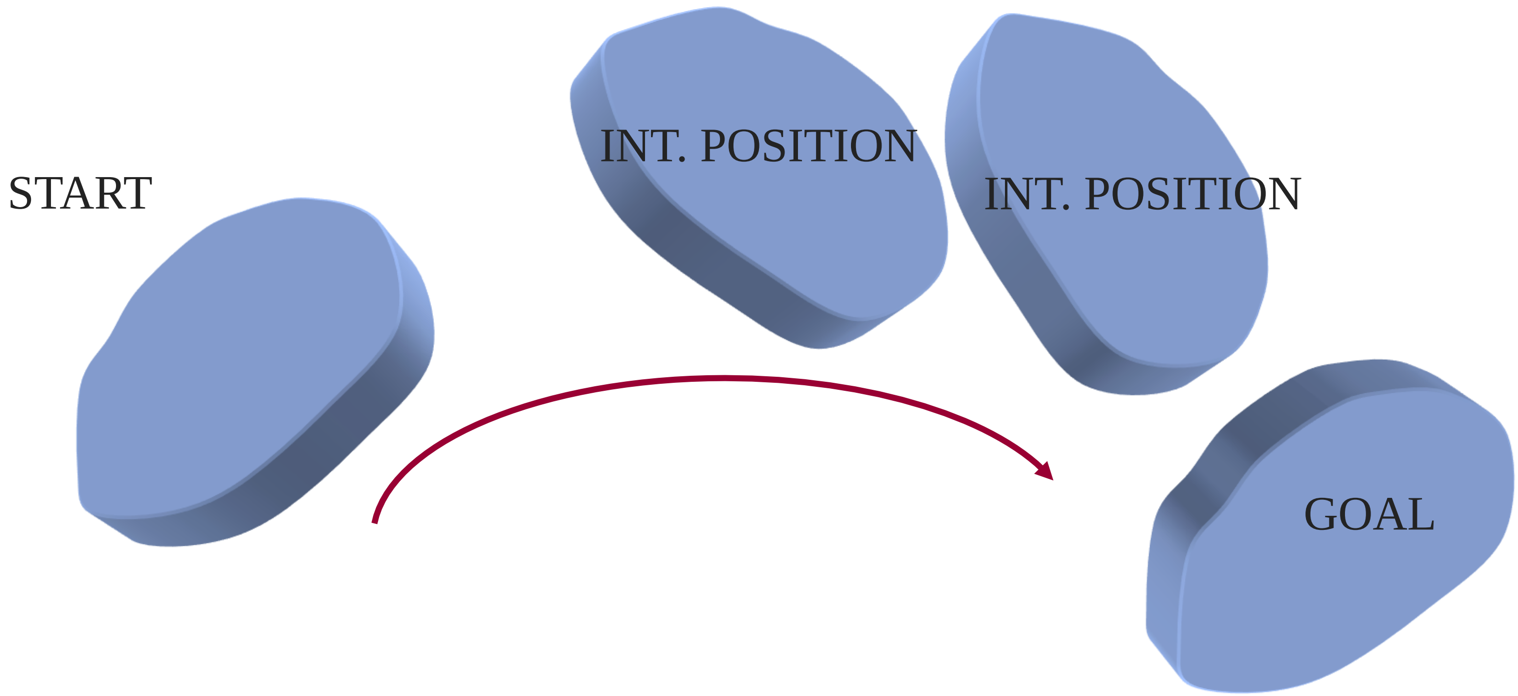
Video 11.1

Vijay Kumar

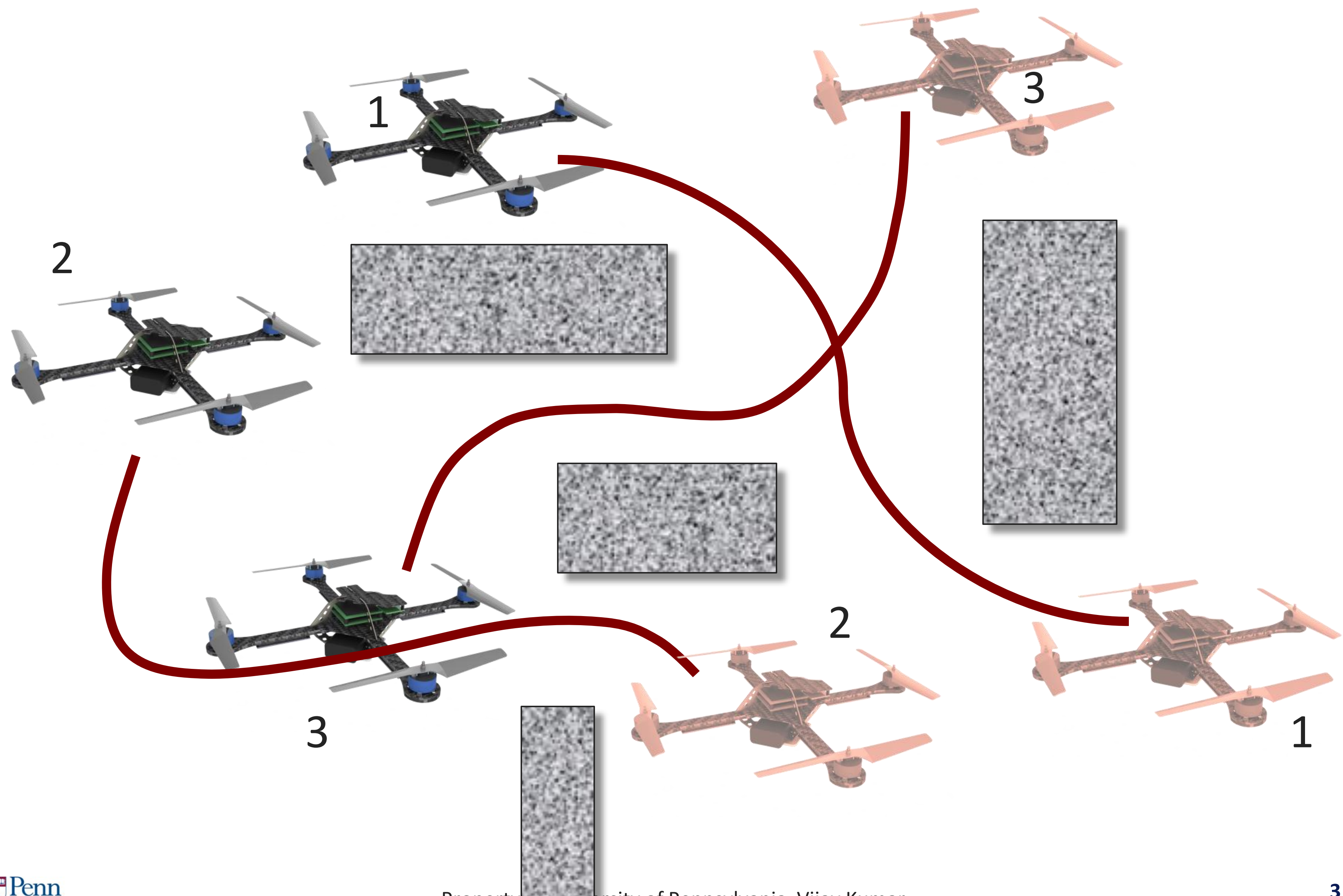
Smooth three dimensional trajectories



Applications

- Trajectory generation in robotics
- Planning trajectories for quad rotors

Motion Planning of Quadrotors



General Set up

- Start, goal positions (orientations)
- Waypoint positions (orientations)
- Smoothness criterion
  - Generally translates to minimizing rate of change of "input"
- Order of the system (n)
  - The input is algebraically related to the *nth* derivative of position (orientation)
  - Boundary conditions on $(n\text{-}1)^{\text{th}}$ order and lower derivatives

Calculus of Variations

running cost

$$x^{\star}(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \boxed{\mathcal{L}(\dot{x}, x, t)}\, dt$$

cost functional

function

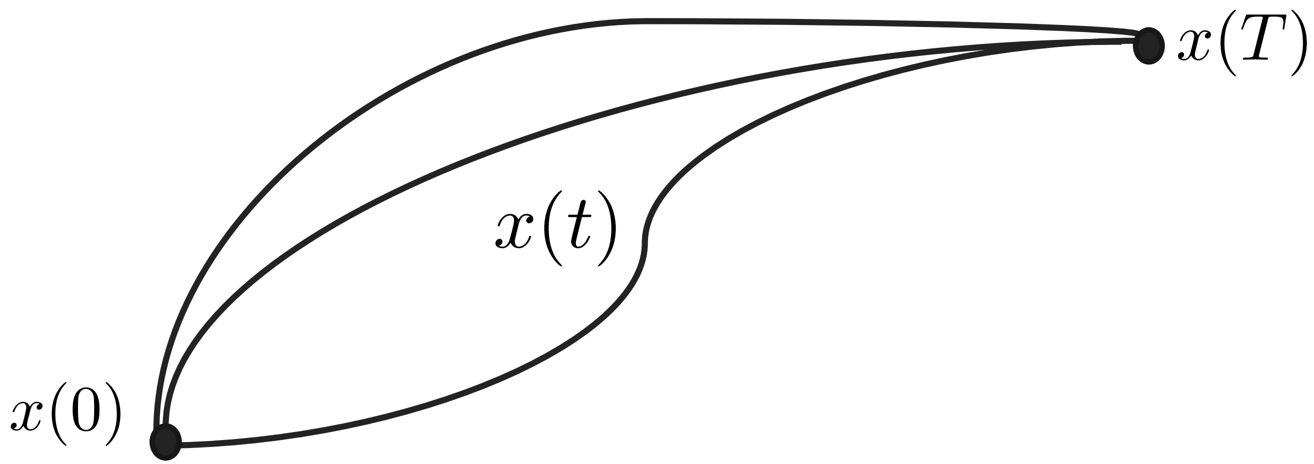
Examples

- Shortest distance path (geometry) $x^{\star}(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \dot{x} dt$
- Fermat's principle (optics) $x^{\star}(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T 1 dt$
- Principle of least action (mechanics) $x^{\star}(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T T(\dot{x}, x, t) - V(\dot{x}, x, t)\, dt$

Calculus of Variations

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \mathcal{L}\left(\dot{x}, x, t\right) dt$$

Consider the set of all differentiable curves, $x(t)$, with a given $x(0)$ and $x(T)$.



