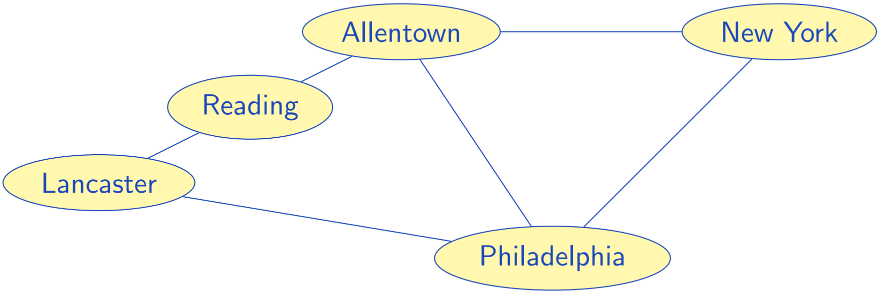
ITMO UNIVERSITY

How to Win Coding Competitions: Secrets of Champions

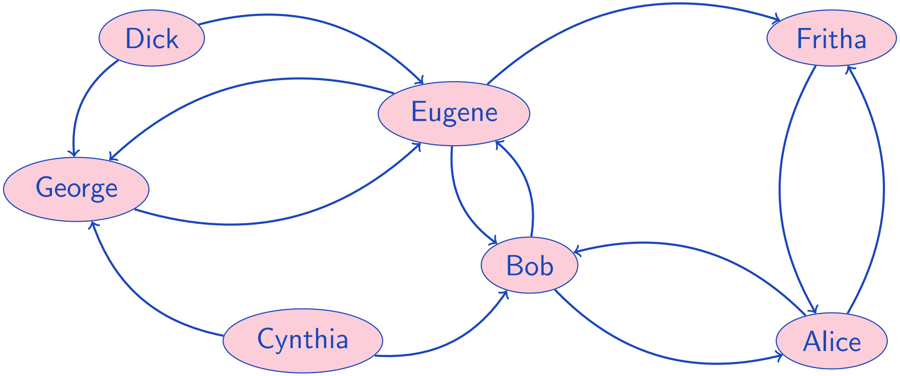
Week 4: Algorithms on Graphs
Lecture 1: Introduction to graphs

Maxim Buzdalov
Saint Petersburg 2016

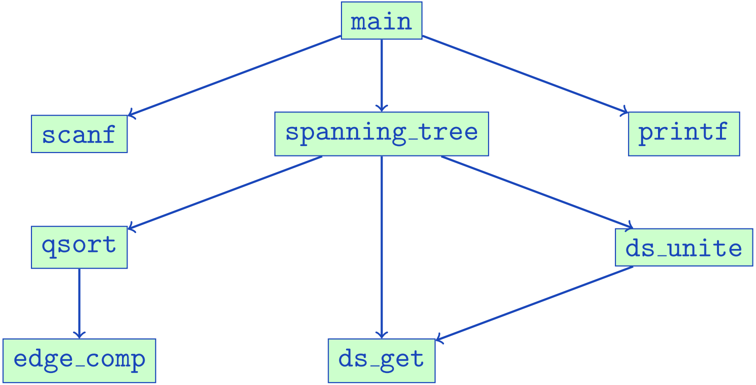
Roads and cities



Social networks



Even computer programs



Undirected graph: an ordered pair $G = \langle V, E \rangle$

- V – set of graph's vertices
- E – set of graph's edges, a multiset of **unordered** pairs of vertices
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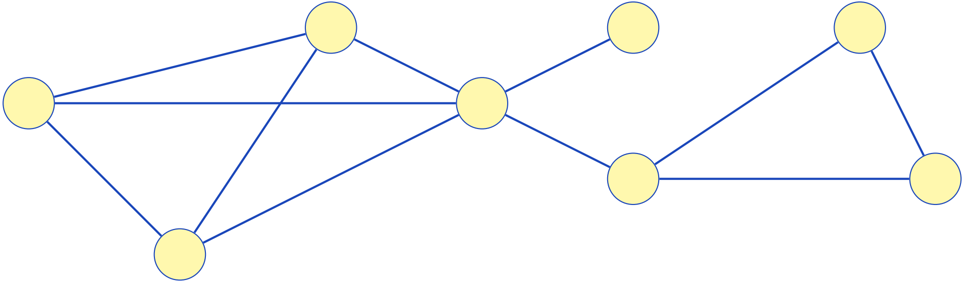

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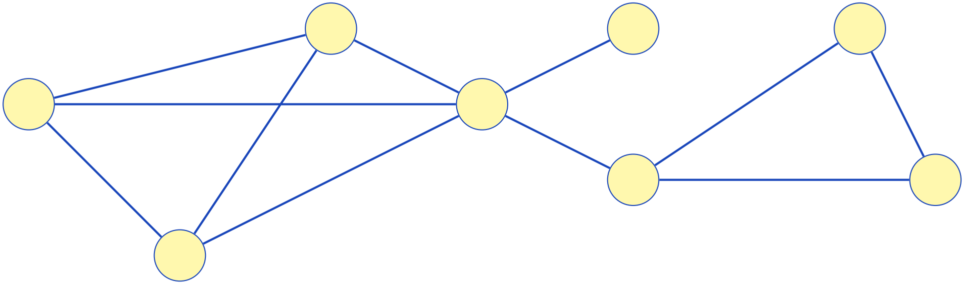

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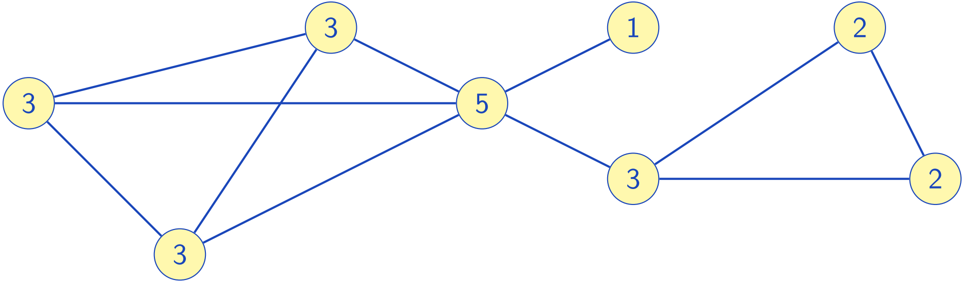

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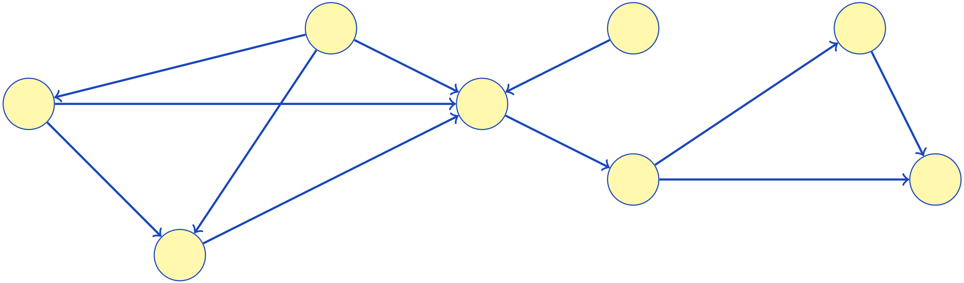

