ITMO UNIVERSITY

How to Win Coding Competitions: Secrets of Champions

Week 4: Algorithms on Graphs
Lecture 6: Eulerian paths and Eulerian tours

Maxim Buzdalov
Saint Petersburg 2016

An Eulerian path is a path in a graph that contains each edge of the graph exactly once

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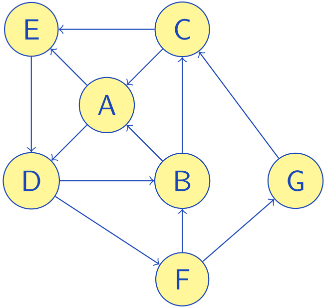

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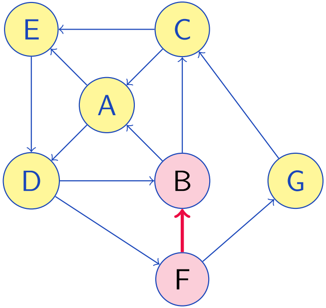

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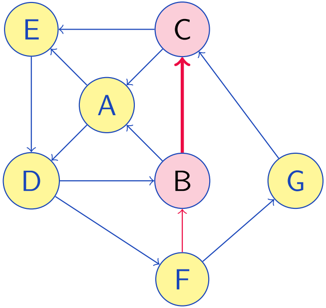

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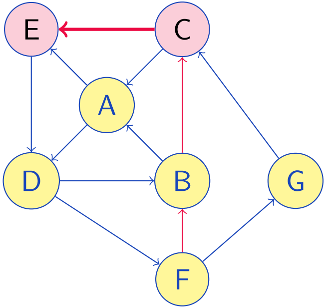

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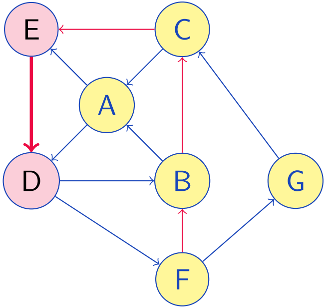

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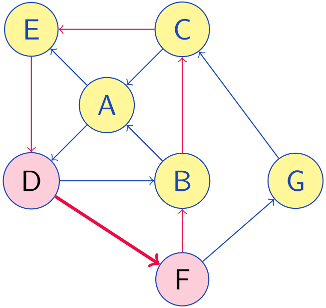

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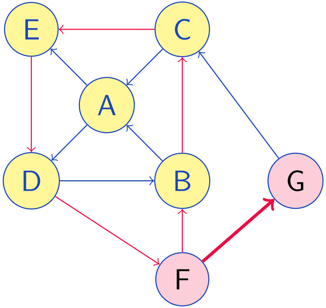

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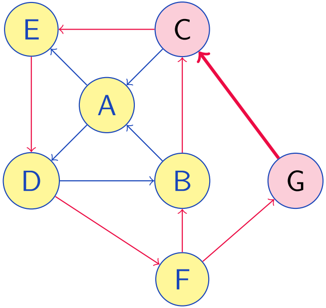

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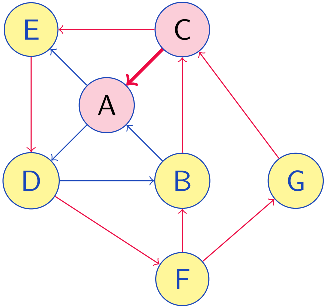

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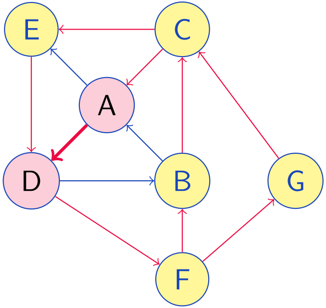

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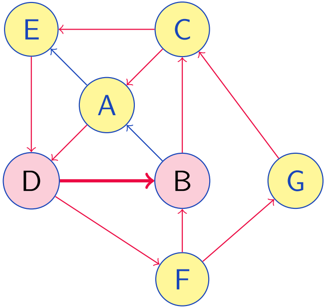