Calculus of Variations

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \mathcal{L}\left(\dot{x}, x, t\right) dt$$

Euler Lagrange Equation

Necessary condition satisfied by the "optimal" function *x*(*t*)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

Courant, R and Hilbert, D. Methods of Mathematical Physics. Vol. I. Interscience Publishers, New York, 1953.

Cornelius Lanczos, The Variational Principles of Mechanics, Dover Publications, 1970

Smooth trajectories (n=1)

$$x^\star(t) = \arg\min_{x(t)} \int_0^T \dot{x}^2 \, dt$$

input

$$u = \dot{x}$$

$$x(0) = x_0, \; x(T) = x_T$$

Smooth trajectories (n=1)

$$x^{\star}(t) = \arg\min_{x(t)} \int_0^T \dot{x}^2 dt$$

Euler Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

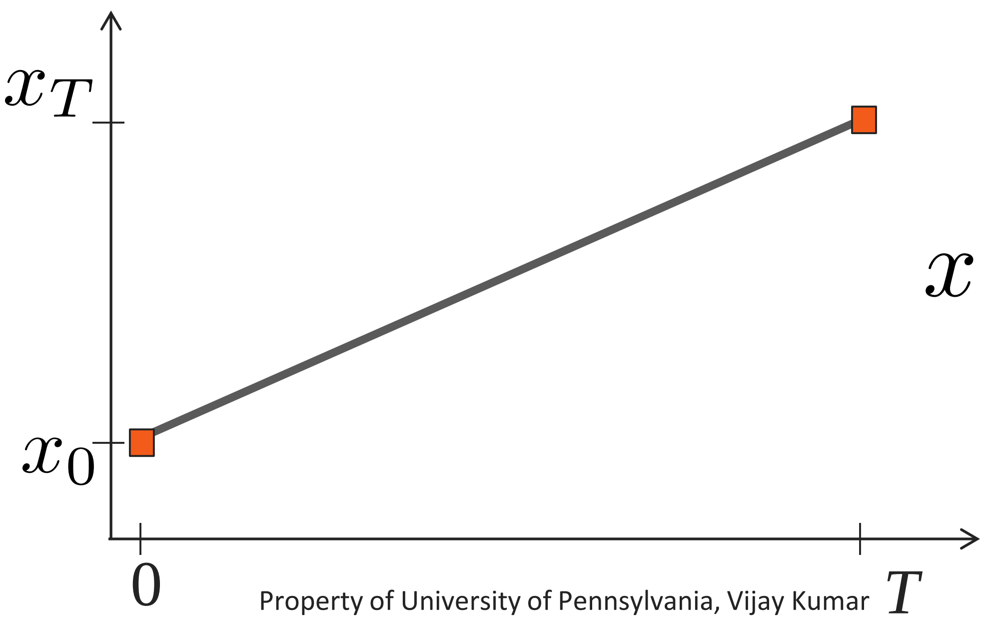
$$\mathcal{L}\left(\dot{x}, x, t\right) = \left(\dot{x}\right)^2 \quad \Rightarrow \quad \ddot{x} = 0$$

$$x = c_1 t + c_0$$

Smooth trajectories (n=1)

$$x^\star(t) = \arg\min_{x(t)} \int_0^T \dot{x}^2 dt$$

$$x(0) = x_0, \; x(T) = x_T$$



$$x = c_1 t + c_0$$

Smooth trajectories (general n)

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \left(x^{(n)}\right)^2 dt$$

input

$$u = x^{(n)}$$

Euler-Lagrange Equation

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \mathcal{L}\left(x^{(n)}, x^{(n-1)}, \ldots, \dot{x}, x, t\right) dt$$

Euler Lagrange Equation

Necessary condition satisfied by the "optimal" function

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial \ddot{x}}\right) + \ldots + (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial \mathcal{L}}{\partial x^{(n)}}\right) = 0$$

Smooth Trajectories

$$x^{\star}(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \left(x^{(n)}\right)^2 dt$$

- *n*=1, shortest distance $\boxed{\text{velocity}}$
- *n*=2, minimum $\boxed{\text{acceleration}}$
- *n*=3, minimum $\boxed{\text{jerk}}$
- *n*=4, minimum $\boxed{\text{snap}}$

n – order of system
nth derivative is an algebraic function of input

Smooth Trajectories

$$x^{\star}(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \left(x^{(n)}\right)^2 dt$$

- n=1, shortest distance $\boxed{\text{velocity}}$
- n=2, minimum $\boxed{\text{acceleration}}$
- n=3, minimum $\boxed{\text{jerk}}$
- n=4, minimum $\boxed{\text{snap}}$

Why is the minimum velocity curve also the shortest distance curve?

Video 11.2
Vijay Kumar

Smooth Trajectories

$$x^{\star}(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \left(x^{(n)}\right)^2 dt$$

- n=1, shortest distance
- n=2, minimum acceleration
- n=3, minimum jerk
- n=4, minimum snap

Minimum Jerk Trajectory

Design a trajectory *x*(*t*) such that *x*(0) = *a*, *x*(*T*) = *b*

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \mathcal{L}\left(\dddot{x}, \ddot{x}, \dot{x}, x, t\right) dt$$

$$\mathcal{L} = \left(\dddot{x}\right)^2$$

Euler-Lagrange:

$$\cancelto{0}{\boxed{\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial \ddot{x}}\right)}}\boxed{-\frac{d^3}{dt^3}\left(\frac{\partial \mathcal{L}}{\partial x^{(3)}}\right)} = 0$$

$$x^{(6)} = 0$$

$$x = c_5 t^5 + c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0$$

Solving for Coefficients

$$x = c_5 t^5 + c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0$$

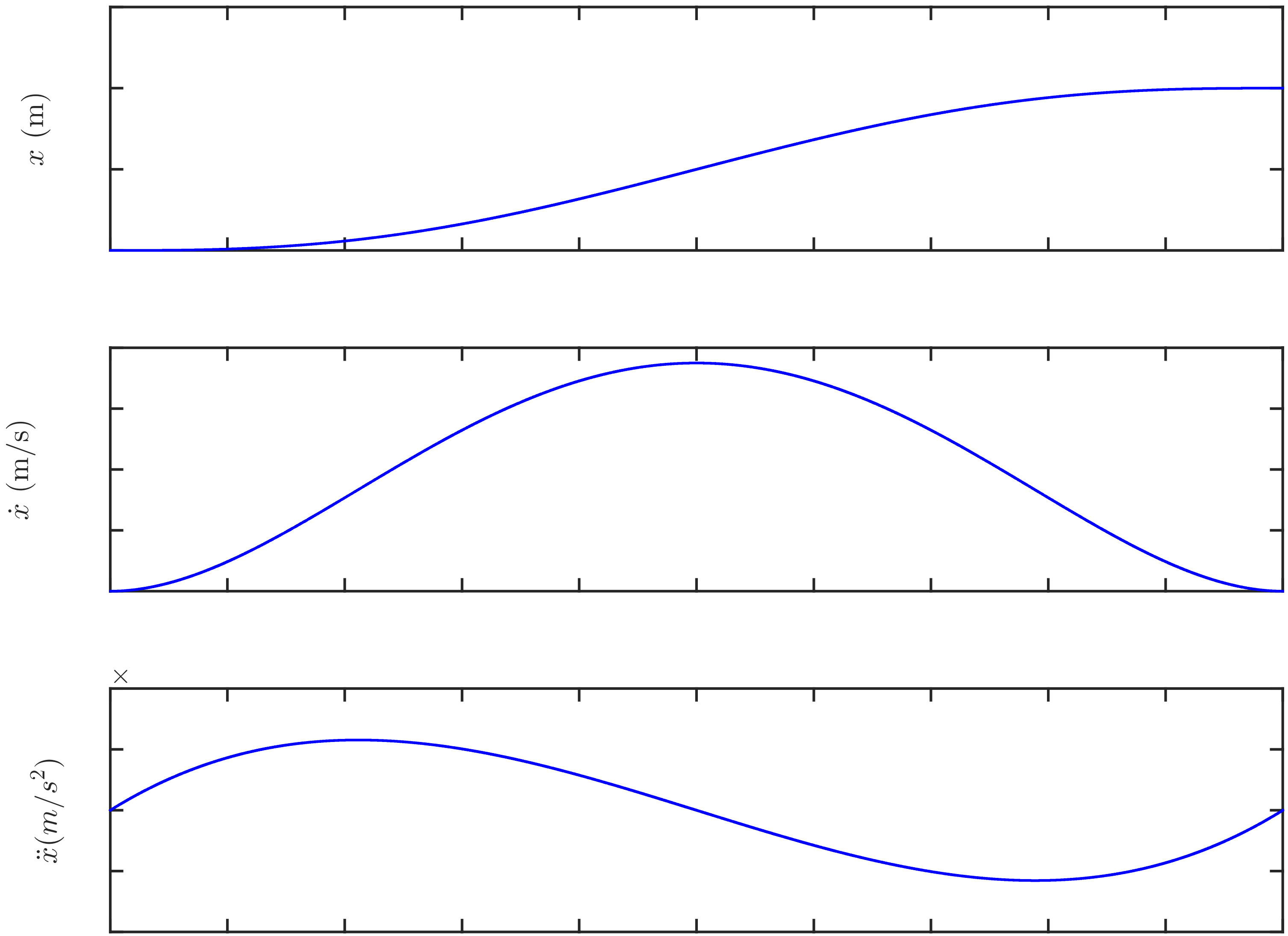
Boundary conditions:

