like in this new basis? This can be expressed by first writing,

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \qquad \text{and} \qquad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

Now we are equipped to rewrite $|\psi\rangle$ in the $\{|+\rangle, |-\rangle\}$-basis,

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle) \\ &= \frac{1}{2}(|+\rangle + |-\rangle) + \frac{e^{i\theta}}{2}(|+\rangle - |-\rangle) \\ &= \frac{1+e^{i\theta}}{2}|+\rangle + \frac{1-e^{i\theta}}{2}|-\rangle \quad . \end{aligned}$$

Recalling the Euler relation, $e^{i\theta} = \cos\theta + i\sin\theta$, we see that the probability of measuring $|+\rangle$ is $\frac{1}{4}((1+\cos\theta)^2 + \sin^2\theta) = \cos^2(\theta/2)$. A similar calculation reveals that the probability of measuring $|-\rangle$ is $\sin^2(\theta/2)$. Measuring in the $(|+\rangle, |-\rangle)$-basis therefore reveals some information about the phase θ.

Later we shall show how to analyze the measurement of a qubit in a general basis.

Measurement example II: General Qubit Bases

What is the result of measuring a general qubit state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, in a general orthonormal basis $|v\rangle\,, \left|v^{\perp}\right\rangle$, where $|v\rangle = a|0\rangle + b|1\rangle$ and $|v^{\perp}\rangle = b^*|0\rangle - a^*|1\rangle$? You should also check that $|v\rangle$ and $\left|v^{\perp}\right\rangle$ are orthogonal by showing that $\left\langle v^{\perp}|v\right\rangle = 0$.

To answer this question, let us make use of our recently acquired bra-ket notation. We first show that the states $|v\rangle$ and $\left|v^{\perp}\right\rangle$ are orthogonal, that is, that their inner product is zero:

$$\begin{aligned}\left\langle v^{\perp}|v\right\rangle &= \left(b^*\,|0\rangle - a^*\,|1\rangle\right)^{\dagger}\left(a\,|0\rangle + b\,|1\rangle\right)\\ &= \left(b\,\langle 0| - a\,\langle 1|\right)^{\dagger}\left(a\,|0\rangle + b\,|1\rangle\right)\\ &= ba\,\langle 0|0\rangle - a^2\,\langle 1|0\rangle + b^2\,\langle 0|1\rangle - ab\,\langle 1|1\rangle\\ &= ba - 0 + 0 - ab\\ &= 0\end{aligned}$$

Here we have used the fact that $\langle i|j\rangle = \delta_{ij}$.

Now, the probability of measuring the state $|\psi\rangle$ and getting $|v\rangle$ as a result is,

$$\begin{aligned}P_{\psi}(v) &= |\langle v|\psi\rangle|^2\\ &= \left|\left(a^*\,\langle 0| + b^*\,\langle 1|\right)\left(\alpha\,|0\rangle + \beta\,|1\rangle\right)\right|^2\\ &= |a^*\alpha + b^*\beta|^2\end{aligned}$$

Similarly,

$$\begin{aligned}P_{\psi}(v^{\perp}) &= \left|\left\langle v^{\perp}|\psi\right\rangle\right|^2\\ &= \left|\left(b\,\langle 0| - a\,\langle 1|\right)\left(\alpha\,|0\rangle + \beta\,|1\rangle\right)\right|^2\\ &= |b\alpha - a\beta|^2\end{aligned}$$