ITMO UNIVERSITY

How to Win Coding Competitions: Secrets of Champions

Week 3: Sorting and Search Algorithms
Lecture 4: Quicksort

Maxim Buzdalov
Saint Petersburg 2016

Previous sorting algorithm: Insertion sort

- Incremental: size of the sorted part increases by one each time
- Can only swap adjacent elements
- Running time: $\Omega(N)$, $O(N^2)$, $\Theta(N^2)$ on average

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- Split the array into two parts L and R, such that $L_i \leq R_j$ for all i and j
- Sort the parts recursively $\rightarrow$ the entire array is sorted!

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- For best results, these parts should be approximately equal

procedure $\textsc{Quicksort}(A, \prec, s, e)$
  $s' \leftarrow s,\ e' \leftarrow e,\ M \leftarrow A[(s+e)/2]$
  while $s' \le e'$ **do**
    while $A[s'] \prec M$ **do** $s' \leftarrow s' + 1$ **end while**
    while $M \prec A[e']$ **do** $e' \leftarrow e' - 1$ **end while**
    if $s' \le e'$ **then**
      $A[s'] \Leftrightarrow A[e']$
      $s' \leftarrow s' + 1,\ e' \leftarrow e' - 1$
    end if
  end while
  if $s \le e'$ **then** $\textsc{Quicksort}(A, \prec, s, e')$ **end if**
  if $s' \le e$ **then** $\textsc{Quicksort}(A, \prec, s', e)$ **end if**
end procedure

procedure $\textsc{Quicksort}(A, \prec, s, e)$
    $s' \leftarrow s$, $e' \leftarrow e$, $M \leftarrow A[(s+e)/2]$ $\triangleright$ M: the pivot value.
    while $s' \le e'$ **do**
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    $s' \leftarrow s$, $e' \leftarrow e$, $M \leftarrow A[(s+e)/2]$ ▷ M: the pivot value. Selection may vary
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    while $s' \le e'$ **do**
        while $A[s'] \prec M$ **do** $s' \leftarrow s' + 1$ **end while** $\triangleright$ If $i \in [s; s')$ then $A[i] \preceq M$
        while $M \prec A[e']$ **do** $e' \leftarrow e' - 1$ **end while**
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end procedure