	Position	Velocity	Acceleration
$t = 0$	a	0	0
$t = T$	b	0	0

Solve:

$$\begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ T^5 & T^4 & T^3 & T^2 & T & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 5T^4 & 4T^3 & 3T^2 & 2T & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 20T^3 & 12T^2 & 6T & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix}$$

Minimum Jerk Trajectory



a=0, b=1, T=50

Extensions to multiple dimensions (first order system, n=1)

$$(x^\star(t), y^\star(t)) = \arg\min_{x(t),y(t)} \int_0^T \mathcal{L}(\dot{x}, \dot{y}, x, y, t)\, dt$$

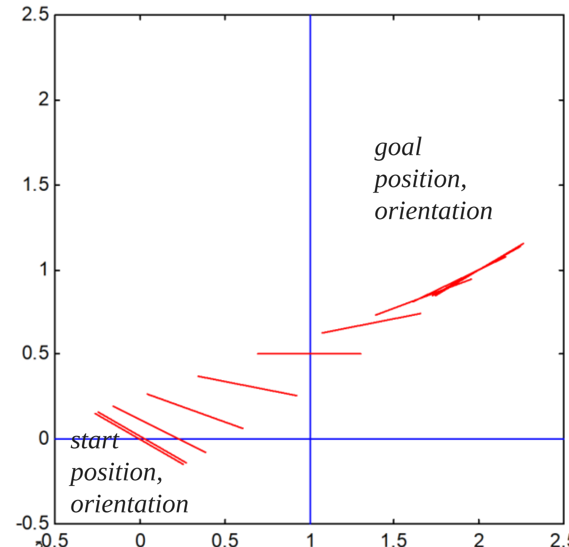
Euler Lagrange Equation

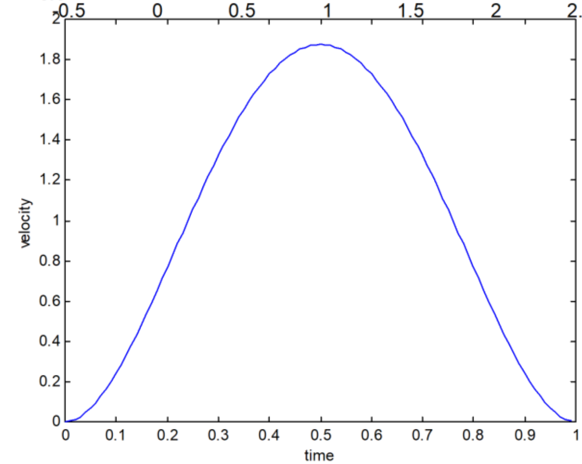
Necessary condition satisfied by the "optimal" function

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} = 0$$

Minimum Jerk for Planar Motions

$$\mathbf{x}^\star(t) : [0,1] \rightarrow SE(2)$$





Minimum-jerk trajectory in (*x*, *y*, θ)

$$\min_{x(t),y(t),\theta(t)} \int_0^1 \left(\dddot{x}^2 + \dddot{y}^2 + \dddot{\theta}^2 \right) dt$$

Human manipulation tasks

- Rate of change of muscle fiber lengths is critical in relaxed, voluntary motions

T. Flash and N. Hogan, The coordination of arm movements: an experimentally confirmed mathematical model, *Journal of neuroscience*, 1985

G.J. Garvin, M. Žefran, E.A. Henis, V. Kumar, Two-arm trajectory planning in a manipulation task, *Biological Cybernetics*, January 1997, Volume 76, Issue 1, pp 53-62

Optimal Trajectories with Constraints

Design a trajectory $x(t)$ such that $x(0) = a$, $x(T) = b$

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \mathcal{L}\left(\dddot{x}, \ddot{x}, \dot{x}, x, t\right) dt$$

