Foundations of Computer Graphics

Online Lecture 2: Review of Basic Math

Vectors and Dot Products

Ravi Ramamoorthi

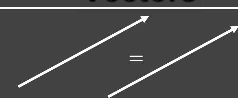
Course: Next Steps

- Complete HW 0
  - Sets up basic compilation issues
  - Verifies you can work with feedback/grading servers
- First few lectures core math ideas in graphics
  - This lecture is a revision of basic math concepts
- HW 1 has few lines of code (but start early)
  - Use some ideas discussed in lecture, create images
- Textbooks: None required
  - OpenGL/GLSL reference helpful (but not required)

Motivation and Outline

- Many graphics concepts need basic math like linear algebra
  - Vectors (dot products, cross products, …)
  - Matrices (matrix-matrix, matrix-vector mult., …)
  - E.g: a point is a vector, and an operation like translating or rotating points on object can be matrix-vector multiply
- Should be refresher on very basic material for most of you
  - Only basic high school math required

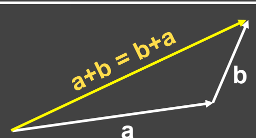
Vectors



Usually written as $\vec{a}$ or in bold. Magnitude written as $\|\vec{a}\|$

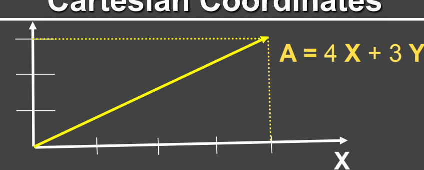
- Length and direction. Absolute position not important
- Use to store offsets, displacements, locations
  - But strictly speaking, positions are not vectors and cannot be added: a location implicitly involves an origin, while an offset does not.

Vector Addition



- Geometrically: Parallelogram rule
- In cartesian coordinates (next), simply add coords

Cartesian Coordinates



$$A = \begin{pmatrix} x \\ y \end{pmatrix} \qquad A^T = \begin{pmatrix} x & y \end{pmatrix} \qquad \|A\| = \sqrt{x^2 + y^2}$$

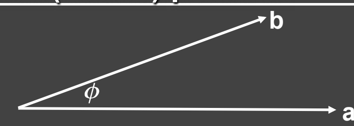
- X and Y can be any (usually orthogonal ***unit***) vectors

Vector Multiplication

- *Dot product*
- Cross product
- Orthonormal bases and coordinate frames

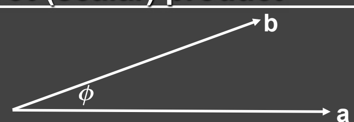
- Note: We use right-handed (standard) coordinates

Dot (scalar) product



$a \bullet b = b \bullet a = ?$

Dot (scalar) product

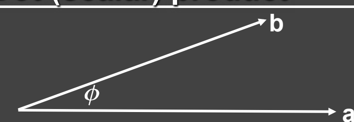


$a \bullet b = b \bullet a = ?$

$$a \bullet b = \|a\|\|b\|\cos\phi$$

$$\phi = \cos^{-1}\left(\frac{a \bullet b}{\|a\|\|b\|}\right)$$

Dot (scalar) product



$a \bullet b = b \bullet a = ?$

$a \bullet (b + c) = a \bullet b + a \bullet c$

$(ka) \bullet b = a \bullet (kb) = k(a \bullet b)$

$$a \bullet b = \|a\|\|b\|\cos\phi$$

$$\phi = \cos^{-1}\left(\frac{a \bullet b}{\|a\|\|b\|}\right)$$

Dot product in Cartesian components

$$a \bullet b = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \bullet \begin{pmatrix} x_b \\ y_b \end{pmatrix} = ?$$

Dot product in Cartesian components

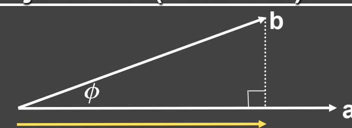
$$a \bullet b = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \bullet \begin{pmatrix} x_b \\ y_b \end{pmatrix} = ?$$

$$a \bullet b = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \bullet \begin{pmatrix} x_b \\ y_b \end{pmatrix} = x_a x_b + y_a y_b$$

Dot product: some applications in CG

- Find angle between two vectors (e.g. cosine of angle between light source and surface for shading)
- Finding projection of one vector on another (e.g. coordinates of point in arbitrary coordinate system)
- Advantage: computed easily in cartesian components

Projections (of b on a)



$$\left\| b \to a \right\| = ?$$

$$b \to a \quad = ?$$

$$\left\| b \to a \right\| = \left\| b \right\| \cos\phi = \frac{a \bullet b}{\left\| a \right\|}$$

$$b \to a = \left\| b \to a \right\| \frac{a}{\left\| a \right\|} = \frac{a \bullet b}{\left\| a \right\|^2} a$$