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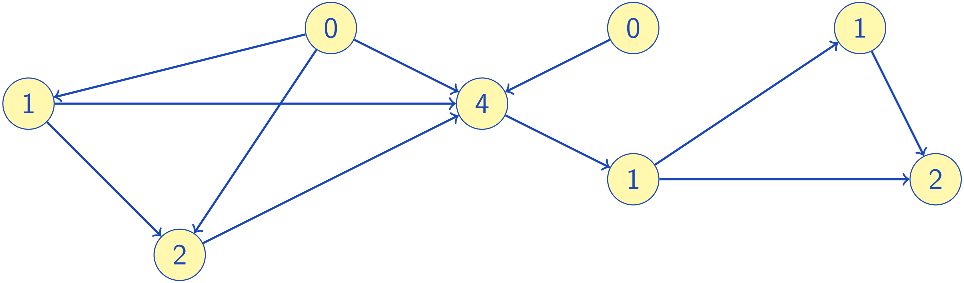

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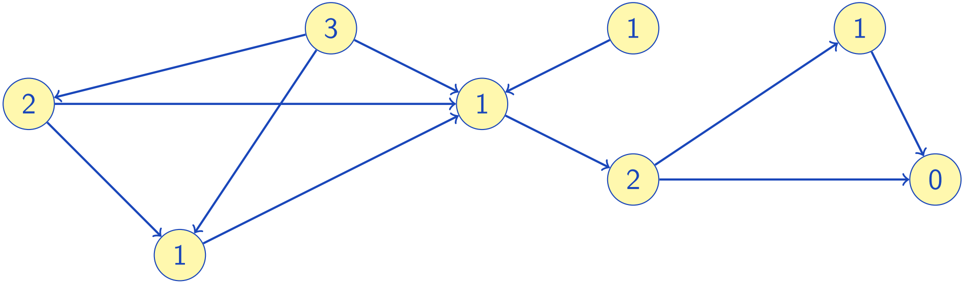

Weighted graph

- Can be either directed or undirected graph
- Has a function $W : E \to X$, where X is the set of weights
- X is typically integers or reals, can also be symbols or strings

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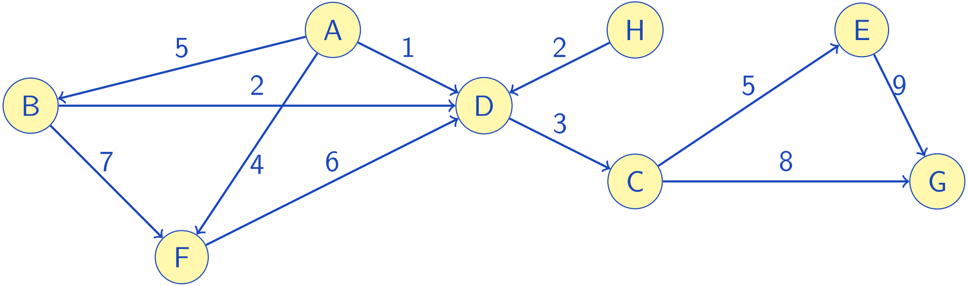

Weighted graph

- Can be either directed or undirected graph
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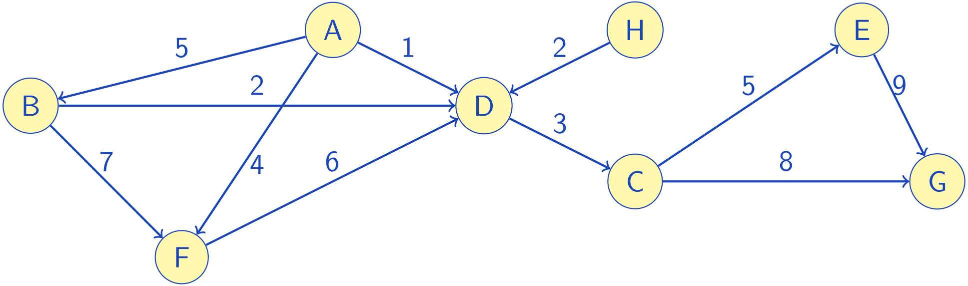
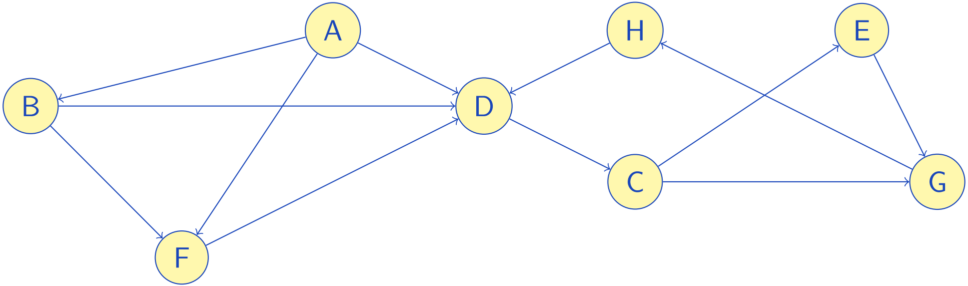
- In some cases, an unweighted graph is the same as a weighted graph with $X = \{1\}$



Path in the graph $G = \langle V, E \rangle$

- A list of vertices $v_1, v_2, \ldots, v_k$, such that $(v_i, v_{i+1}) \in E$ for all $1 \leq i < k$



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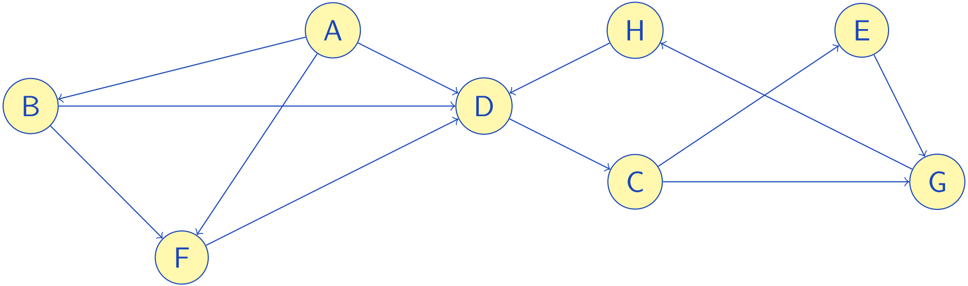

