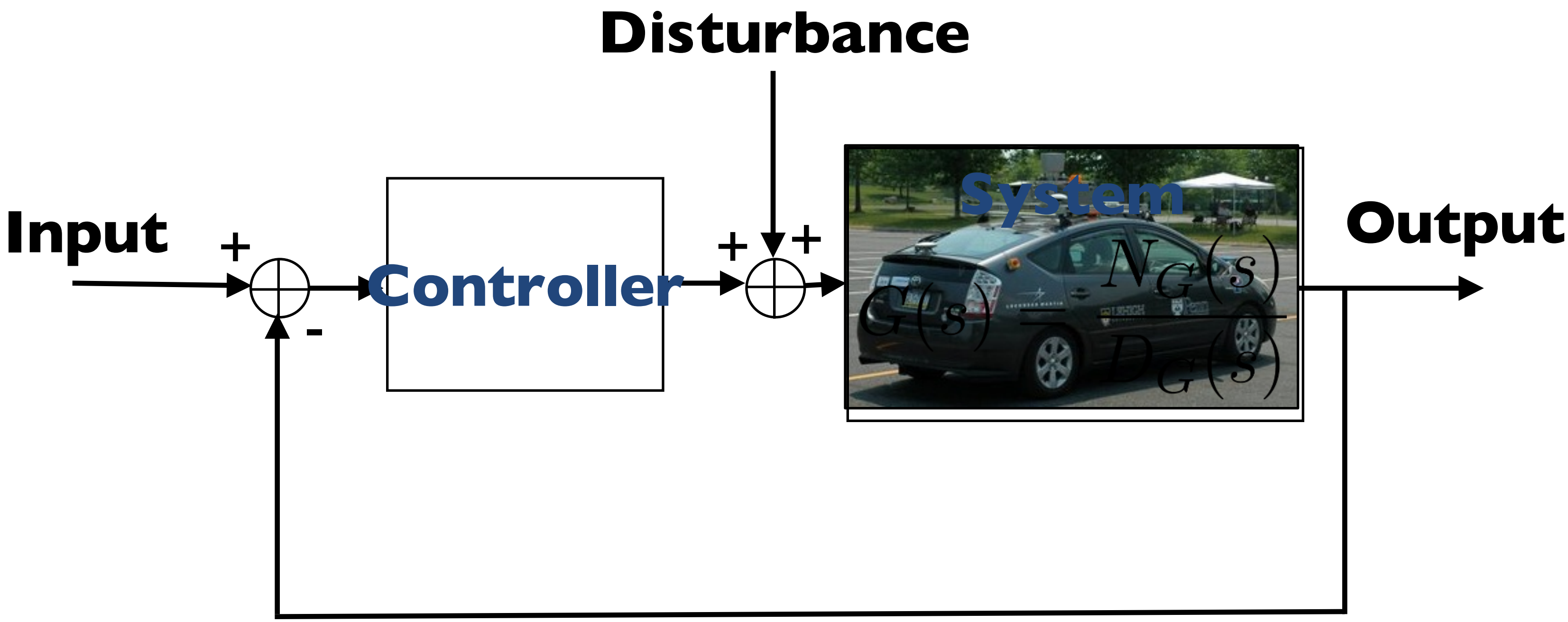
Penn Engineering
ONLINE LEARNING

Video 6.1

Vijay Kumar and Ani Hsieh

In General


Disturbance
Input
+
-
Controller
+
+
System
$G(s) = \frac{N_G(s)}{D_G(s)}$
Output

Learning Objectives for this Week

- State Space Notation
  - Modeling in the time domain
  - Solutions in the time domain
- From Frequency Domain to Time Domain and Back
- Design in the Time Domain
- Linearization

State-Space Representation

- Converts N-th order differential equation into N simultaneous FIRST-ORDER differential equations

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \ldots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \ldots + b_0 r(t)$$