$$|\dot{x}| \leq \dot{x}_{\max}$$

$$|\ddot{x}| \leq \ddot{x}_{\max}$$

$$|\dddot{x}| \leq \dddot{x}_{\max}$$

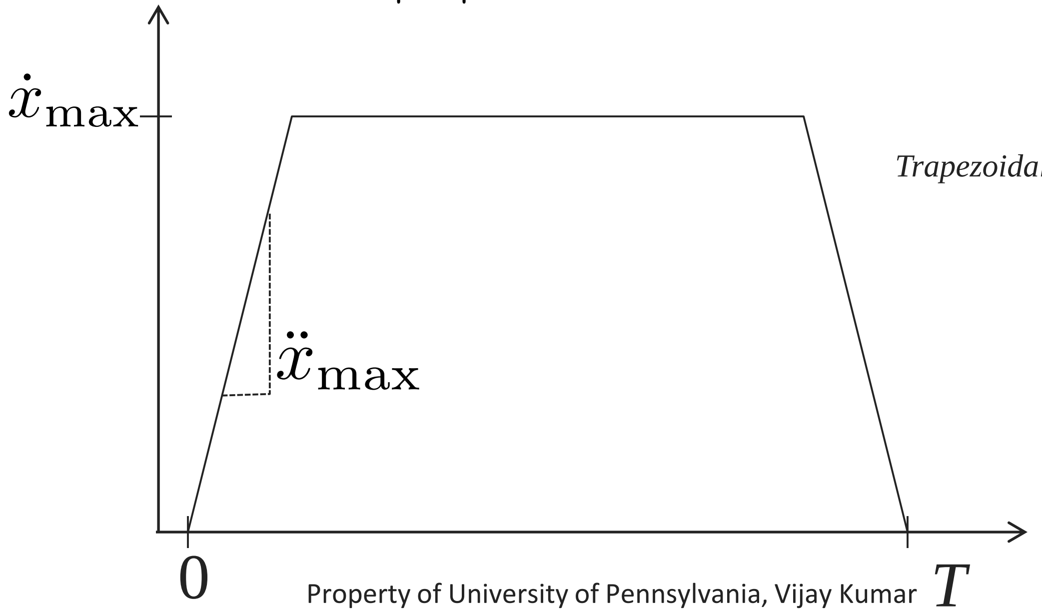
Minimum Time Trajectories

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T 1 dt$$

$$x(0) = x_0,\ \dot{x}(0) = 0,\ x(T) = x_T,\ \dot{x}(T) = 0$$

$$|\dot{x}| \leq \dot{x}_{\max}$$

$$|\ddot{x}| \leq \ddot{x}_{\max}$$

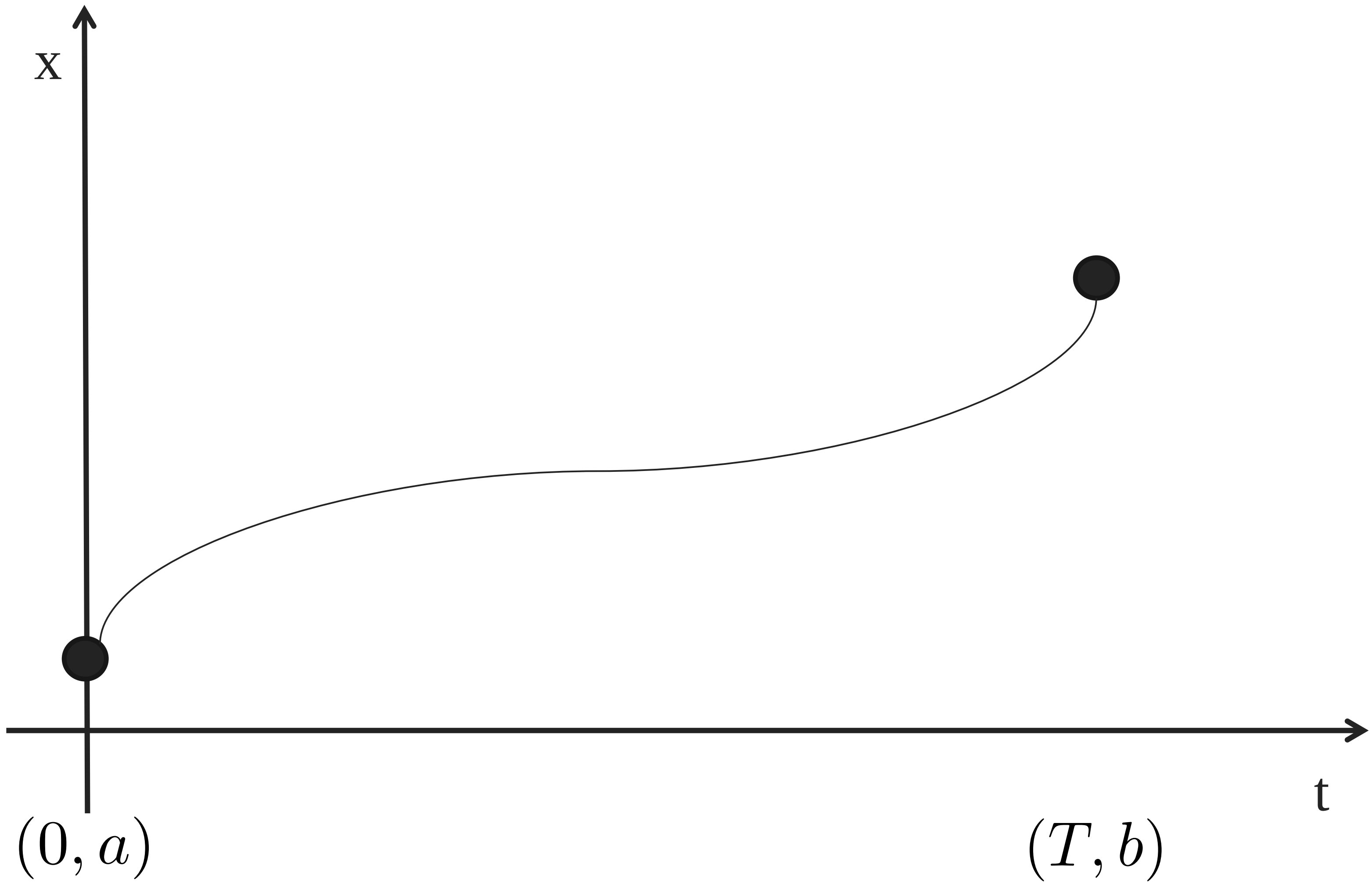


Trapezoidal Velocity Profile

Video 11.3
Vijay Kumar

Smooth 1D Trajectories

Design a trajectory $x(t)$ such that $x(0) = a$, $x(T) = b$

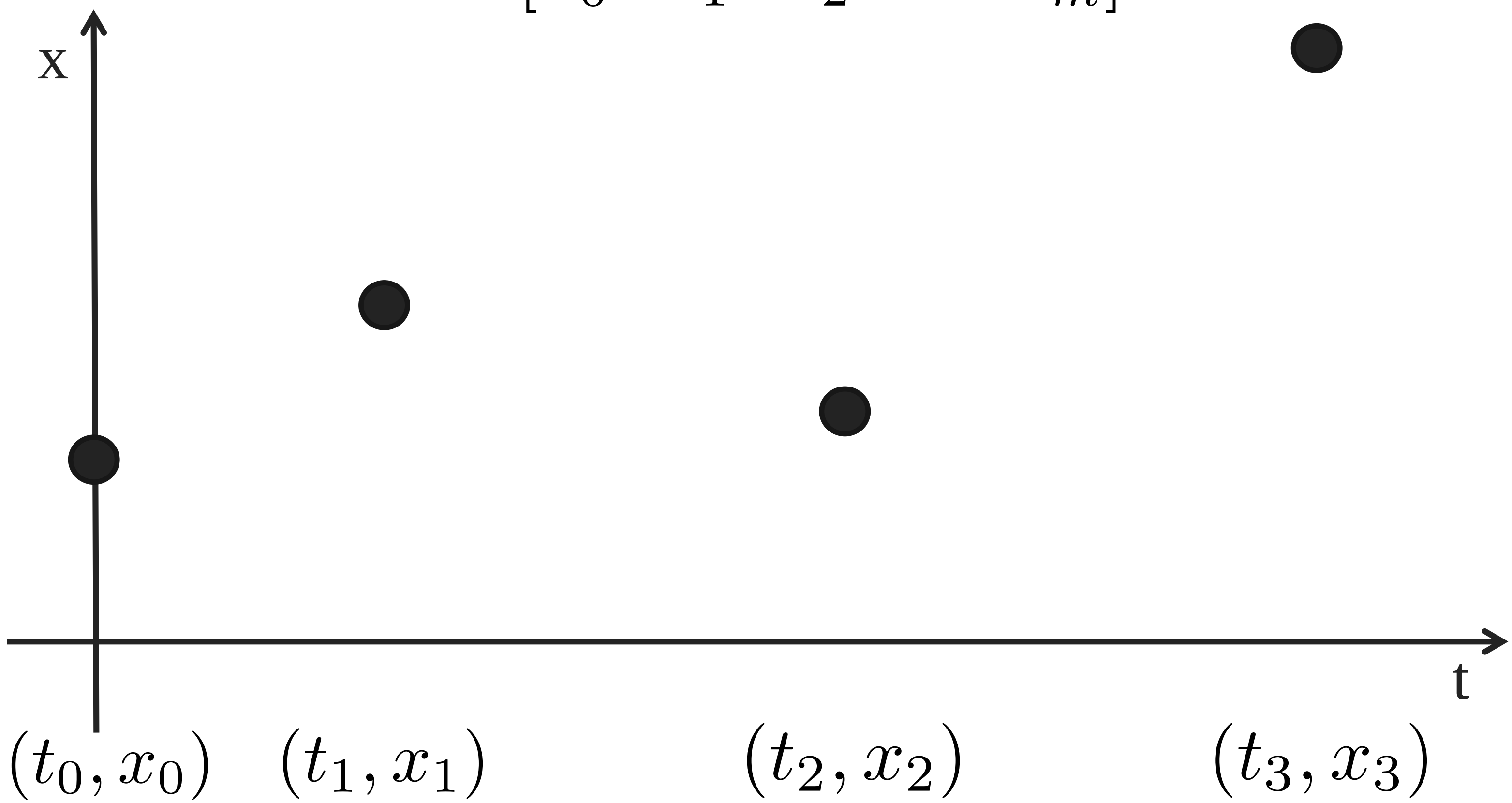


Multi-Segment 1D Trajectories

Design a trajectory $x(t)$ such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \ldots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_m]^T$$



Multi-Segment 1D Trajectories

Design a trajectory *x*(*t*) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \ldots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_m]^T$$

Define piecewise continuous trajectory:

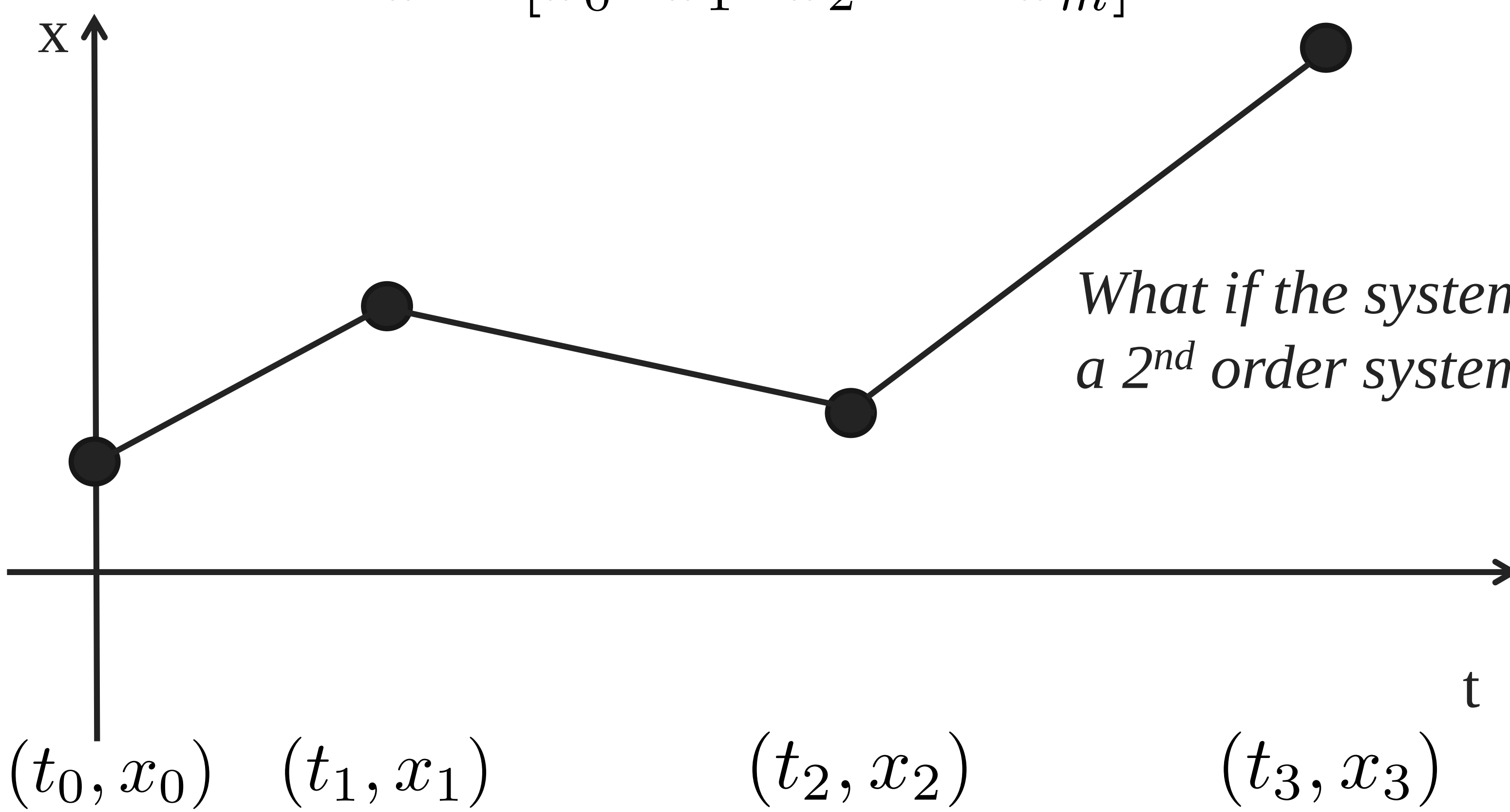
$$x(t) = \begin{cases} x_1(t), & t_0 \leq t < t_1 \\ x_2(t), & t_1 \leq t < t_2 \\ \ldots & \\ x_m(t), & t_{m-1} \leq t < t_m \end{cases}$$