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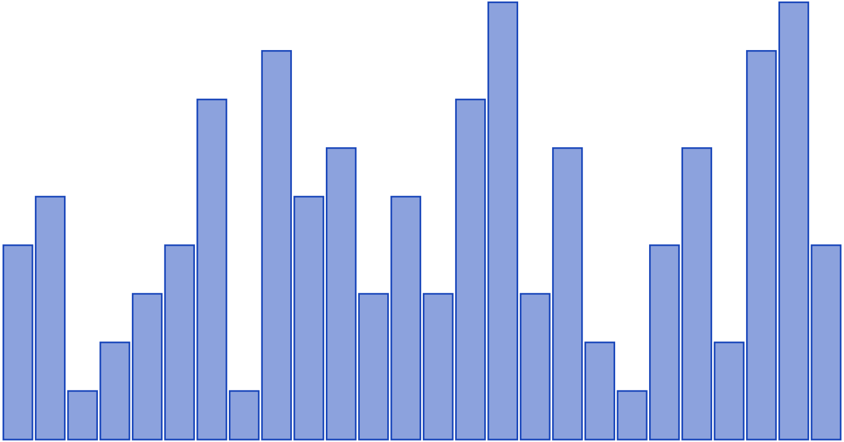
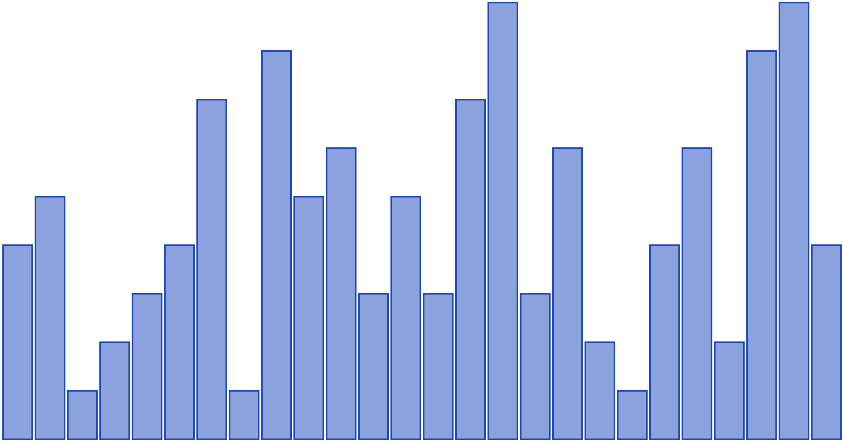
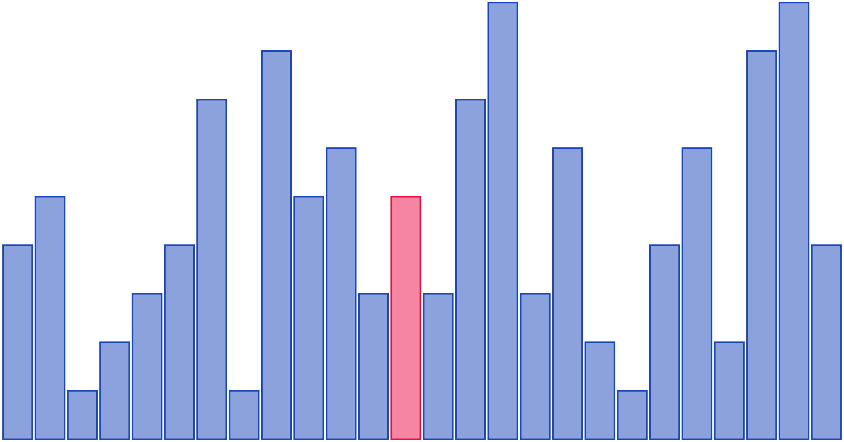
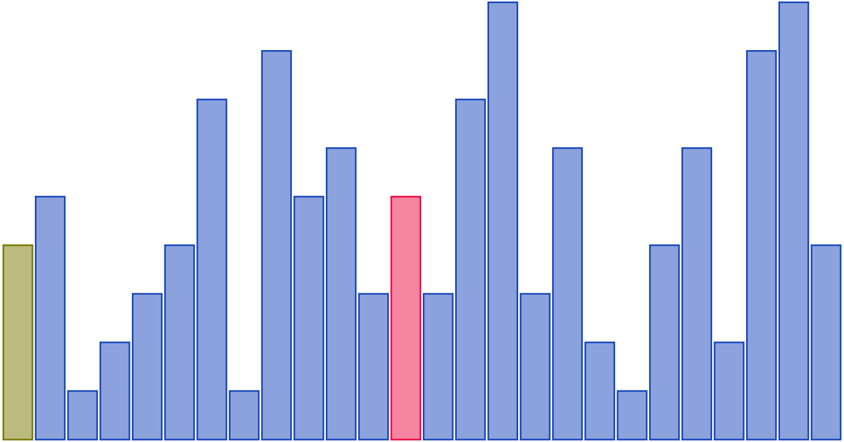
        if $s' \le e'$ **then** ▷ If the array is not yet split completely...
            $A[s'] \Leftrightarrow A[e']$ ▷ swap the elements and continue splitting
            $s' \leftarrow s' + 1,\ e' \leftarrow e' - 1$
        end if
    end while
    if $s \le e'$ **then** $\textsc{Quicksort}(A, \prec, s, e')$ **end if**
    if $s' \le e$ **then** $\textsc{Quicksort}(A, \prec, s', e)$ **end if**
end procedure