- Allows for multiple inputs and/or outputs
- Versatility – our initial conditions DO NOT have to be 0

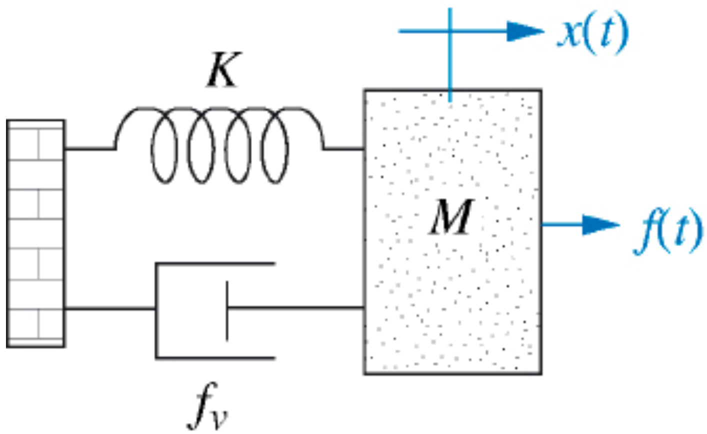
State Variables

Smallest set of *linearly independent* system variables s.t. state_variables(t_0) + known input (or forcing) functions *completely* determines the system.

# of state variables = Dimension of the State Space

# of state variables = order of the original diff eqn

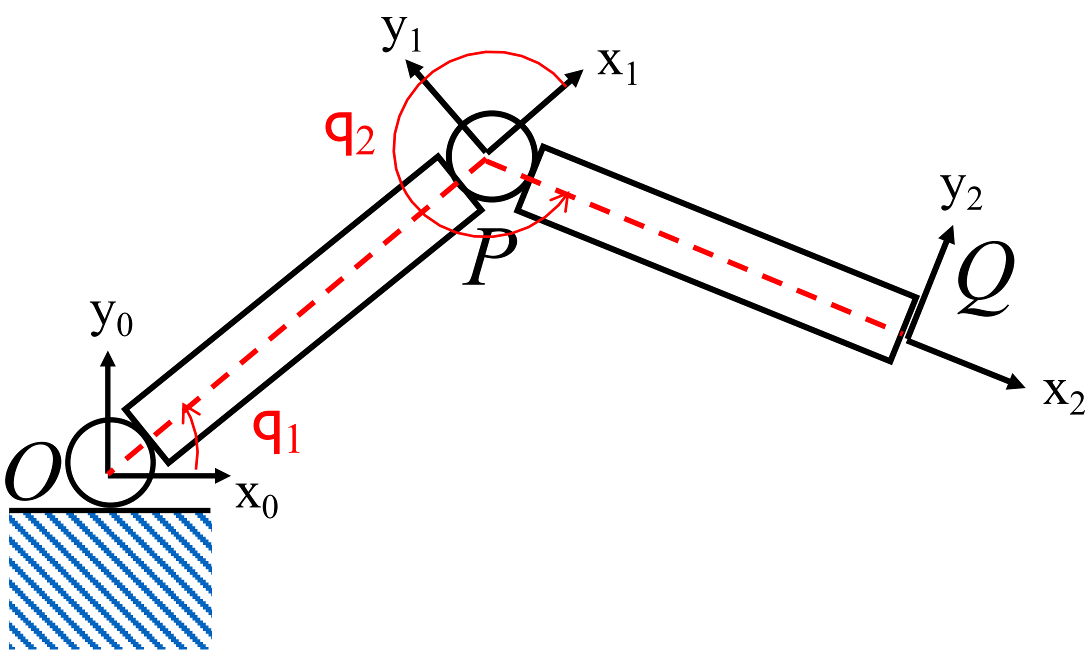
Example



$$f(t) = M\frac{d^2x}{dt^2} + f_v\frac{dx}{dt} + kx(t)$$

Another Example (I)

$$
\begin{aligned}
d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 + g_1 &= \tau_1 \\
d_{21}\ddot{q}_2 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + g_2 &= \tau_2
\end{aligned}
$$