Continuous but not Differentiable

Design a trajectory *x*(*t*) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \dots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \dots \quad x_m]^T$$



What if the system is a 2nd order system?

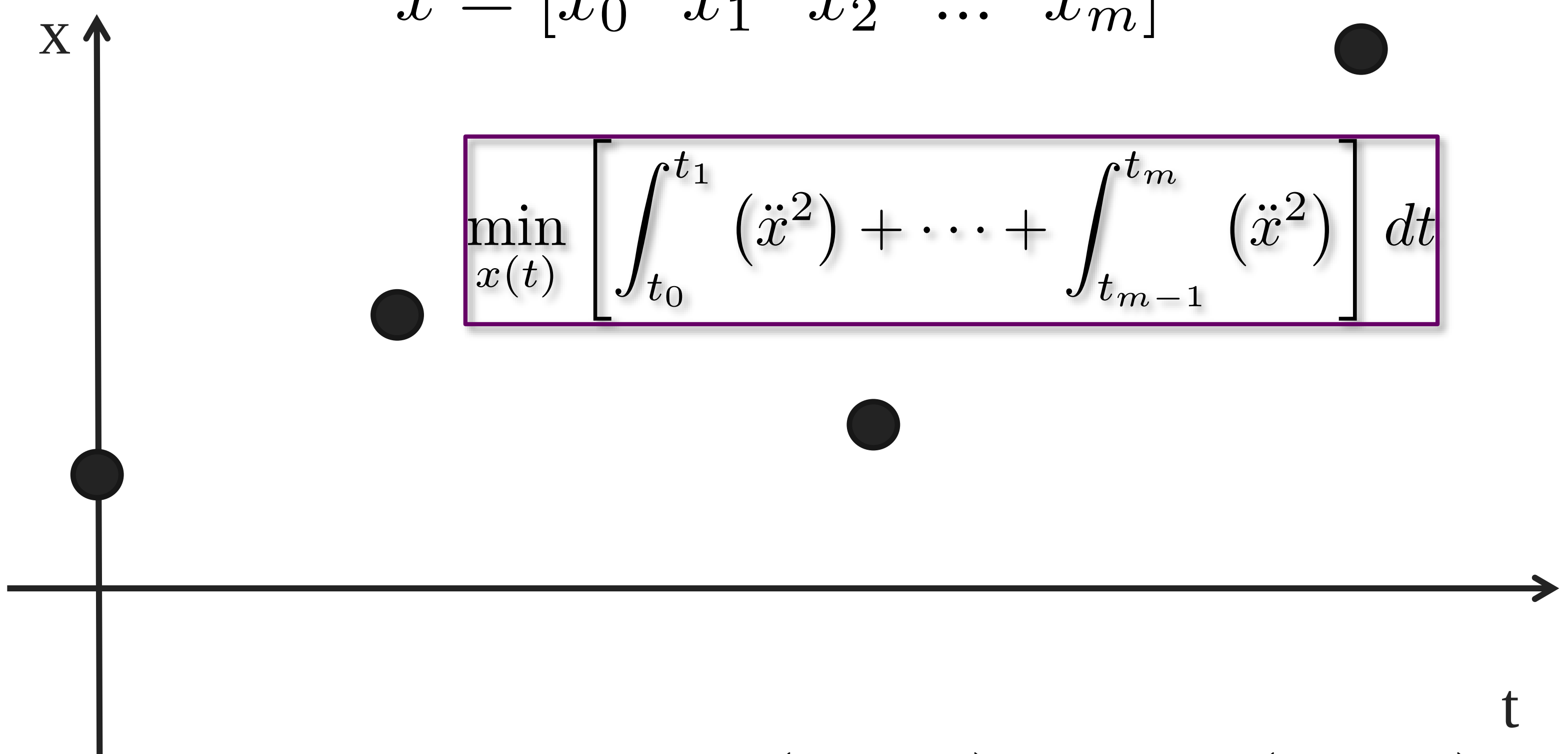
Minimum Acceleration Curve for 2nd Order Systems

Design a trajectory *x*(*t*) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \ldots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_m]^T$$

$$\boxed{\min_{x(t)} \left[\int_{t_0}^{t_1} (\ddot{x}^2) + \cdots + \int_{t_{m-1}}^{t_m} (\ddot{x}^2) \right] dt}$$



$(t_0, x_0) \quad (t_1, x_1) \quad (t_2, x_2) \quad (t_3, x_3)$

Minimum Acceleration Curve for 2nd Order Systems

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \dots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \dots \quad x_m]^T$$

$$x(t) = \begin{cases} x_1(t) = c_{1,3}t^3 + c_{1,2}t^2 + c_{1,1}t + c_{1,0}, & t_0 \leq t < t_1 \\ x_2(t) = c_{2,3}t^3 + c_{2,2}t^2 + c_{2,1}t + c_{2,0}, & t_1 \leq t < t_2 \\ \dots \\ x_m(t) = c_{m,3}t^3 + c_{m,2}t^2 + c_{m,1}t + c_{m,0}, & t_{m-1} \leq t < t_m \end{cases}$$

$4m$ degrees of freedom

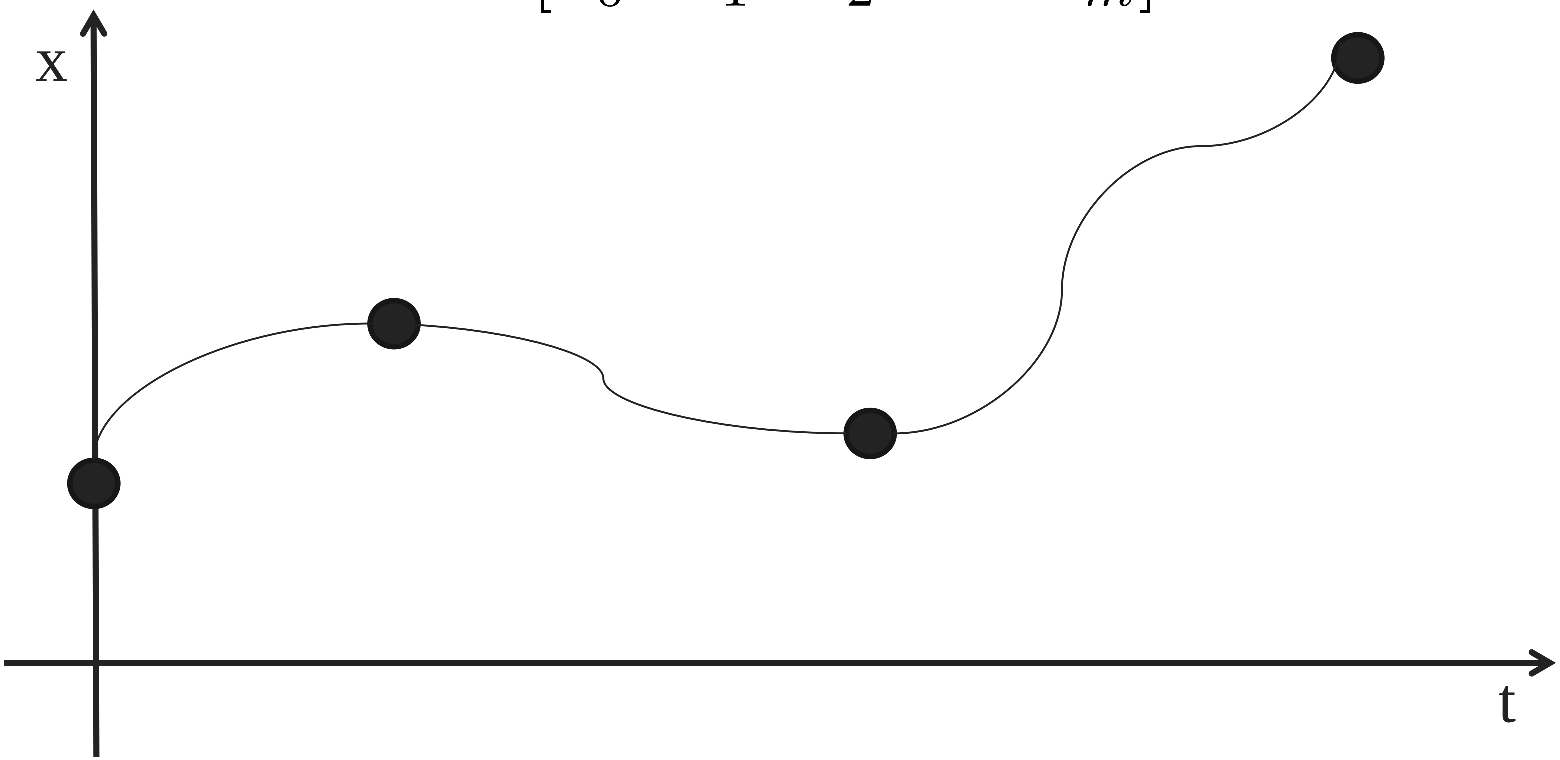
Cubic spline

Cubic Spline

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \dots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \dots \quad x_m]^T$$