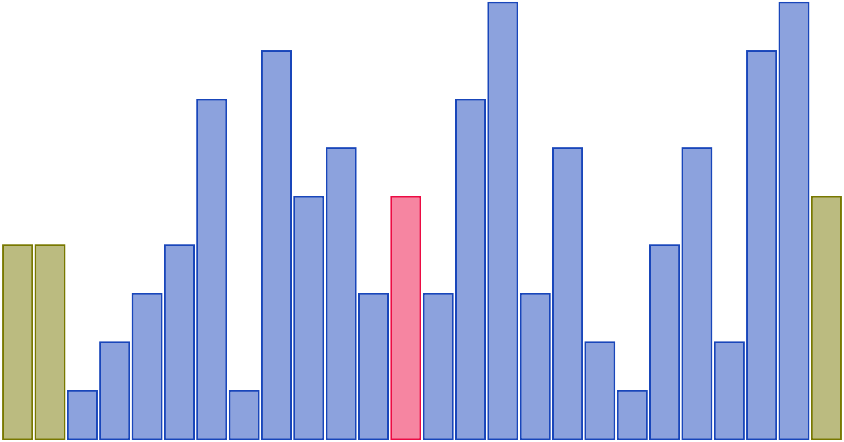
procedure $\textsc{Quicksort}(A, \prec, s, e)$
    $s' \leftarrow s,\ e' \leftarrow e,\ M \leftarrow A[(s+e)/2]$ ▷ M: the pivot value. Selection may vary
    while $s' \le e'$ **do**
        while $A[s'] \prec M$ **do** $s' \leftarrow s'+1$ **end while** ▷ If $i \in [s; s')$ then $A[i] \preceq M$
        while $M \prec A[e']$ **do** $e' \leftarrow e'-1$ **end while** ▷ If $i \in (e'; e]$ then $M \preceq A[i]$
        if $s' \le e'$ **then** ▷ If the array is not yet split completely...
            $A[s'] \Leftrightarrow A[e']$ ▷ swap the elements and continue splitting
            $s' \leftarrow s'+1,\ e' \leftarrow e'-1$
        end if
    end while
    if $s \le e'$ **then** $\textsc{Quicksort}(A, \prec, s, e')$ **end if** ▷ If $i \in (e'; s')$, $A[i] = M$
    if $s' \le e$ **then** $\textsc{Quicksort}(A, \prec, s', e)$ **end if** ▷ and is in the right place
end procedure