Another Example (2)

$$\begin{aligned} d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 + g_1 &= \tau_1 \\ d_{21}\ddot{q}_2 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + g_2 &= \tau_2 \end{aligned}$$

Video 6.2

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Transfer Function → State Space (1)

Given $$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

Transfer Function → State Space (2)

Given $$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

Transfer Function → State Space (3)

Given $$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State Space → Transfer Function (1)

Given
$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

State Space → Transfer Function (2)

Given
$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Solutions in the Time Domain

Given
$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned} \qquad \mathbf{x}(0) = \mathbf{x}_0$$

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\
(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x}) &= \mathbf{B}\mathbf{u} \\
e^{-\mathbf{A}t}(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x}) &= e^{-\mathbf{A}t}\mathbf{B}\mathbf{u} \\
\frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{x}(t)\right) &= e^{-\mathbf{A}t}\mathbf{B}\mathbf{u} \\
\int_0^t \frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{x}(t)\right)dt &= \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau
\end{aligned}$$

Time Domain Solution (2)

$$e^{-\mathbf{A}t}\mathbf{x}\big|_0^t = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}_0 = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$e^{-\mathbf{A}t}\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Output of the System

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Natural Response — Particular Response

Thus, output of the system is given by

$$\mathbf{y}(t) = \mathbf{C}\left(e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\right) + \mathbf{D}\mathbf{u}$$

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The Matrix Exponential

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \dots$$

Properties of $e^{\mathbf{A}t}$

- $e^{\mathbf{A}t}\mathbf{A} = \mathbf{A}e^{\mathbf{A}t}$
- If $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{\Lambda}$ then $e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{\Lambda}t}\mathbf{P}^{-1}$

Characterizing System Response

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Since $e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{\Lambda}t}\mathbf{P}^{-1}$, for constant $\mathbf{A}$, then $\mathbf{\Lambda} = \mathbf{I}\lambda(\mathbf{A})$ w/ constant $\mathbf{P}$ and $\mathbf{P}^{-1}$

Then,

$$e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{bmatrix}$$

Steady-State Performance

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Since $e^{\mathbf{A}t} = \mathbf{P}\begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \ddots & 0 & \dots & 0 \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}\mathbf{P}^{-1}$

System is stable if and only if

$$Re(\lambda_i(\mathbf{A})) < 0 \quad \forall i = 1, \dots, n$$

Transient Performance

What about complex eigenvalues?

Recall $e^{j\theta} = \cos\theta + j\sin\theta e^{\lambda_i t}$, then

with $\lambda_i = \sigma_i + j\omega_i$

results in

$$e^{\lambda_i t} = e^{\sigma_i t} e^{j\omega_i t} = e^{\sigma_i t}\left(\cos(\omega_i t) + j\sin(\omega_i t)\right)$$

$$j\sin(\omega_i t)$$

Complex eigenvalues come in pairs, terms will cancel out.

Example (I)

Given
$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 2 \\ -2 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$$
w/ $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
$$\mathbf{y} = \begin{bmatrix} 2 & 1 \end{bmatrix} \mathbf{x}$$

Eigenvalues and eigenvectors of $\begin{bmatrix} 0 & 2 \\ -2 & -5 \end{bmatrix}$ are

$\lambda_{1,2} = -1, -4$ and $\mathbf{p}_{1,2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Example (2)

$$\int_0^t \Phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau = \left[\begin{array}{c} \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t} - e^{-2t} \\ \frac{-1}{3}e^{-t} - \frac{2}{3}e^{-4t} + e^{-2t} \end{array} \right]$$