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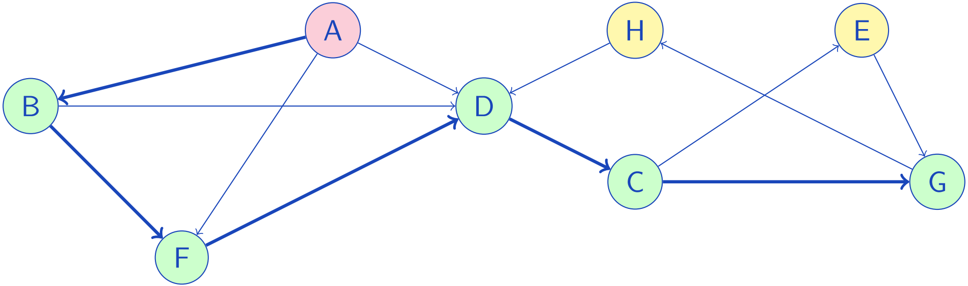

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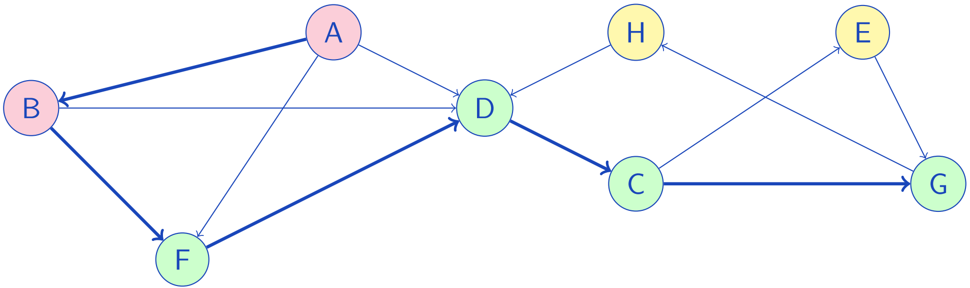

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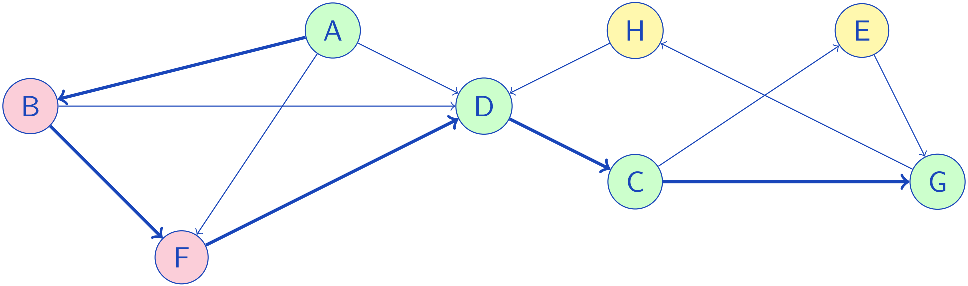

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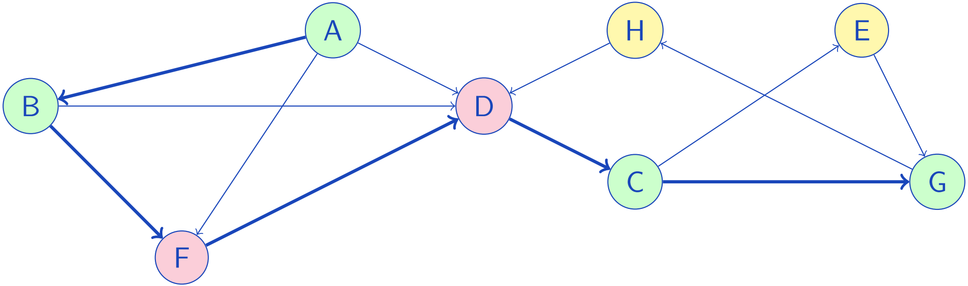

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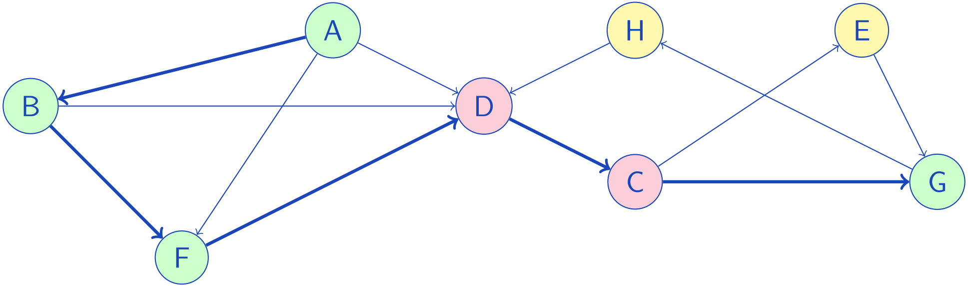

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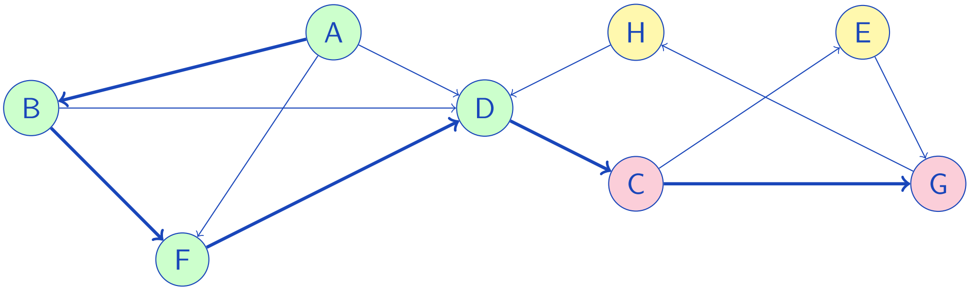

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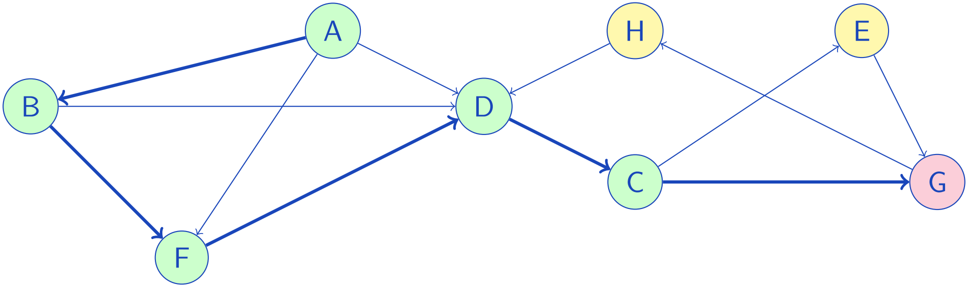