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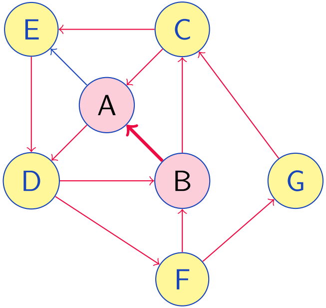

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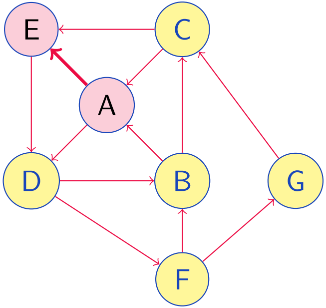

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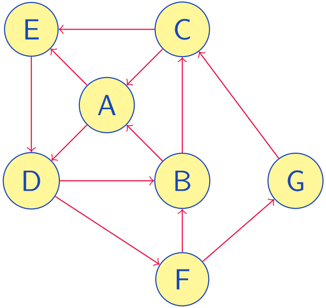

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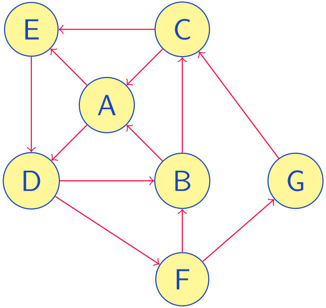

An Eulerian tour is a Eulerian path which starts and ends on the same vertex

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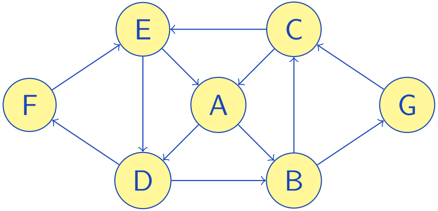

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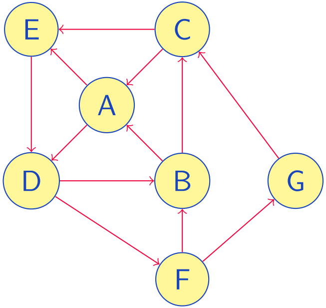

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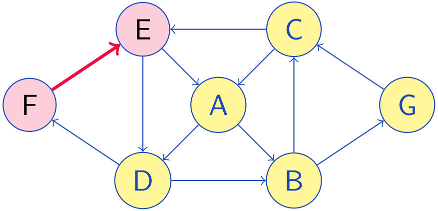

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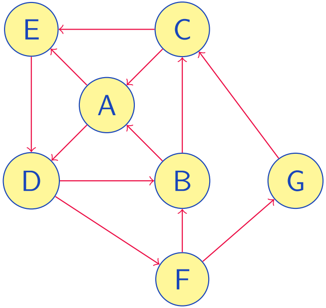

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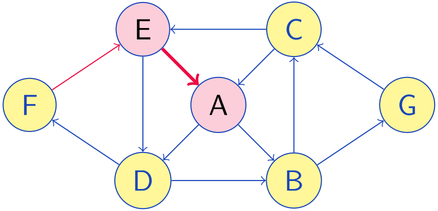

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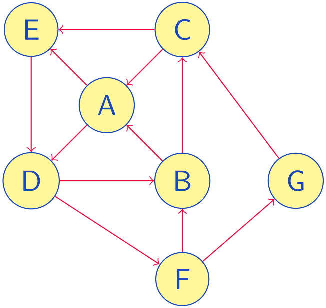

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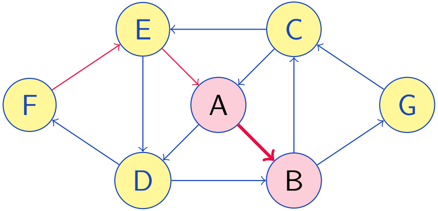

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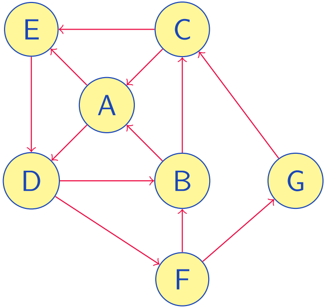