Example (3)

$$\mathbf{x}(t) = e^{\mathbf{A}t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \int_0^t \Phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$= \begin{bmatrix} \frac{10}{3}e^{-t} + \frac{4}{3}e^{-4t} - e^{-2t} \\ \frac{-5}{3}e^{-t} + \frac{8}{3}e^{-4t} + e^{-2t} \end{bmatrix}$$

State Space Design

Given
$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \text{w/ } \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

Linear State Feedback Control Law

$$\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{r}$$

Closed-loop system
$$\begin{aligned}\dot{\mathbf{x}} &= (\mathbf{A} - \mathbf{B}\mathbf{K})\,\mathbf{x} \\ \mathbf{y} &= (\mathbf{C} - \mathbf{D}\mathbf{K})\,\mathbf{x}\end{aligned}$$

Choose **K** such that CL response is stable.

Advantages of State Space Design

Given
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \text{w/ } \mathbf{x}(0) = \mathbf{x}_0$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

with $\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{r}$

1. Not restricted to 2^{nd} order approximations
2. Access to a larger range of closed-loop poles
3. Allows for full state feedback

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A Caveat – Controllability

A linear system is *controllable* if for each $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(t_f) = \mathbf{x}_f$, there exists a $\mathbf{u}(t)$ that can get the system from $\mathbf{x}_0$ to $\mathbf{x}_f$ at time t_f .

Such a linear system is controllable *if and only if*

$$det\left(\begin{bmatrix} | & | & | & & | \\ \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \ldots & \mathbf{A}^{n-1}\mathbf{B} \\ | & | & | & & | \end{bmatrix}\right) \neq 0$$

Controllability Matrix

Feedback Law and Controllability

Let $\alpha(s) = s^n + \alpha_n s^{n-1} + \ldots + \alpha_2 s + \alpha_1$

s.t.

α_i are real coefficients. Then there exists

$\mathbf{u} = -\mathbf{K}\mathbf{x}$ such that $det(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}) = \alpha(s)$

if and only if $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ is *controllable*.

State feedback enables any controllable linear system to have arbitrary closed-loop poles!

Linear Quadratic (LQ) Control

Find optimal feedback control strategy that

$$\min_{\mathbf{u}(t),\mathbf{x}(t)} \quad J = \int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt$$

Cost Function

$$\text{subject to} \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

Constraints

- Constrained Optimization Problem
- Optimal Control

Optimal LQ Controller

$$\min_{\mathbf{u}(t),\mathbf{x}(t)} J = \int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt$$

$$\text{subject to} \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

w/ symmetric, positive definite matrix **Q** and **R**

$$\mathbf{u}_* = -\mathbf{K}_*\mathbf{x} \quad \text{w/}\, \mathbf{K}_* = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$$

where $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{R}^{-1}\mathbf{B}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = 0$

A Few More Words

The Algebraic Riccati Equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{PA} - \mathbf{R}^{-1}\mathbf{BB}^T\mathbf{P} + \mathbf{Q} = 0$$