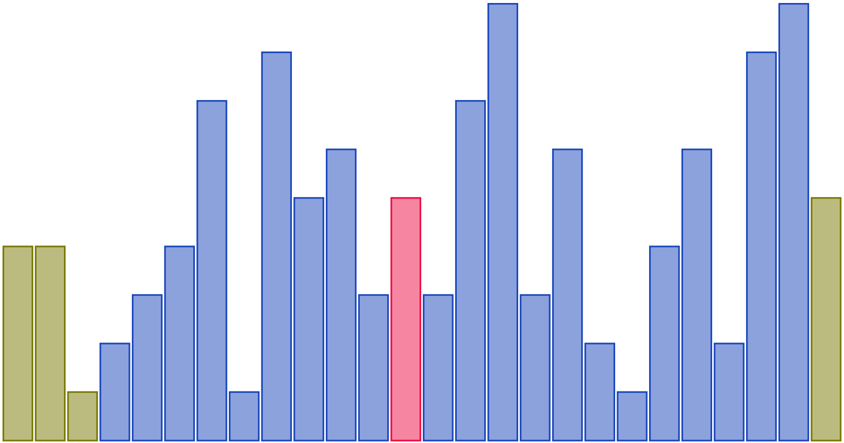


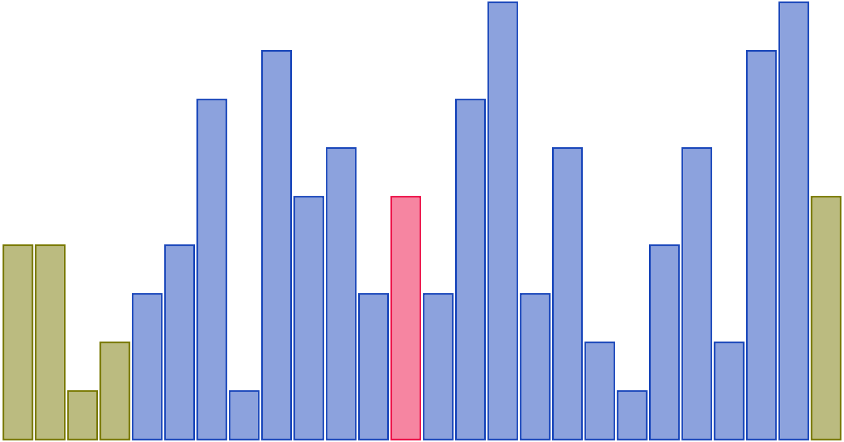


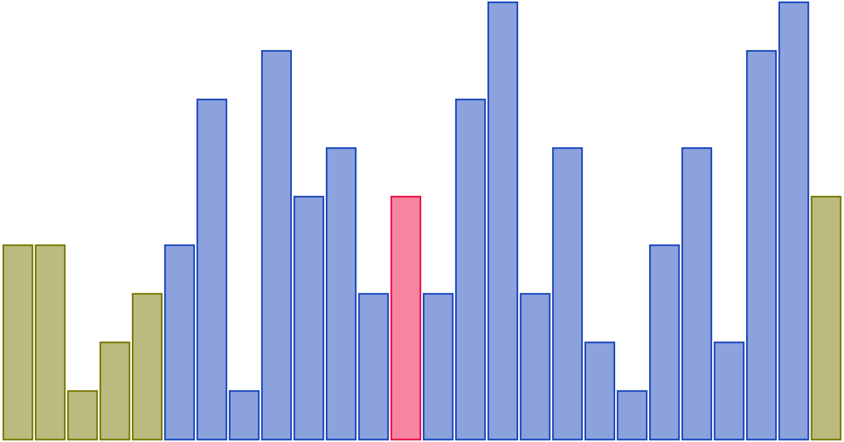


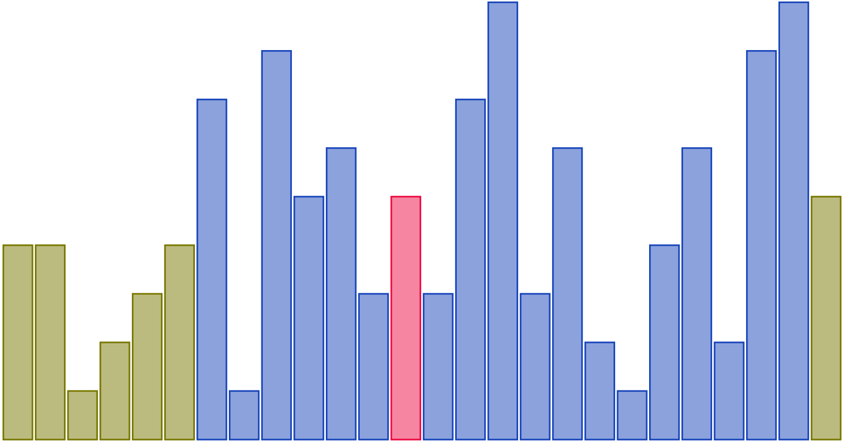


