ITMO UNIVERSITY

How to Win Coding Competitions: Secrets of Champions

Week 3: Sorting and Search Algorithms
Lecture 8: Integer sorting

Maxim Buzdalov
Saint Petersburg 2016

How can we speed up sorting, if we know that we are sorting integers?

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In this video:

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In this video:

- Counting sort

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In this video:

- Counting sort
- Bucket sort
- Radix sort

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5	1	2	5	4	3	2	1	4	3	1	5

1	2	3	4	5
0	0	0	0	0

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1	2	3	4	5
0	0	0	0	1

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				4	3	2	1	4	3	1	5

1	2	3	4	5
1	1	0	0	2

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					3	2	1	4	3	1	5

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1	1	0	1	2

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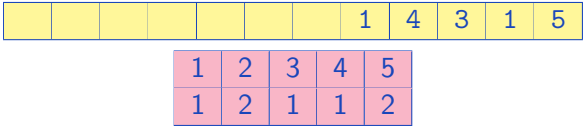

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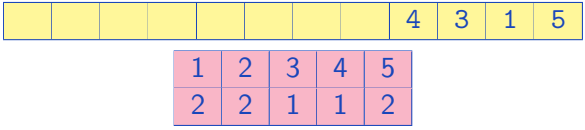

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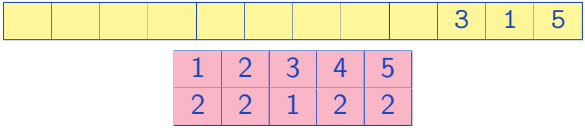

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1	2	3	4	5
2	2	2	2	2

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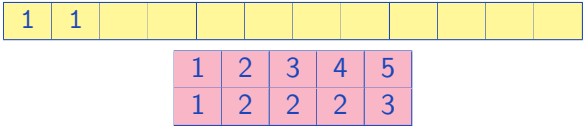

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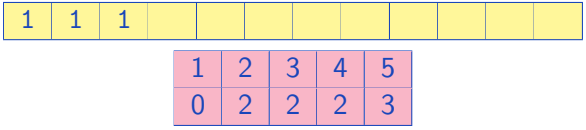

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1	1	1	2								

1	2	3	4	5
0	1	2	2	3

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Running time: $\Theta(N + M)$, additional space: $\Theta(M)$

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Running time: $\Theta(N + M)$, additional space: $\Theta(M)$

- Faster than comparison-based sorting algorithms when M is small

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5/Z	1/K	2/E	5/W	4/J	3/U	2/K	1/Z	4/A	3/Q	1/X	5/E

1				
2				
3				
4				
5				

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2				
3				
4				
5	Z			

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1	K			
2				
3				
4				
5	Z			

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2	E			
3				
4				
5	Z	W		

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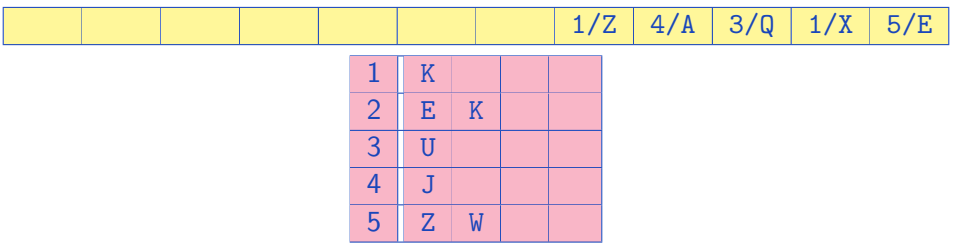

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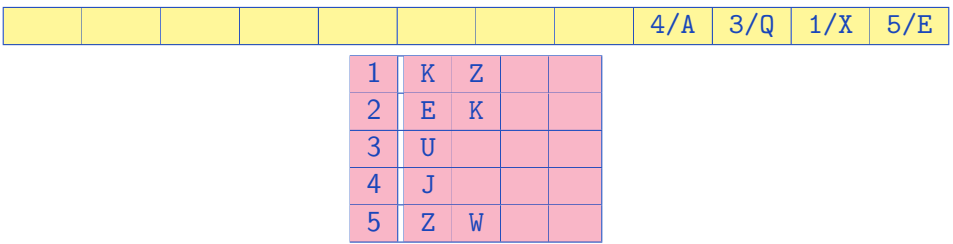

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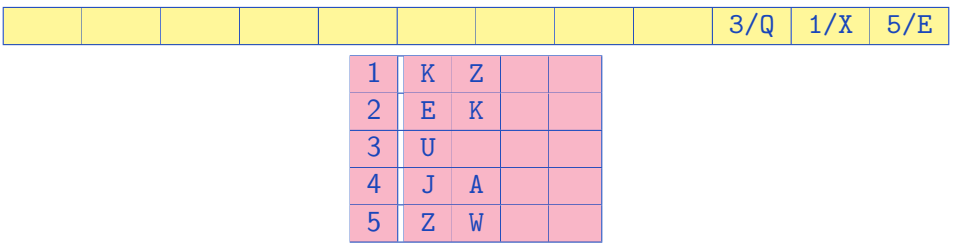

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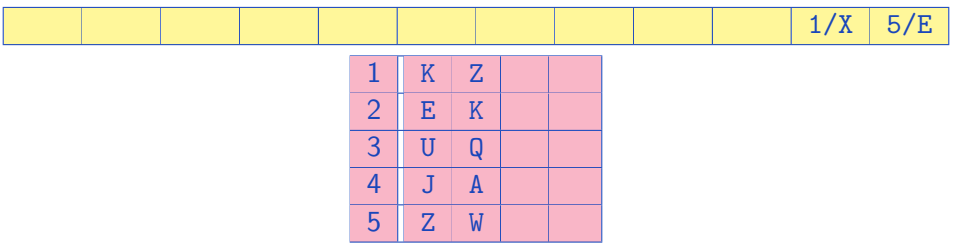

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1/K	1/Z	1/X	2/E								

1				
2		K		
3	U	Q		
4	J	A		
5	Z	W	E	

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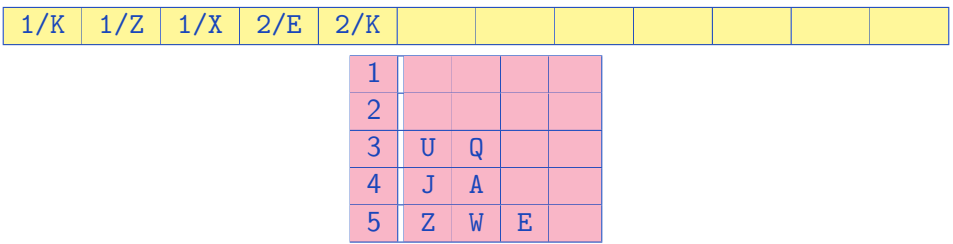

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1/K	1/Z	1/X	2/E	2/K	3/U	3/Q	4/J	4/A	5/Z	5/W	5/E

1				
2				
3				
4				
5				

Given: N key-value pairs, keys are integers from $[1; M]$, M is quite small.
How to sort them efficiently?

- We cannot "just count" integers anymore
- But we still can profit from using the keys as array indices
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This is Bucket sort! $\Theta(N + M)$ time, $\Theta(N + M)$ space.

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This is Radix sort! $\Theta((N + M)L)$ time, $\Theta(N + M)$ space.