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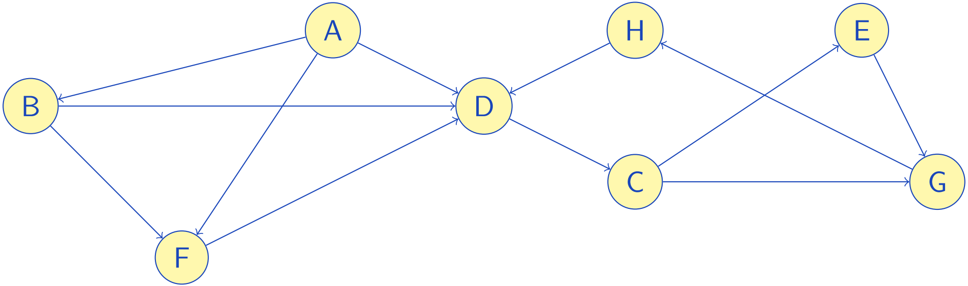

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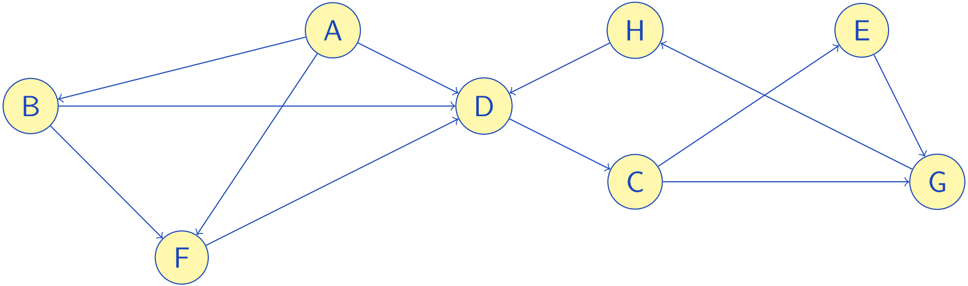

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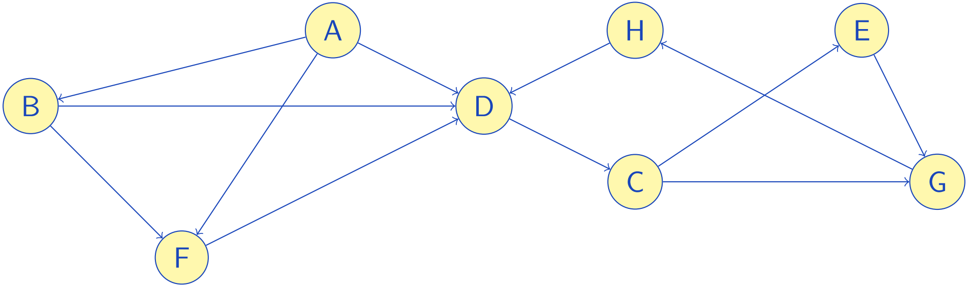

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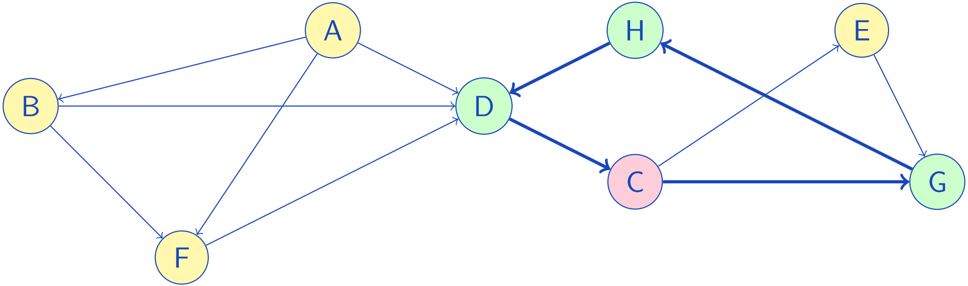

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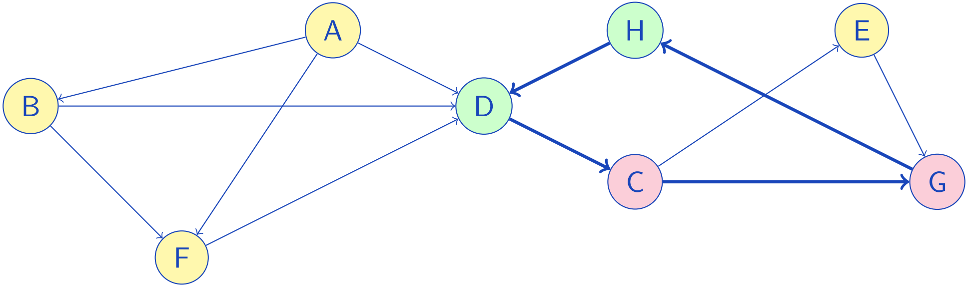

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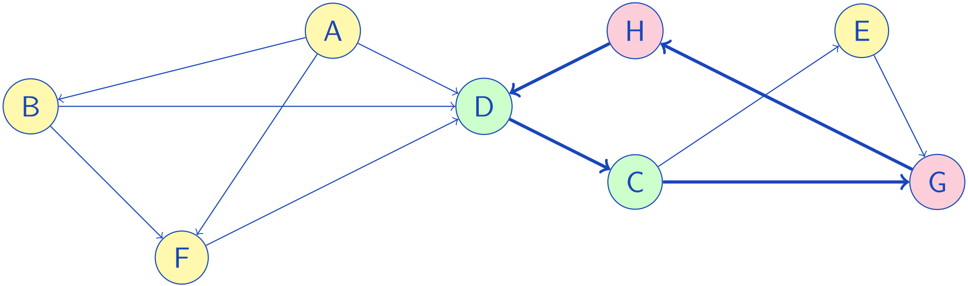

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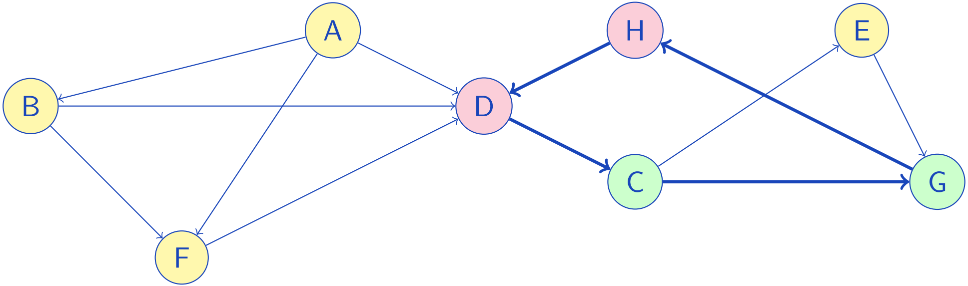

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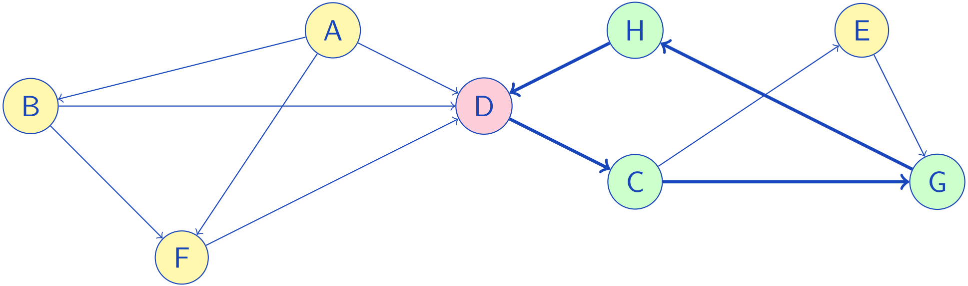