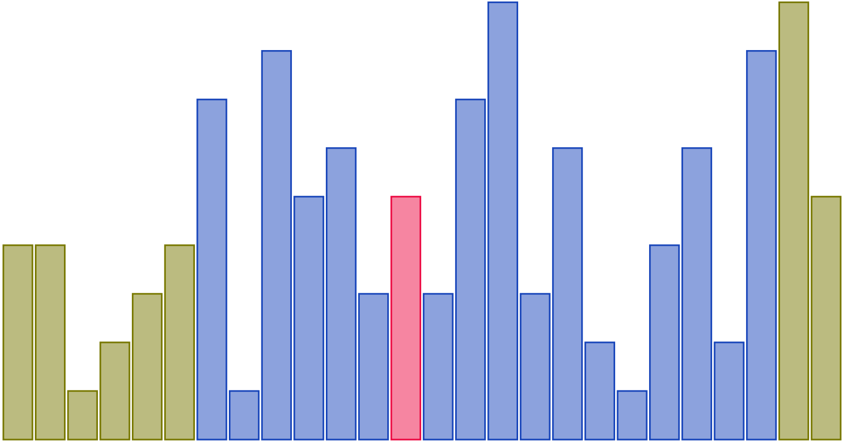


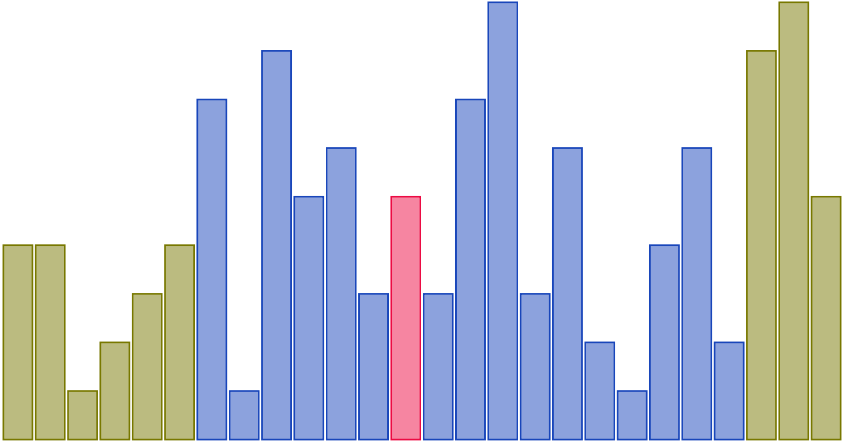


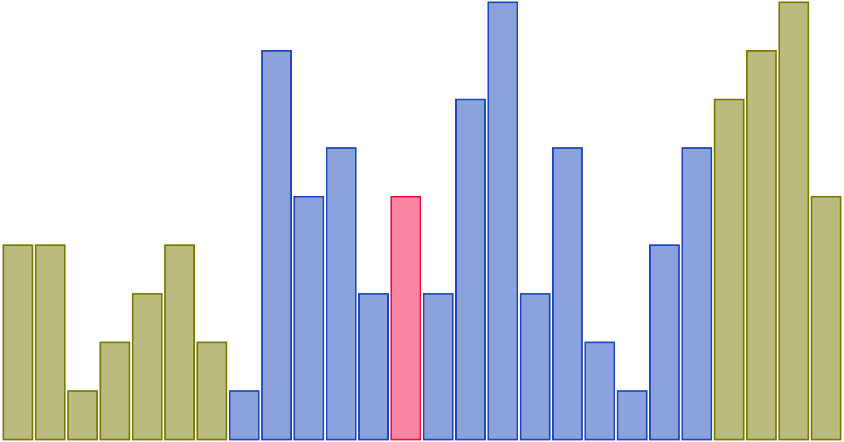


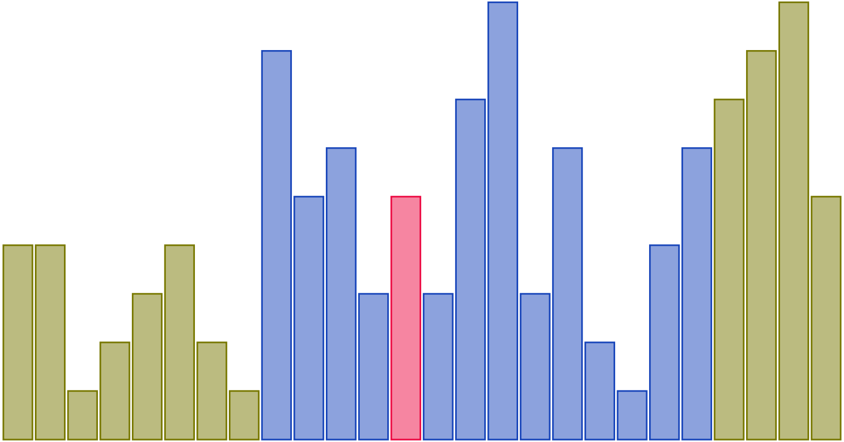


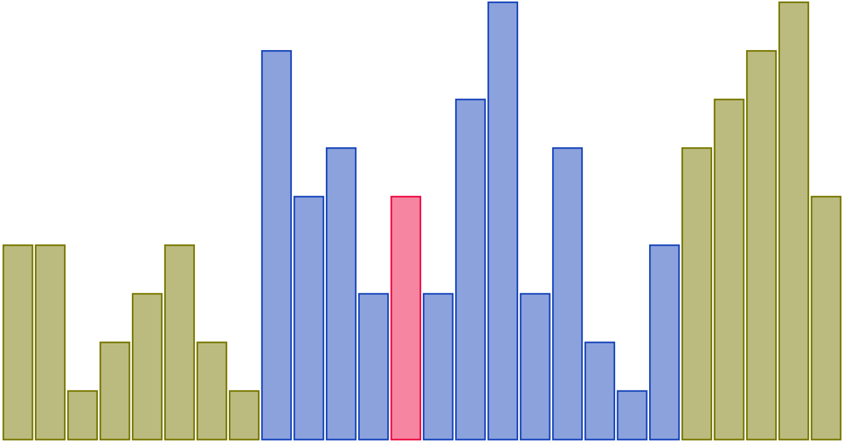