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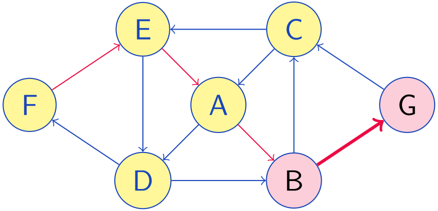

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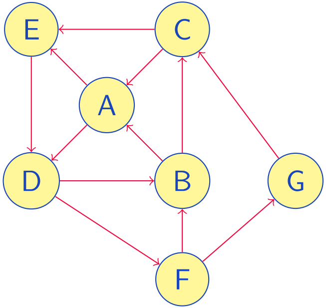

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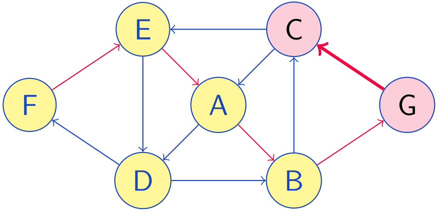

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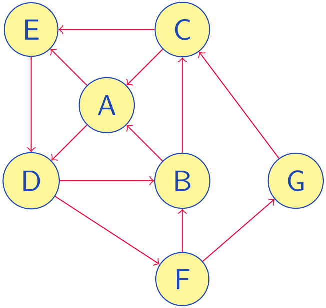

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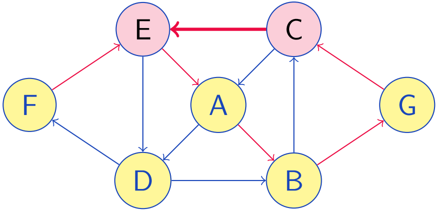

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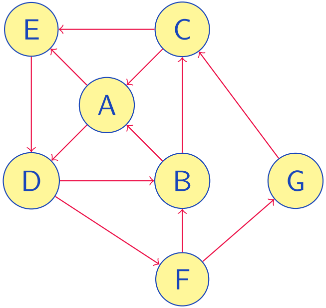

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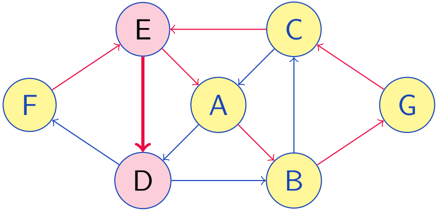

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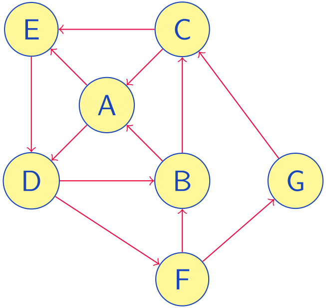

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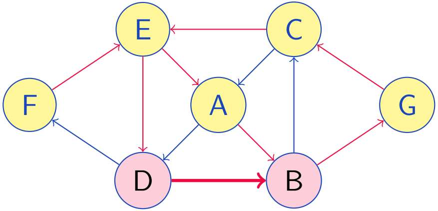

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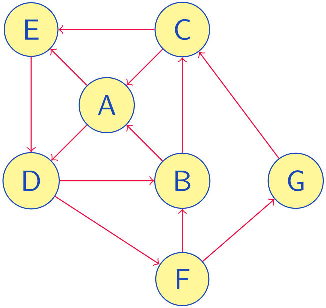

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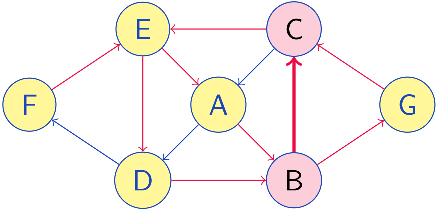

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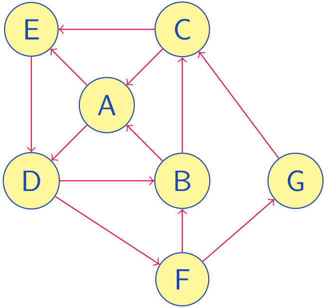