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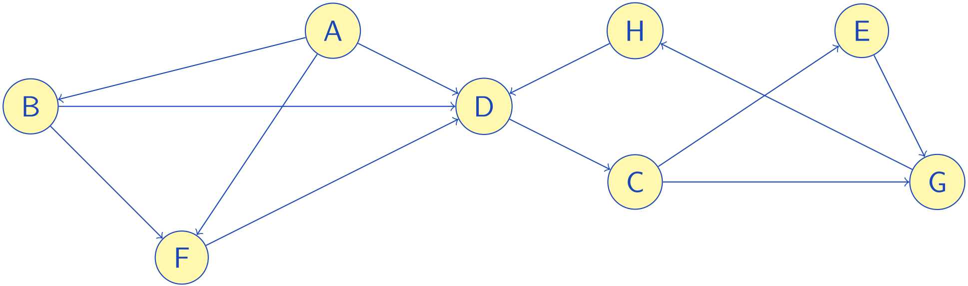

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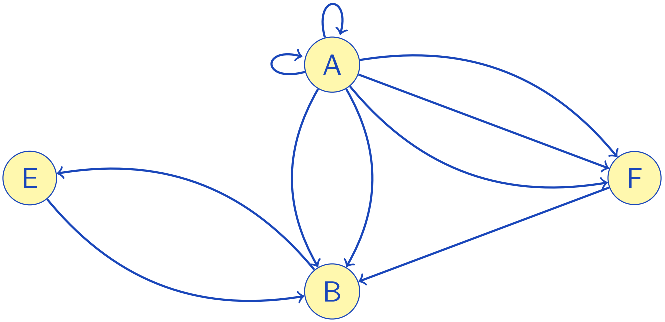

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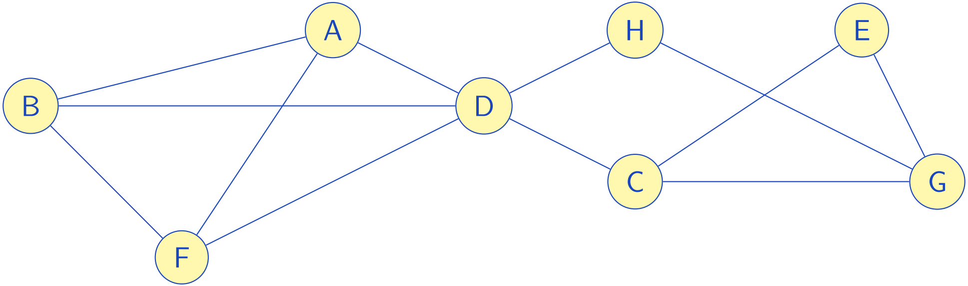
Example of connected graph



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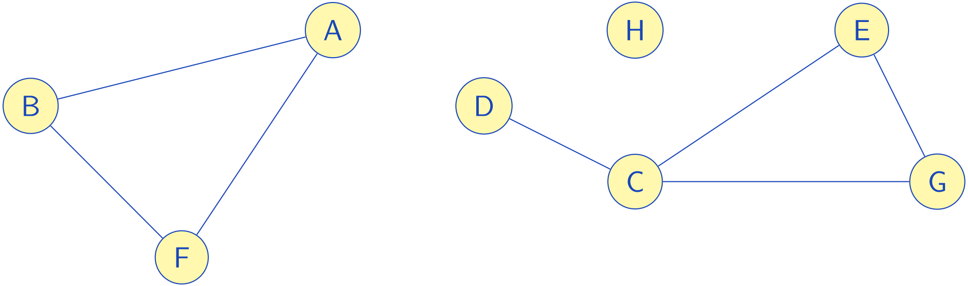
Example of graph which is not connected



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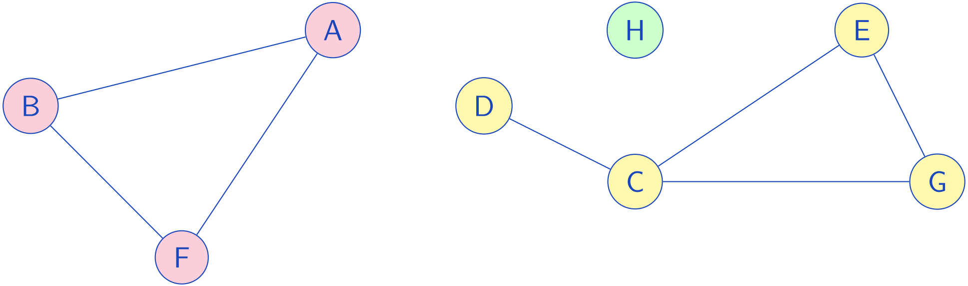
Example of graph which is not connected – three connected components



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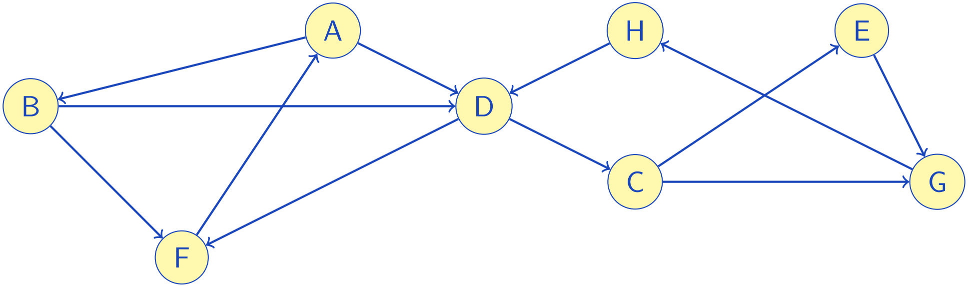

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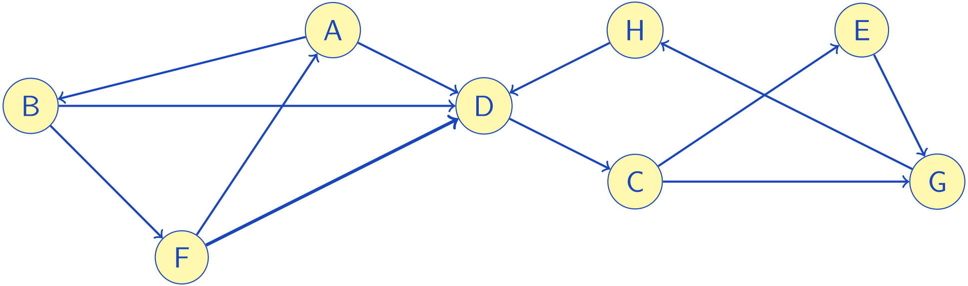

