Foundations of Computer Graphics

Online Lecture 3: Transformations 1

Basic 2D Transforms

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Motivation

- Many different coordinate systems in graphics
  - World, model, body, arms, …
- To relate them, we must transform between them
- Also, for modeling objects. I have a teapot, but
  - Want to place it at correct location in the world
  - Want to view it from different angles (HW 1)
  - Want to scale it to make it bigger or smaller
- Demo of HW 1

Goals

- This unit is about the math for these transformations
  - Represent transformations using matrices and matrix-vector multiplications.
- Demos throughout lecture: HW 1 and Applet
- Transformations Game Applet
  - Brown University Exploratories of Software
  - http://www.cs.brown.edu/exploratories/home.html
  - Credit: Andries Van Dam and Jean Laleuf

General Idea

- Object in model coordinates
- Transform into world coordinates
- Represent points on object as vectors
- Multiply by matrices
- Demos with applet

Outline

- *2D transformations: rotation, scale, shear*
- Composing transforms
- 3D rotations
- Translation: Homogeneous Coordinates (next time)
- Transforming Normals (next time)

(Nonuniform) Scale

$$Scale(s_x, s_y) = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \qquad S^{-1} = \begin{pmatrix} s_x^{-1} & 0 \\ 0 & s_y^{-1} \end{pmatrix}$$

$$\begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s_x x \\ s_y y \\ s_z z \end{pmatrix}$$

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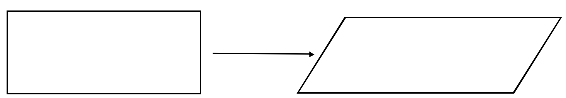
Shear

$$Shear = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \qquad S^{-1} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$$



Rotations

2D simple, 3D complicated. [Derivation? Examples?]

2D?

R(X+Y)=R(X)+R(Y)

- Linear
- Commutative

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2D Rotations

Rotations

2D simple, 3D complicated. [Derivation? Examples?]

2D?

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

R(X+Y)=R(X)+R(Y)

- Linear
- Commutative

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Composing Transforms

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Outline

- 2D transformations: rotation, scale, shear
- *Composing transforms*
- 3D rotations
- Translation: Homogeneous Coordinates
- Transforming Normals

Composing Transforms

- Often want to combine transforms
- E.g. first scale by 2, then rotate by 45 degrees
- Advantage of matrix formulation: All still a matrix
- Not commutative!! Order matters

E.g. Composing rotations, scales

$$x_3 = Rx_2 \qquad x_2 = Sx_1$$
$$x_3 = R(Sx_1) = (RS)x_1$$
$$x_3 \neq SRx_1$$

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Inverting Composite Transforms

- Say I want to invert a combination of 3 transforms
- Option 1: Find composite matrix, invert
- Option 2: Invert each transform ***and swap order***
- Obvious from properties of matrices, demo

$$M = M_1 M_2 M_3$$
$$M^{-1} = M_3^{-1} M_2^{-1} M_1^{-1}$$
$$M^{-1}M = M_3^{-1}(M_2^{-1}(M_1^{-1}M_1)M_2)M_3$$

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3D Rotations

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Outline

- 2D transformations: rotation, scale, shear
- Composing transforms
- *3D rotations*
- Translation: Homogeneous Coordinates
- Transforming Normals

Rotations

Review of 2D case

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Orthogonal?, $R^T R = I$

Rotations in 3D

Rotations about coordinate axes simple

$$R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Always linear, orthogonal

- Rows/cols orthonormal

$$R^T R = I$$

R(X+Y)=R(X)+R(Y)

Geometric Interpretation 3D Rotations

- Rows of matrix are 3 unit vectors of new coord frame
- Can construct rotation matrix from 3 orthonormal vectors

$$R_{uvw} = \begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{pmatrix} \quad u = x_u X + y_u Y + z_u Z$$

$$Rp = \begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{pmatrix} \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} = ?$$

Geometric Interpretation 3D Rotations

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Geometric Interpretation 3D Rotations

$$Rp = \begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{pmatrix} \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} = \begin{pmatrix} u \bullet p \\ v \bullet p \\ w \bullet p \end{pmatrix}$$

- Rows of matrix are 3 unit vectors of new coord frame
- Can construct rotation matrix from 3 orthonormal vectors
- Effectively, projections of point into new coord frame
- New coord frame uvw taken to cartesian components xyz
- Inverse or transpose takes xyz cartesian to uvw

Non-Commutativity

- Not Commutative (unlike in 2D)!!
- Rotate by x, then y is not same as y then x
- Order of applying rotations does matter
- Follows from matrix multiplication not commutative
  - R1 * R2 is not the same as R2 * R1
- Demo: HW1, order of right or up will matter

Arbitrary rotation formula

- Rotate by an angle θ about arbitrary axis **a**
  - Homework 1: must rotate eye, up direction
  - Somewhat mathematical derivation but useful formula
- Problem setup: Rotate vector **b** by θ about **a**
- Helpful to relate **b** to X, **a** to Z, verify does right thing
- For HW1, you probably just need final formula

Axis-Angle formula

- Step 1: **b** has components parallel to **a**, perpendicular
  - Parallel component unchanged (rotating about an axis leaves that axis unchanged after rotation, e.g. rot about z)

Axis-Angle formula

- Step 2: Define **c** orthogonal to both **a** and **b**
  - Analogous to defining Y axis
  - Use cross products and matrix formula for that

Axis-Angle formula

- Step 3: With respect to the perpendicular comp of **b**
  - Cos θ of it remains unchanged
  - Sin θ of it projects onto vector **c**

Axis-Angle: Putting it together

$$(b \setminus a)_{ROT} = (I_{3\times 3}\cos\theta - aa^T\cos\theta)b + (A^*\sin\theta)b$$

$$(b \rightarrow a)_{ROT} = (aa^T)b$$

$$R(a,\theta) = I_{3\times 3}\cos\theta + aa^T(1-\cos\theta) + A^*\sin\theta$$

- $I_{3\times 3}\cos\theta$: Unchanged (cosine)
- $aa^T(1-\cos\theta)$: Component along a (hence unchanged)
- $A^*\sin\theta$: Perpendicular (rotated comp)

Axis-Angle: Putting it together

$$(b \setminus a)_{ROT} = (I_{3\times 3}\cos\theta - aa^T\cos\theta)b + (A^*\sin\theta)b$$

$$(b \rightarrow a)_{ROT} = (aa^T)b$$

$$R(a,\theta) = I_{3\times 3}\cos\theta + aa^T(1-\cos\theta) + A^*\sin\theta$$

$$R(a,\theta) = \cos\theta\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (1-\cos\theta)\begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix} + \sin\theta\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

(x y z) are cartesian components of a