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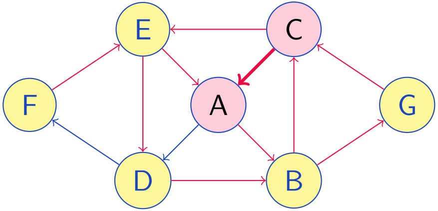

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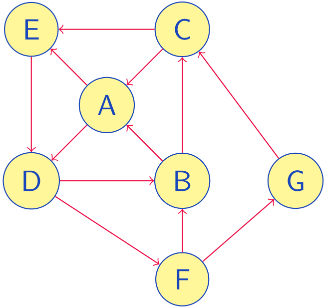

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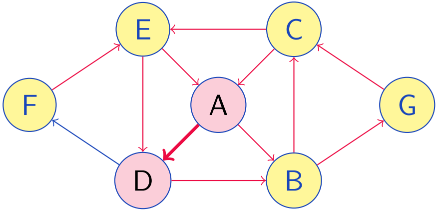

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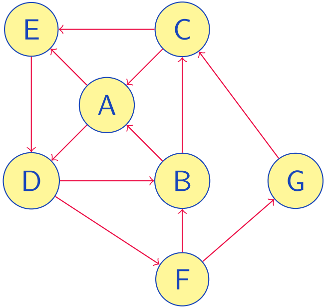

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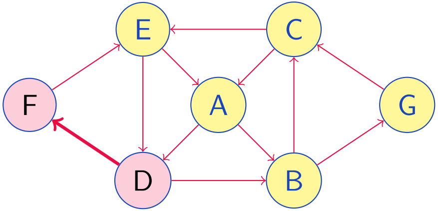

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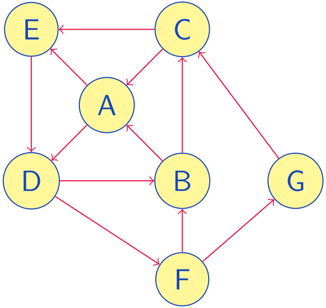

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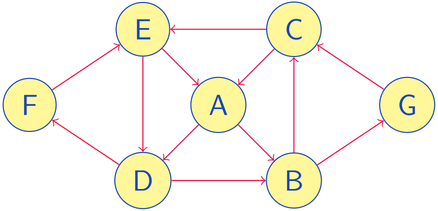

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- That's what Depth First Search can do!

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$A(v) = \{u \mid (v, u) \in E\}$
$R \leftarrow [\,]$
procedure $\mathrm{DFS}(v)$
    for $u \in A(v)$ **do**
        Remove u from $A(v)$
        if graph is undirected **then**
            Remove v from $A(u)$
        end if
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    end for
    $R \leftarrow [v] + R$
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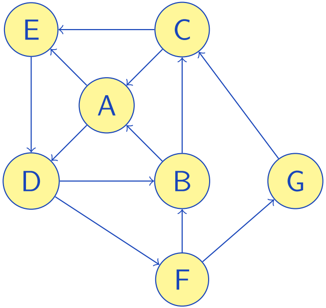
    $R \leftarrow [v] + R$ ▷ Prepend v to the answer
end procedure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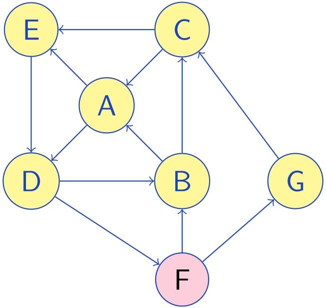
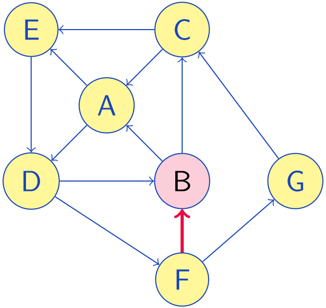
Example: DFS finds Eulerian path