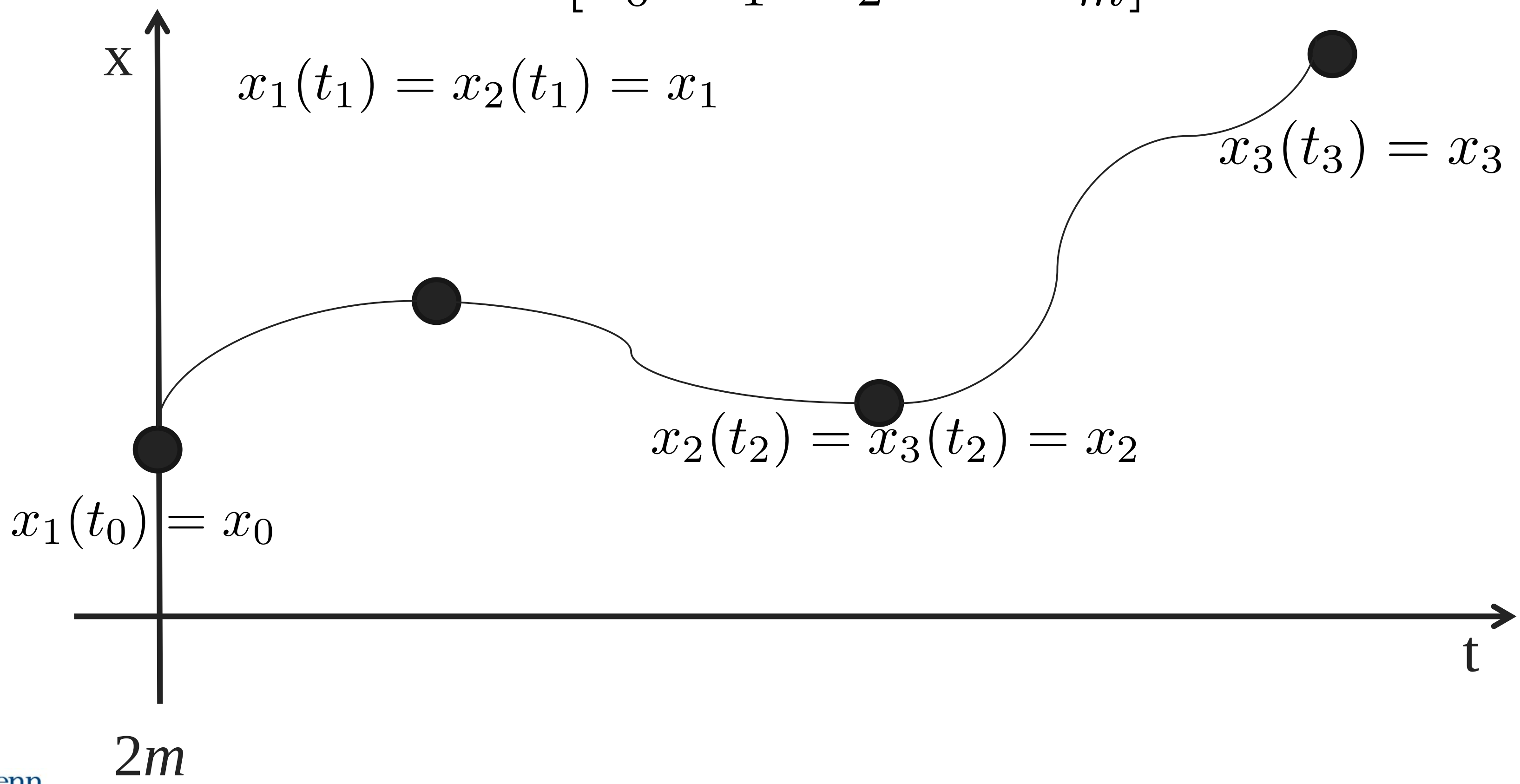


Cubic Spline

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \dots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \dots \quad x_m]^T$$

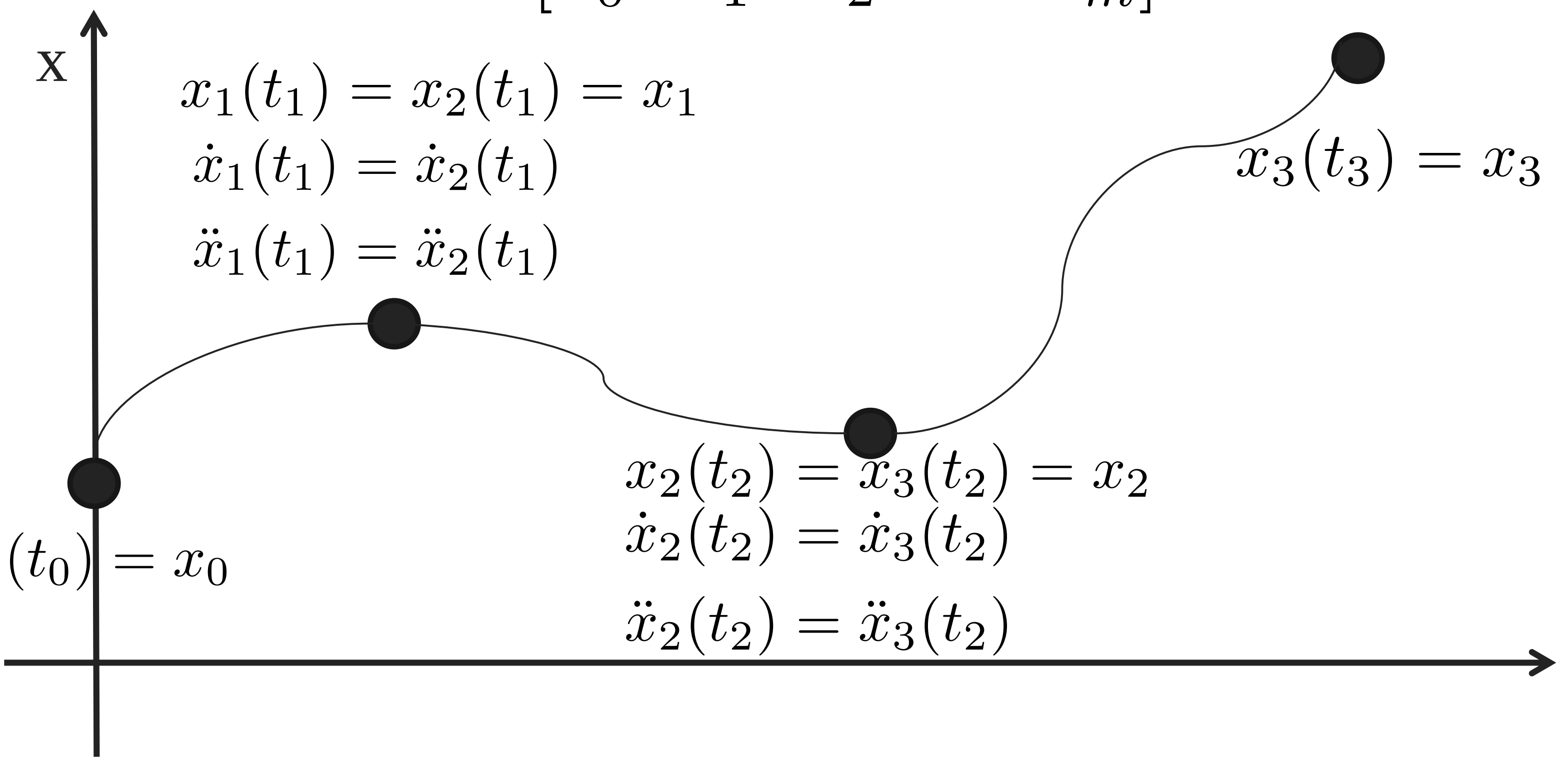


Cubic Spline

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \dots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \dots \quad x_m]^T$$