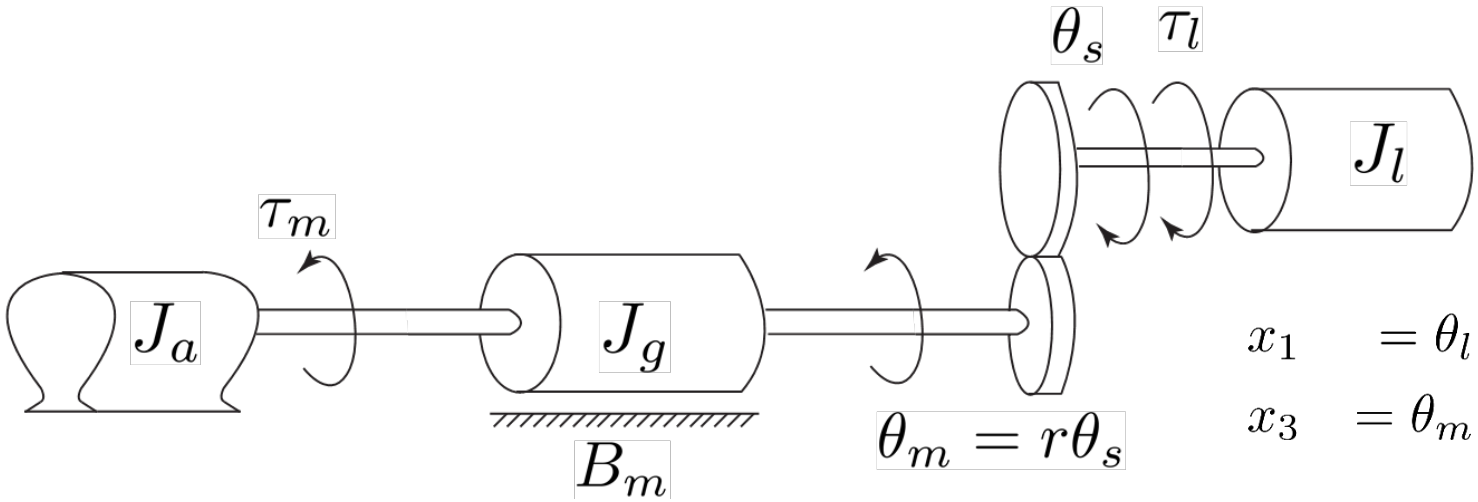
- **P** is n x n matrix
- **P** is unique
- **P** is symmetric and positive definite
- $\mathbf{u}_* = -\mathbf{K}_*\mathbf{x}$ w/ $\mathbf{K}_* = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$ is *optimal* w.r.t. the cost function J

Caveat for the Caveat – Observability

Recall: If $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ is controllable, then **K** can be chosen s.t. $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$ achieves arbitrary CL poles.

Assumption: $\mathbf{u} = -\mathbf{K}\mathbf{x}$

Example – Robot Joint Control (I)



$$x_1 = \theta_l \quad x_2 = \dot{\theta}_l$$
$$x_3 = \theta_m \quad x_4 = \dot{\theta}_m$$

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k}{J_l}x_1 - \frac{B_l}{J_l}x_2 + \frac{k}{J_l}x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{k}{J_m}x_1 - \frac{k}{J_m}x_3 + \frac{B_m}{J_m}x_4 + \frac{1}{J_m}u
\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_L} & -\frac{B_l}{J_L} & \frac{k}{J_L} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_m} & 0 & -\frac{k}{J_m} & \frac{B_m}{J_m} \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_m} \end{bmatrix}$$

Example – Robot Joint Control (2)

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_L} & -\frac{B_l}{J_L} & \frac{k}{J_L} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_m} & 0 & -\frac{k}{J_m} & \frac{B_m}{J_m} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_m} \end{bmatrix} \mathbf{u}$$

With $y = \mathbf{C}\ \mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = x_1$

Note:

1. Only θ_m is can be measured
2. $\mathbf{u} = -\mathbf{K}\mathbf{x}$ requires θ_l

Observers

- State estimators
- Use system model and measured output to estimate the full state

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_L} & -\frac{B_l}{J_L} & \frac{k}{J_L} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_m} & 0 & -\frac{k}{J_m} & \frac{B_m}{J_m} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_m} \end{bmatrix} \mathbf{u}$$

$$y = \mathbf{C}\ \mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = x_1$$

$$\Bigg\} \ \hat{\mathbf{x}}(t)$$

- Also a dynamical system $\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t)$

$$\mathbf{y}(t) = \mathbf{C}\hat{\mathbf{x}}(t) + \mathbf{D}\mathbf{u}(t)$$

Obtaining $\hat{\mathbf{x}}(t)$

Let $$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}})$$

Error between plant & estimate output

- **A**, **B**, **C** are known
- Solve for $\hat{\mathbf{x}}(t)$ from any initial condition
- Use $\hat{\mathbf{x}}(t)$ in feedback law $\mathbf{u} = -\mathbf{K}\hat{\mathbf{x}}$
- Pick **L** s.t. $\hat{\mathbf{x}} \rightarrow \mathbf{x}$ as $t \rightarrow \infty$