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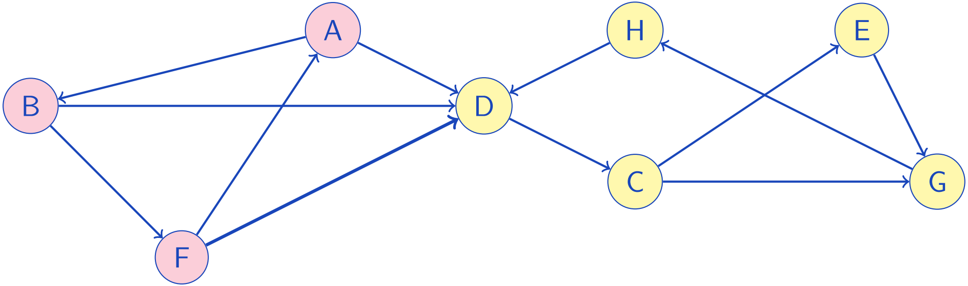
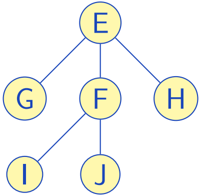
Example of graph which is not strongly connected – two strongly connected components



There exist several important types of graphs. Some of these frequently appear in programming competitions. Here they are:

- Trees, rooted trees and forests
- Cactuses
- Complete graphs and tournaments
- Bipartite graphs
- Planar graphs

Tree: an undirected graph in which any two vertices are connected by exactly one path



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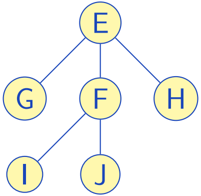

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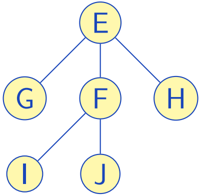

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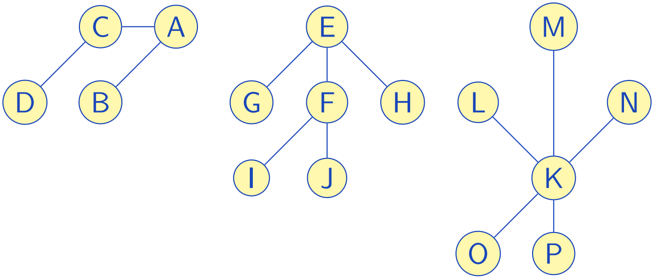
Forest: a disjoint union of trees



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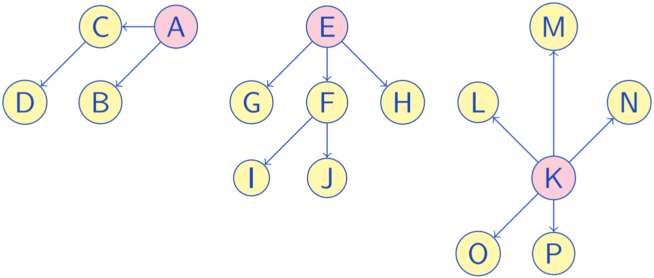
Forest: a disjoint union of trees

Rooted tree: a tree where one of the vertices is a root.

All edges have direction either from or towards root



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  - $V' \subseteq V$, $E' \subseteq E$
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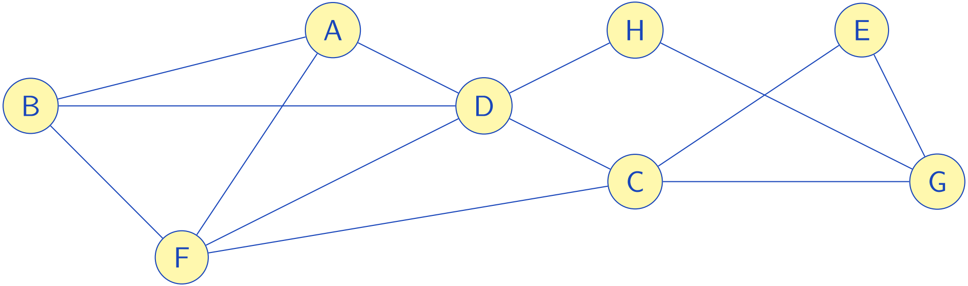

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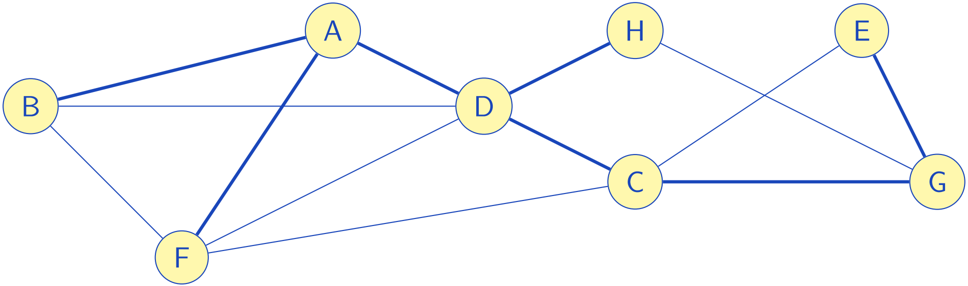

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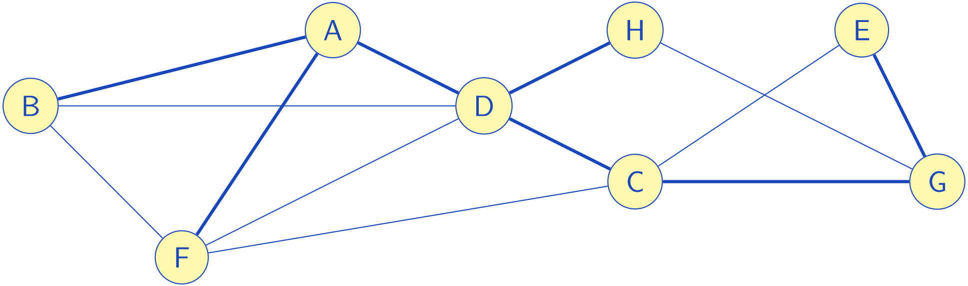

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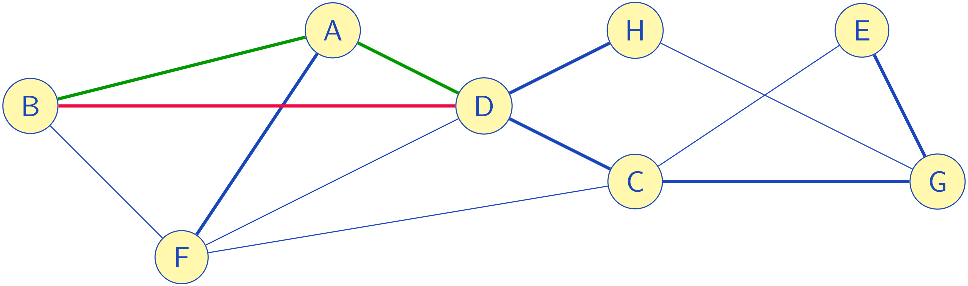