$$x_1(t_1) = x_2(t_1) = x_1$$

$$\dot{x}_1(t_1) = \dot{x}_2(t_1)$$

$$\ddot{x}_1(t_1) = \ddot{x}_2(t_1)$$

$$x_3(t_3) = x_3$$

$$x_1(t_0) = x_0$$

$$x_2(t_2) = x_3(t_2) = x_2$$

$$\dot{x}_2(t_2) = \dot{x}_3(t_2)$$

$$\ddot{x}_2(t_2) = \ddot{x}_3(t_2)$$

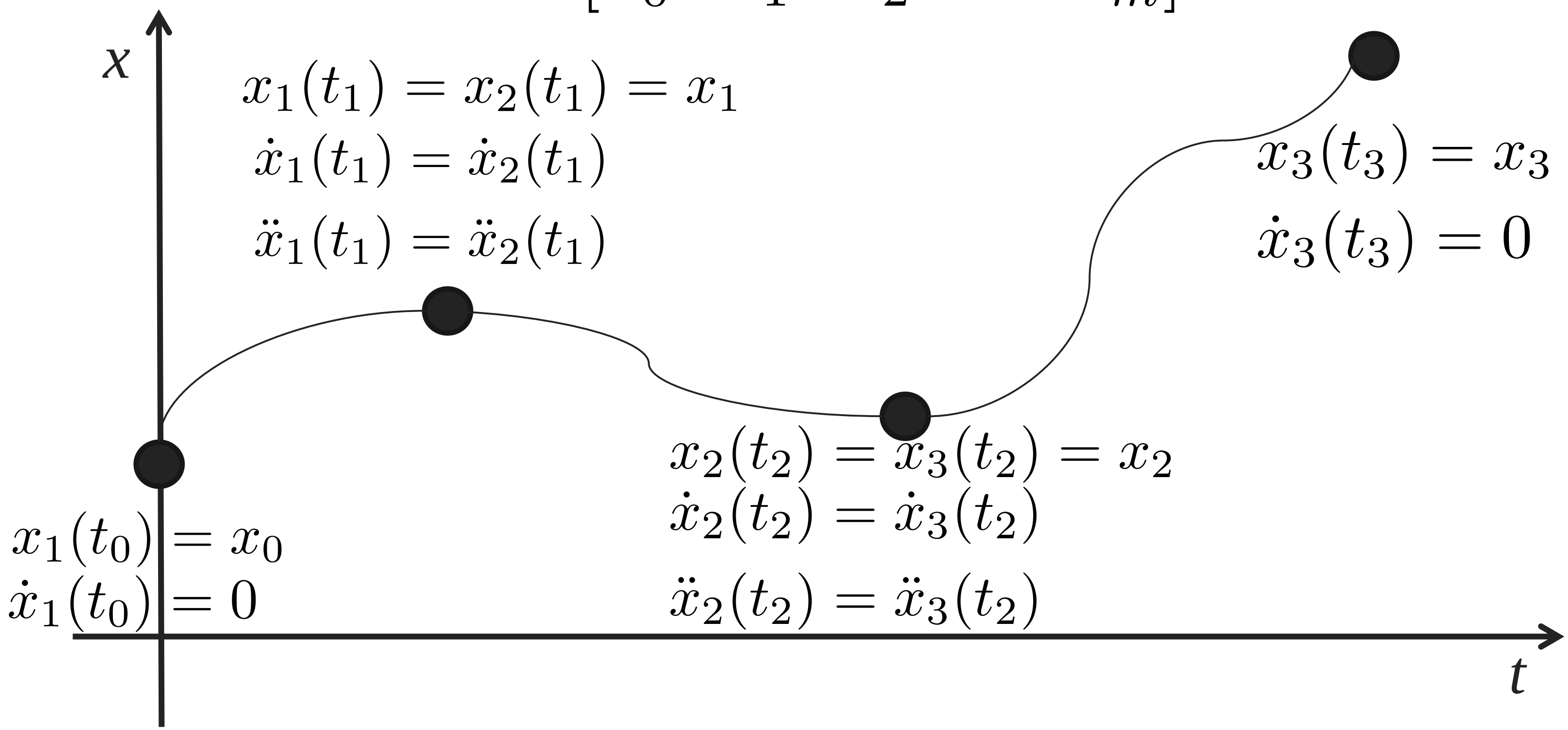
$2m+2(m\text{-}1)$

Cubic Spline

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \dots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \dots \quad x_m]^T$$



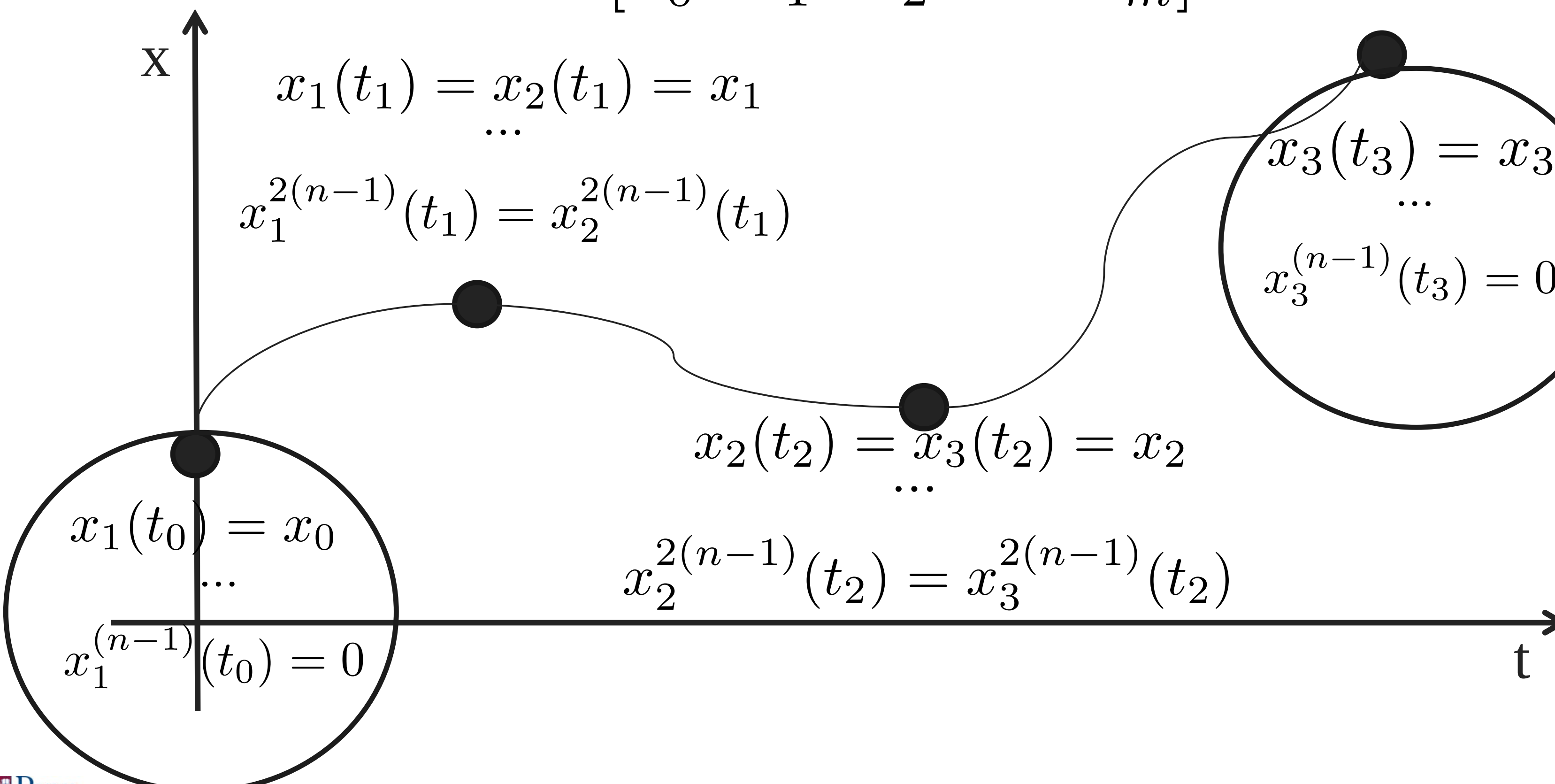
2*m*+2(*m*-1)+2 = 4*m* constraints

Spline for nth order system

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \ldots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_m]^T$$