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Performance of the Estimator

Let $\mathbf{e}(t) = \mathbf{x} - \hat{\mathbf{x}}$ be the *estimation error*

Then, $\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{LC})\mathbf{e}$

1. Dynamics determined by $\lambda(\mathbf{A} - \mathbf{LC})$
2. Pick L s. t. $\mathbf{e} \to \mathbf{0}$ as $t \to \infty$

Eigenvalues of $(\mathbf{A} - \mathbf{LC})$ can be arbitrarily set if and only if system is *observable*.

Observability

A linear system is *observable* if every $\mathbf{x}(0) = \mathbf{x}_0$ can be exactly determined from $\mathbf{y}(t)$ and $\mathbf{u}(t)$ in a finite time interval $t_0 \leq t \leq t_f$.

The pair $(\mathbf{A}, \mathbf{C})$ is observable *if and only if*

$$det(\begin{bmatrix} \mathbf{C} \\ \mathbf{A}^T\mathbf{C} \\ \mathbf{A}^{T^2}\mathbf{C} \\ \vdots \\ \mathbf{A}^{T^{n-1}}\mathbf{C} \end{bmatrix}) \neq 0$$

Observability Matrix

Example – Robot Joint Control (3)

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_L} & -\frac{B_l}{J_L} & \frac{k}{J_L} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_m} & 0 & -\frac{k}{J_m} & \frac{B_m}{J_m} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_m} \end{bmatrix} \mathbf{u}$$

$$y = \mathbf{C}^T \mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = x_1$$

With $\mathbf{u} = -\mathbf{K}\hat{\mathbf{x}}$

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

Separation Principle: Allows us to separately design the feedback control and the state estimator

For nonlinear systems

In general, given

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

where **f** is a nonlinear function in **x**, possibly **u**

What if …

1. We want to analyze system behavior around $\mathbf{x} = \mathbf{x}_e$?
2. We want to control system behavior around $\mathbf{x} = \mathbf{x}_e$?

Linearization

Given $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ and $\mathbf{x} = \mathbf{x}_e$,$\mathbf{u} = \mathbf{u}_e$

- Let $\bar{\mathbf{x}} = \mathbf{x} - \mathbf{x}_e$ and $\bar{\mathbf{u}} = \mathbf{u} - \mathbf{u}_e$, substitute into **f(x,u)**

$$\dot{\bar{\mathbf{x}}} = \mathbf{f}(\bar{\mathbf{x}}, \bar{\mathbf{u}})$$

- Apply Taylor series expansion about $(\mathbf{x}_e, \mathbf{u}_e)$

$$\dot{\bar{\mathbf{x}}} \approx \mathbf{f}(\mathbf{x}_e, \mathbf{u}_e) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}\bar{\mathbf{x}} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}\bar{\mathbf{u}}$$

Then $A \triangleq \frac{\partial \mathbf{f}}{\partial \mathbf{x}}|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}$ $B \triangleq \frac{\partial \mathbf{f}}{\partial \mathbf{u}}|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}$

Linearization

$$\dot{\bar{\mathbf{x}}} \approx \mathbf{f}(\mathbf{x}_e, \mathbf{u}_e) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\Big|_{\bar{\mathbf{x}},\bar{\mathbf{u}}} \bar{\mathbf{x}} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\Big|_{\bar{\mathbf{x}},\bar{\mathbf{u}}} \bar{\mathbf{u}}$$

Let $A \triangleq \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\Big|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}$ and $B \triangleq \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\Big|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}$

$$\dot{\bar{\mathbf{x}}} = \mathbf{A}\bar{\mathbf{x}} + \mathbf{B}\bar{\mathbf{u}}$$

Proceed …