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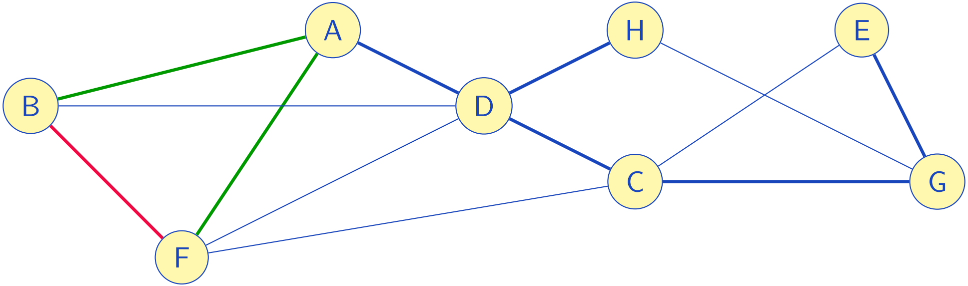

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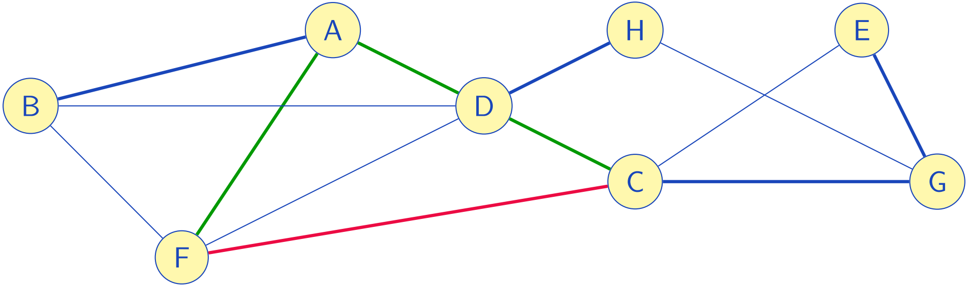

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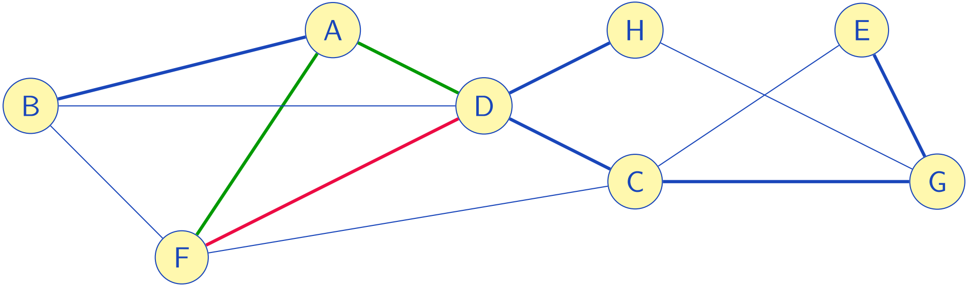

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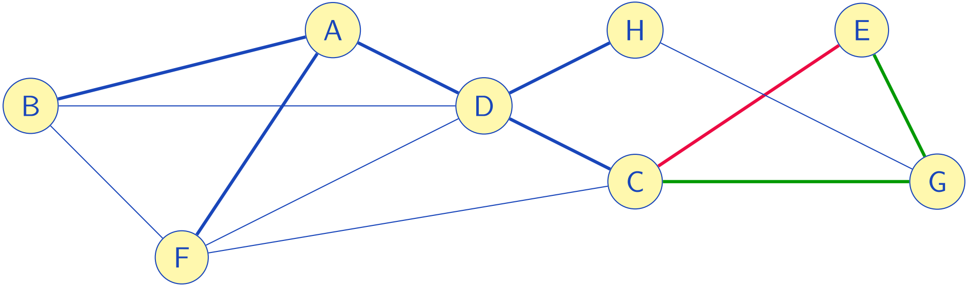

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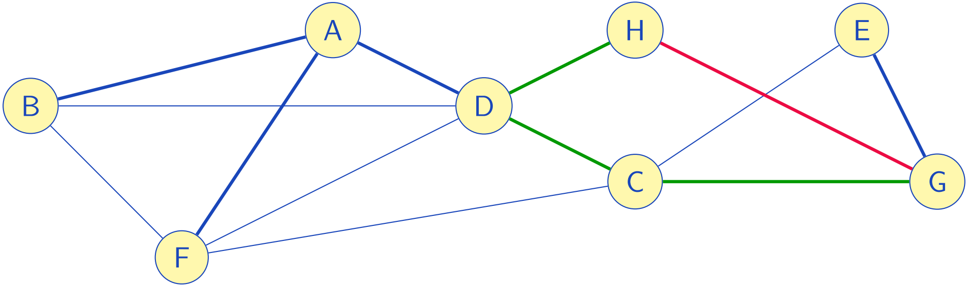

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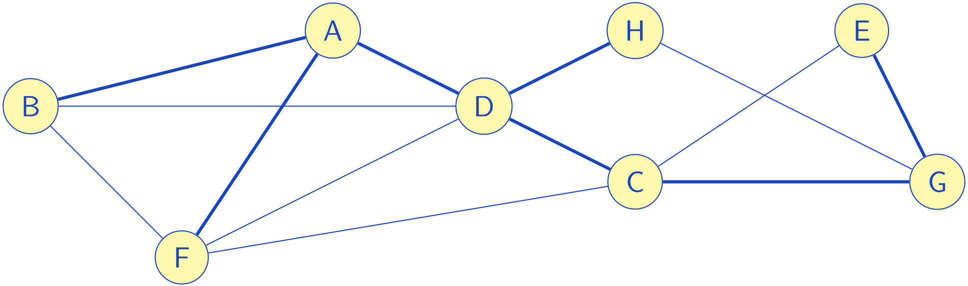
These are the fundamental cycles with regards to T.
Any cycle can be uniquely expressed as a combination of fundamental cycles



Edge-cactus: connected, undirected, any two simple cycles share at most one vertex

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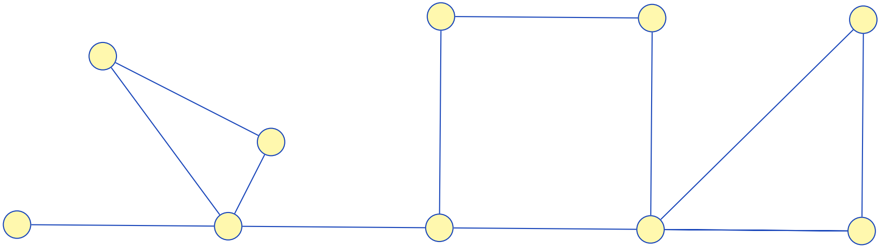