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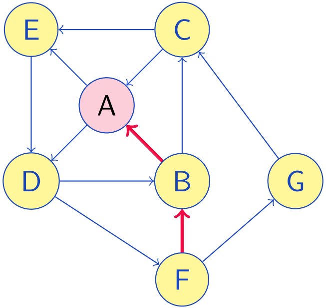
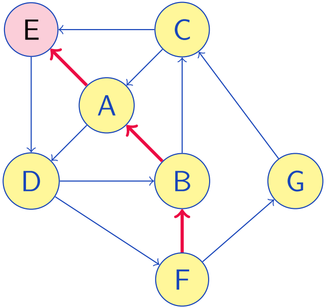
Example: DFS finds Eulerian path

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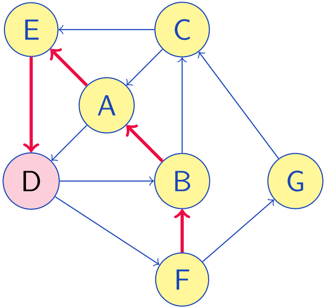
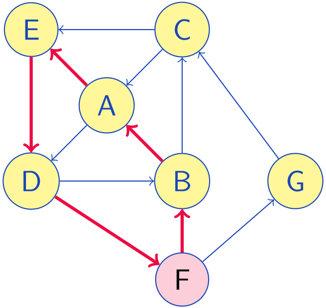
Example: DFS finds Eulerian path

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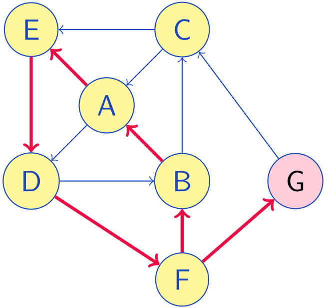
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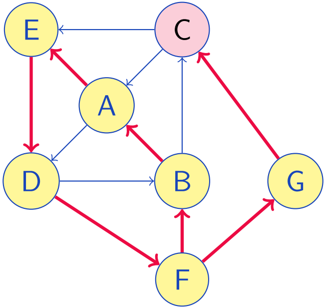
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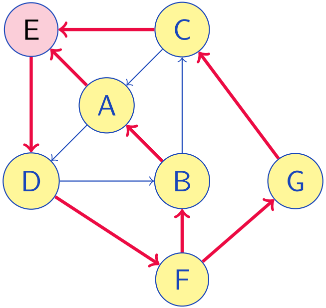
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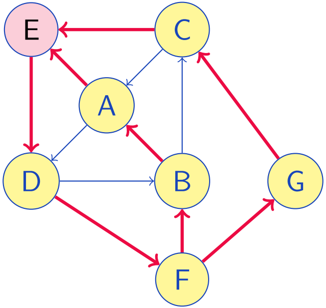

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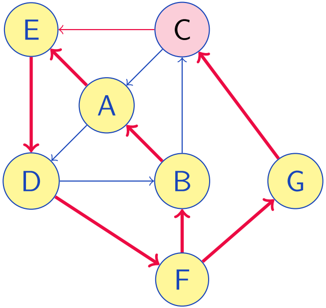

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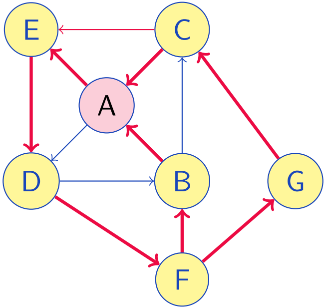

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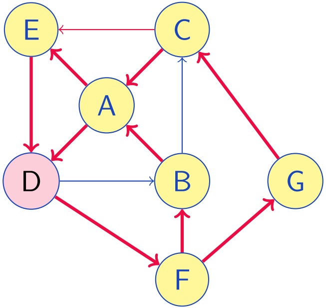

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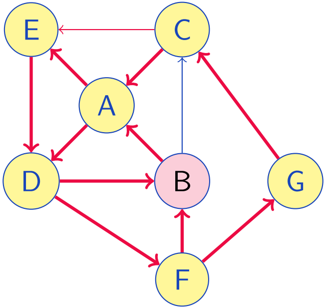