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Vectors: Cross Products

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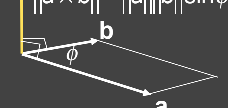
Vector Multiplication

- Dot *product*
- *Cross product*
- Orthonormal bases and coordinate frames
- Note: We use right-handed (standard) coordinates

Cross (vector) product

$$a \times b = -b \times a$$

$$\left\| a \times b \right\| = \left\| a \right\| \left\| b \right\| \sin\phi$$



- Cross product orthogonal to two initial vectors
- Direction determined by right-hand rule
- Useful in constructing coordinate systems (later)

Cross product: Properties

$$x \times y = +z$$
$$y \times x = -z$$
$$y \times z = +x$$
$$z \times y = -x$$
$$z \times x = +y$$
$$x \times z = -y$$

$$a \times b = -b \times a$$
$$a \times a = 0$$
$$a \times (b + c) = a \times b + a \times c$$
$$a \times (kb) = k(a \times b)$$

Cross product: Cartesian formula?

Cross product: Cartesian formula?

$$a \times b = \begin{vmatrix} x & y & z \\ x_a & y_a & z_a \\ x_b & y_b & z_b \end{vmatrix} = \begin{pmatrix} y_a z_b - y_b z_a \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{pmatrix}$$

$$a \times b = A^* b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

Dual matrix of vector a

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Vectors: Orthonormal Basis Frames

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Vector Multiplication

- Dot *product*
- Cross product
- *Orthonormal bases and coordinate frames*

- Note: We use right-handed (standard) coordinates

Orthonormal bases/coordinate frames

- Important for representing points, positions, locations

- Often, many sets of coordinate systems (not just X, Y, Z)
  - Global, local, world, model, parts of model (head, hands, …)

- Critical issue is transforming between these systems/bases
  - Topic of next 3 lectures

Coordinate Frames

- Any set of 3 vectors (in 3D) so that

$$\|u\| = \|v\| = \|w\| = 1$$

$$u \bullet v = v \bullet w = u \bullet w = 0$$

$$w = u \times v$$

$$p = (p \bullet u)u + (p \bullet v)v + (p \bullet w)w$$

Constructing a coordinate frame

- Often, given a vector **a** (viewing direction in HW1), want to construct an orthonormal basis
- Need a second vector **b** (up direction of camera in HW1)
- Construct an orthonormal basis (for instance, camera coordinate frame to transform world objects into in HW1)

Constructing a coordinate frame?

We want to associate **w** with **a**, and **v** with **b**

- But **a** and **b** are neither orthogonal nor unit norm
- And we also need to find **u**

Constructing a coordinate frame?

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$$w = \frac{a}{\|a\|}$$

Constructing a coordinate frame?

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$$w = \frac{a}{\|a\|}$$

$$u = \frac{b \times w}{\|b \times w\|}$$

Constructing a coordinate frame?

We want to associate **w** with **a**, and **v** with **b**

- But **a** and **b** are neither orthogonal nor unit norm
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$$w = \frac{a}{\|a\|}$$

$$u = \frac{b \times w}{\|b \times w\|}$$

$$v = w \times u$$

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Matrices

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Matrices

- Can be used to transform points (vectors)
  - Translation, rotation, shear, scale
    (more detail next lecture)

What is a matrix

- Array of numbers (m×n = m rows, n columns)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix}$$

- Addition, multiplication by a scalar simple: element by element

Matrix-matrix multiplication

- Number of columns in first must = rows in second

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix}$$

- Element (i,j) in product is dot product of row i of first matrix and column j of second matrix

Matrix-matrix multiplication

- Number of columns in first must = rows in second

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix}=\begin{pmatrix} 9 & 27 & 33 & 13 \\ 19 & 44 & 61 & 26 \\ 8 & 28 & 32 & 12 \end{pmatrix}$$

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$$\begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix} \text{ NOT EVEN LEGAL!!}$$

- Non-commutative (AB and BA are different in general)
- Associative and distributive
  - A(B+C) = AB + AC
  - (A+B)C = AC + BC

Matrix-Vector Multiplication

- Key for transforming points (next lecture)
- Treat vector as a column matrix (m×1)
- E.g. 2D reflection about y-axis

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

Transpose of a Matrix (or vector?)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

$$(AB)^T = B^T A^T$$

Identity Matrix and Inverses

$$I_{3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$AA^{-1} = A^{-1}A = I$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Vector multiplication in Matrix form

- Dot product? $a \bullet b = a^T b$

$$\begin{pmatrix} x_a & y_a & z_a \end{pmatrix}\begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = \begin{pmatrix} x_a x_b + y_a y_b + z_a z_b \end{pmatrix}$$

- Cross product? $a \times b = A^* b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix}\begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$

Dual matrix of vector a