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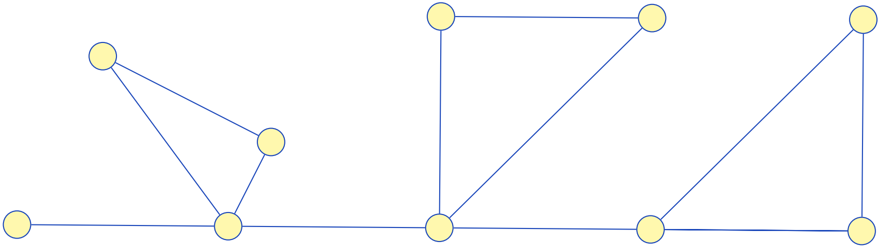

Bipartite graph: a undirected graph where $V = L \cup R$, $L \cap R = \emptyset$, such that for any edge (u, v) it holds that $u \in L$, $v \in R$

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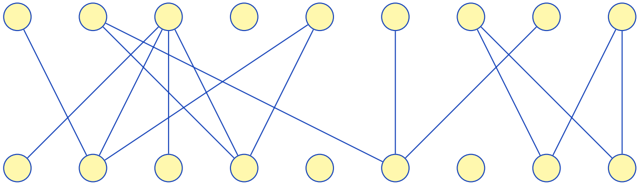

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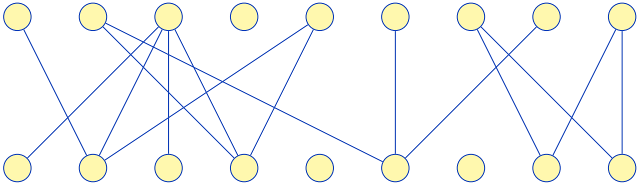
- In any bipartite graph, every loop has an even length



Complete graph K_n: an undirected graph with edges between all pairs of vertices

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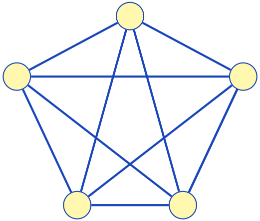

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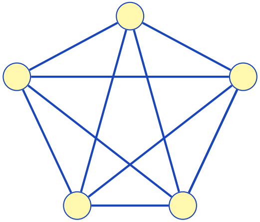
Tournament T_n: a directed graph with either (u, v) or (v, u) for all u and v



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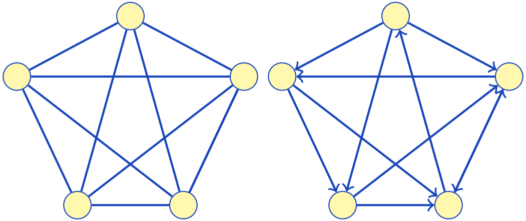

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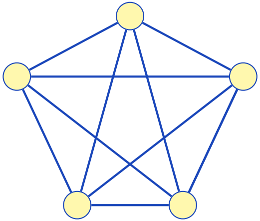

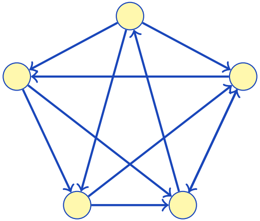

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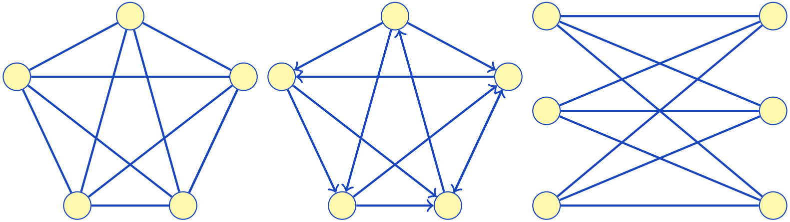
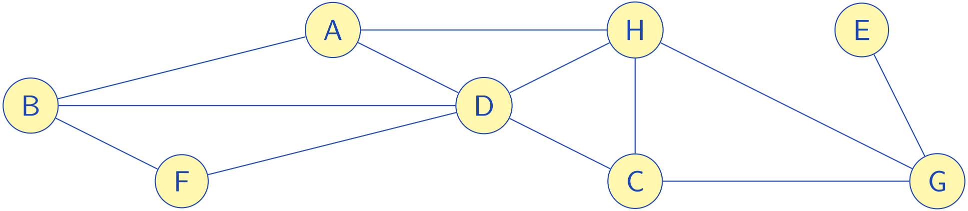


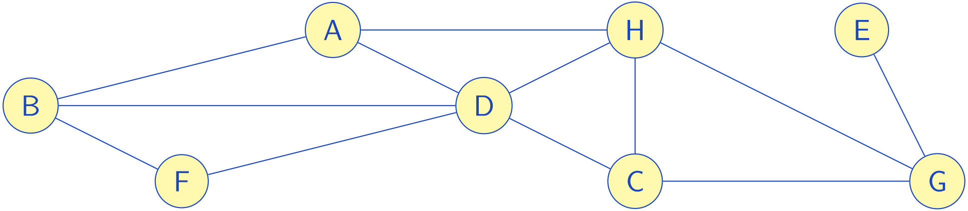
Planar graph: a graph which can be laid out on the plane without edges intersections

Planar graph: a graph which can be laid out on the plane without edges intersections



Planar graph: a graph which can be laid out on the plane without edges intersections

- Face: a piece of plane bounded by a loop in the graph never intersected by edges



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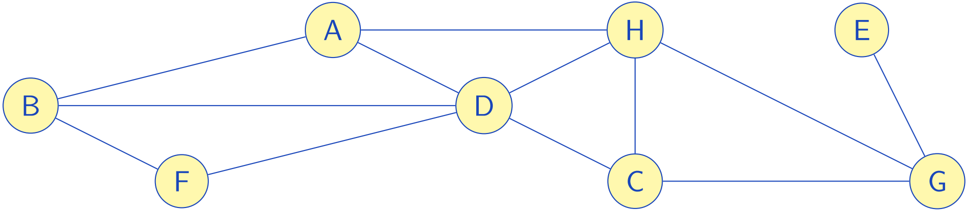

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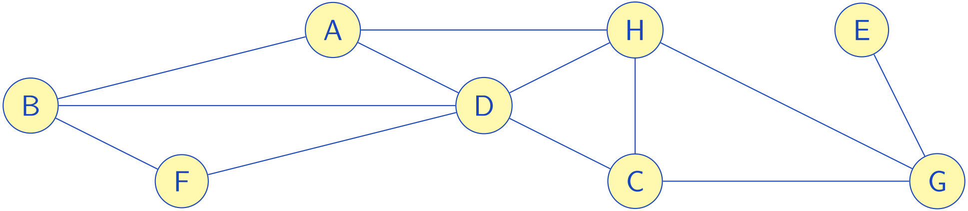

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- Fact 3 (Kuratowski): The graph is planar if and only if it doesn't contain subgraphs which are subdivision of K_5 or $K_{3,3}$. Subdivision of the graph is introducing some number of vertices in each edge.