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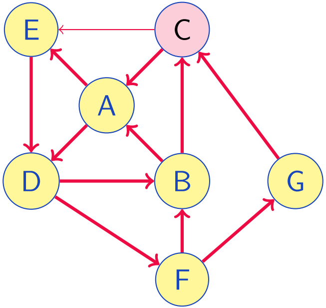

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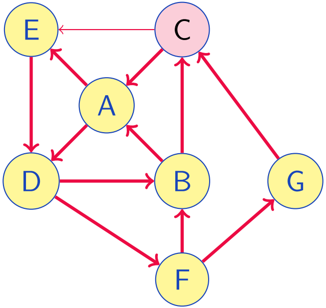

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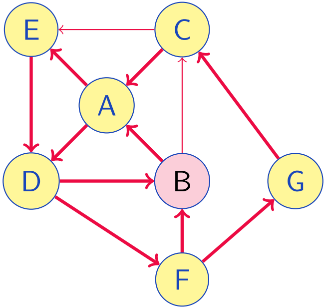

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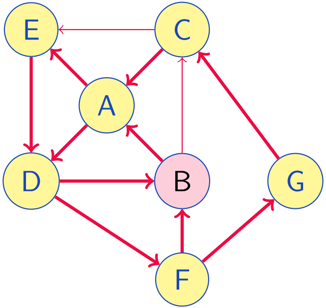

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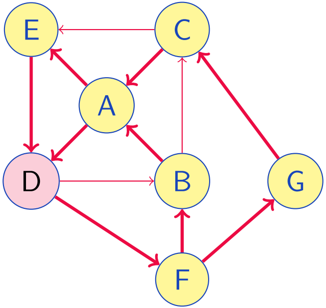

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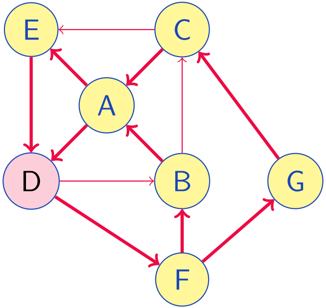

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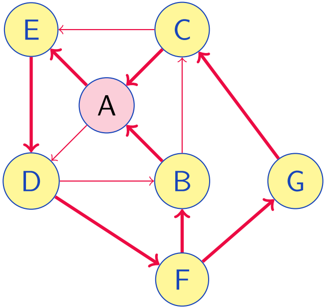

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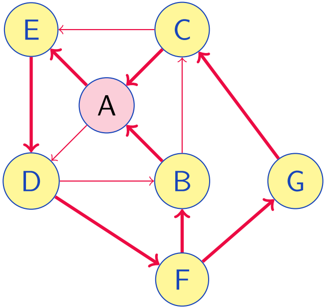

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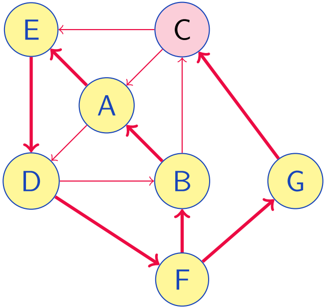

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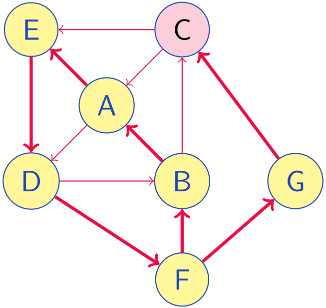

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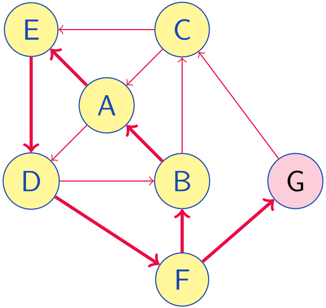

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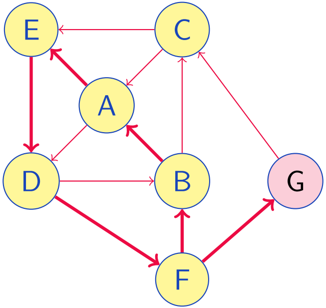