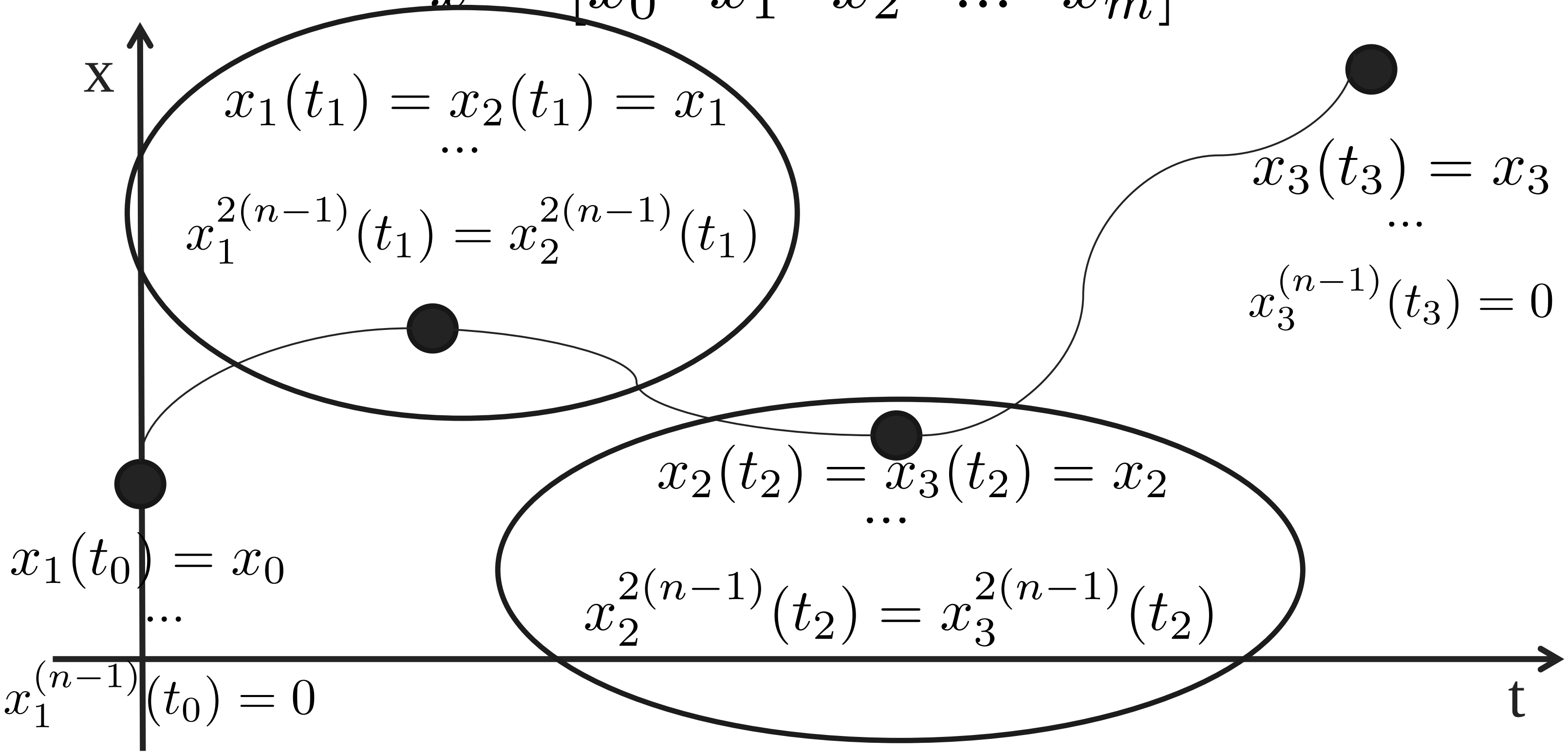


Spline for nth order system

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \ldots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_m]^T$$



Summary

- Polynomial interpolants
- Boundary conditions at intermediate points
- Splines
  - Smooth polynomial functions defined piecewise (degree *n*)
  - Smooth connections at in between "knots" (match values of functions and *n*-1 derivatives)

Video 11.4
Vijay Kumar

Minimum Snap Trajectory

When working with quadrotors, we want to find a trajectory that minimizes the cost function:

$$x^{\star}(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \|x^{(4)}\|^2 dt$$

From the Euler-Lagrange equations, a necessary condition for the optimal trajectory is:

$$x^{(8)} = 0$$

The minimum-snap trajectory is a 7th order polynomial.

Trajectory with 3 waypoints

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2]^T$$

$$x = [x_0 \quad x_1 \quad x_2]^T$$

The trajectory will be a 7th-order piecewise polynomial with 2 segments:

$$x(t) = \begin{cases} c_{1,7}t^7 + c_{1,6}t^6 + c_{1,5}t^5 + c_{1,4}t^4 + c_{1,3}t^3 + c_{1,2}t^2 + c_{1,1}t + c_{1,0}, & t_0 \leq t < t_1 \\ c_{2,7}t^7 + c_{2,6}t^6 + c_{2,5}t^5 + c_{2,4}t^4 + c_{2,3}t^3 + c_{2,2}t^2 + c_{2,1}t + c_{2,0}, & t_1 \leq t < t_2 \end{cases}$$

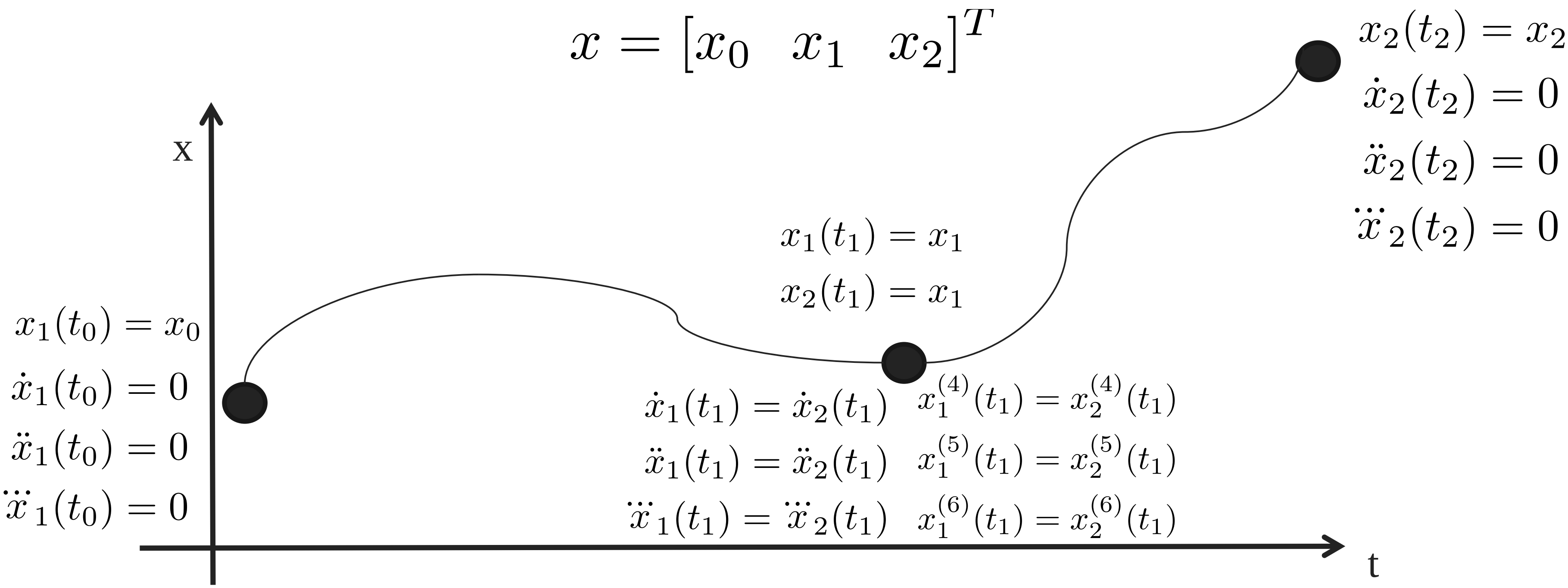
This trajectory has 16 unknowns.

Trajectory with 3 waypoints

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2]^T$$

$$x = [x_0 \quad x_1 \quad x_2]^T$$



Trajectory with 3 waypoints

$$\mathbf{x} = [c_{1,7} \quad c_{1,6} \quad c_{1,5} \quad c_{1,4} \quad c_{1,3} \quad c_{1,2} \quad c_{1,1} \quad c_{1,0}$$
$$c_{2,7} \quad c_{2,6} \quad c_{2,5} \quad c_{2,4} \quad c_{2,3} \quad c_{2,2} \quad c_{2,1} \quad c_{2,0}]^T$$

Position constraints in matrix form:

$$\begin{bmatrix} t_0^7 & t_0^6 & t_0^5 & t_0^4 & t_0^3 & t_0^2 & t_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_1^7 & t_1^6 & t_1^5 & t_1^4 & t_1^3 & t_1^2 & t_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1^7 & t_1^6 & t_1^5 & t_1^4 & t_1^3 & t_1^2 & t_1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_2^7 & t_2^6 & t_2^5 & t_2^4 & t_2^3 & t_2^2 & t_2 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_1 \\ x_2 \end{bmatrix}$$

Trajectory with 3 waypoints

$$\mathbf{x} = [c_{1,7} \quad c_{1,6} \quad c_{1,5} \quad c_{1,4} \quad c_{1,3} \quad c_{1,2} \quad c_{1,1} \quad c_{1,0} \quad c_{2,7} \quad c_{2,6} \quad c_{2,5} \quad c_{2,4} \quad c_{2,3} \quad c_{2,2} \quad c_{2,1} \quad c_{2,0}]^T$$

Endpoint derivative constraints at t_0 in matrix form:

$$\begin{bmatrix} 7t_0^6 & 6t_0^5 & 5t_0^4 & 4t_0^3 & 3t_0^2 & 2t_0^1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 42t_0^5 & 30t_0^4 & 20t_0^3 & 12t_0^2 & 6t_0^1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 210t_0^4 & 120t_0^3 & 60t_0^2 & 24t_0^1 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} \dot{x}(t_0) \\ \ddot{x}(t_0) \\ \dddot{x}(t_0) \end{bmatrix}$$