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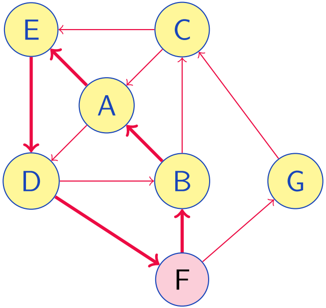

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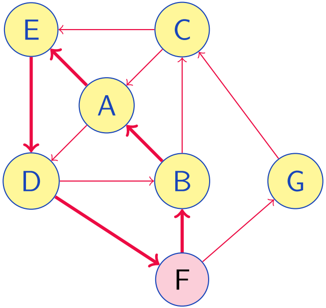

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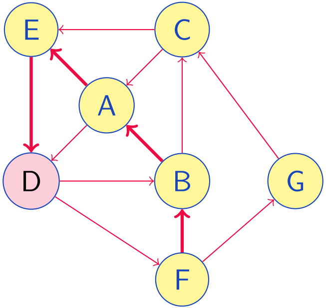

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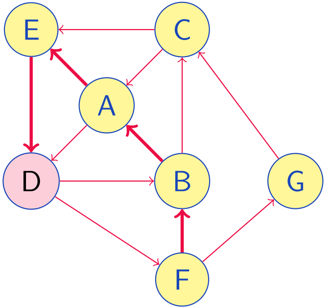

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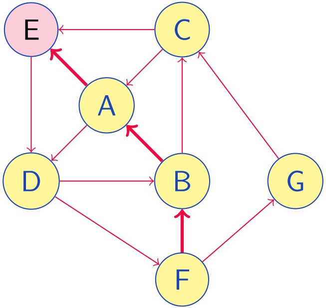

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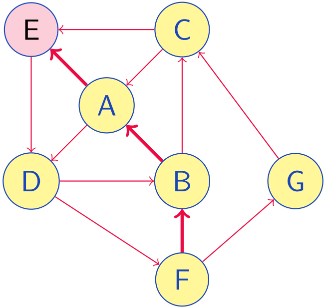

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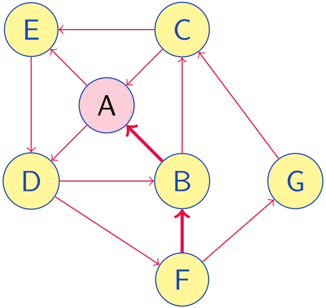

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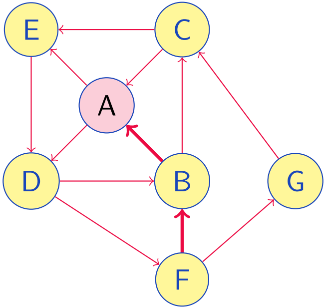

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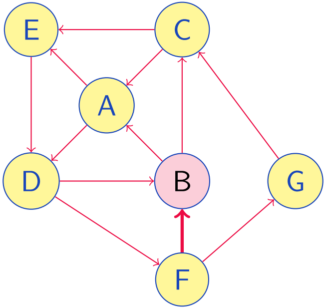

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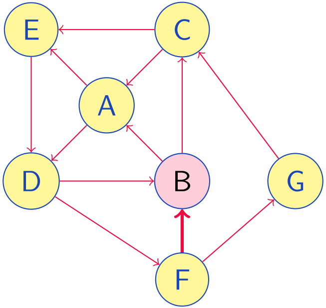

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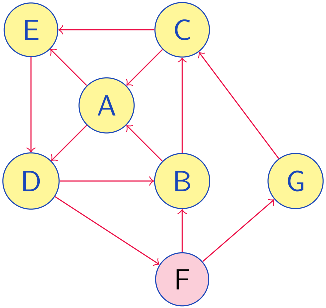

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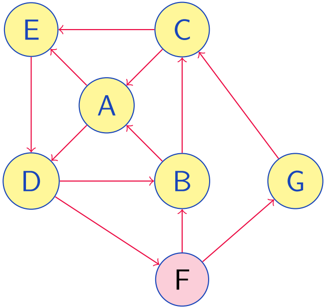

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