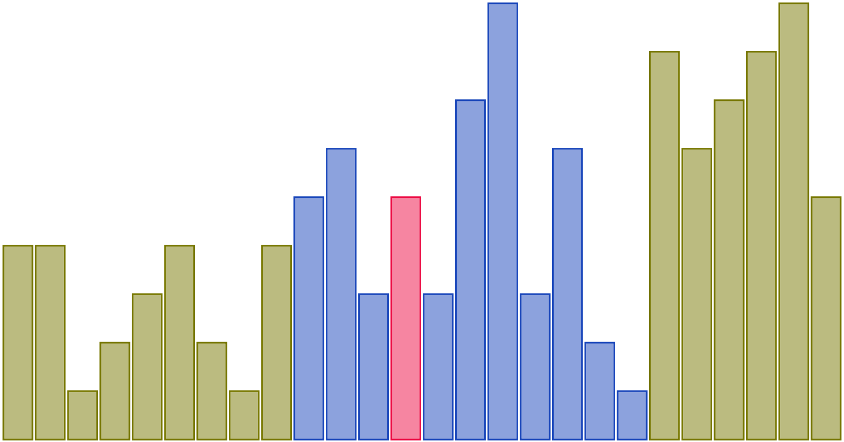


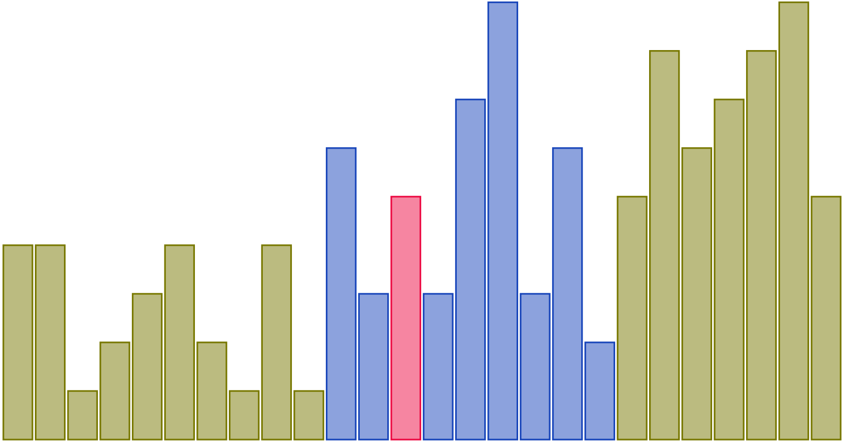


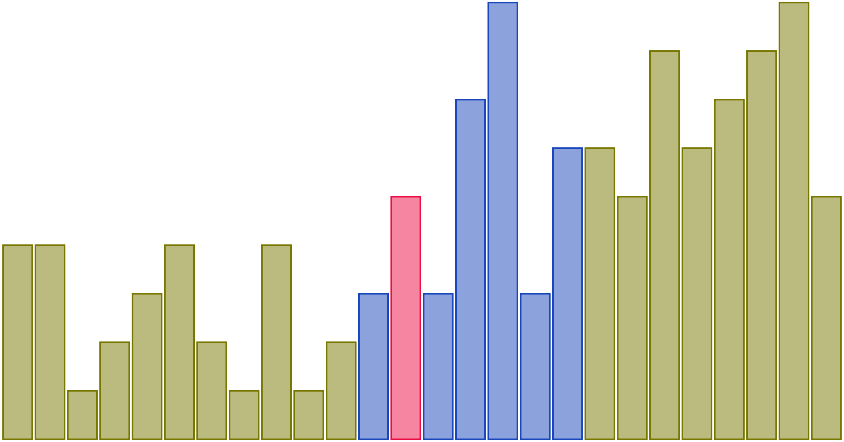


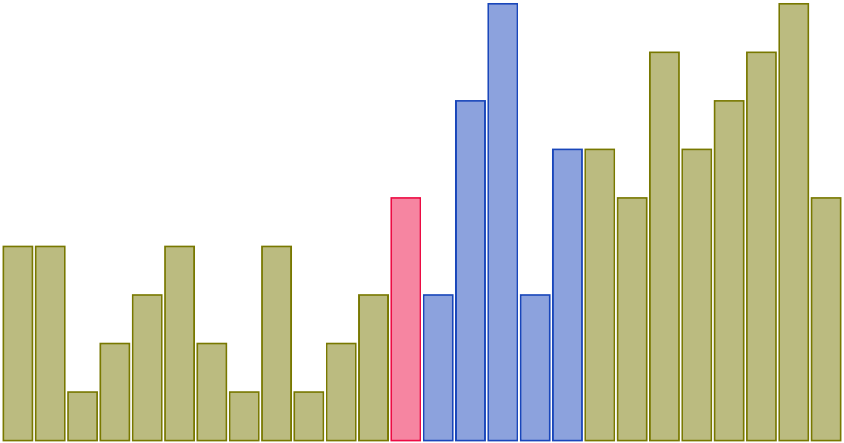


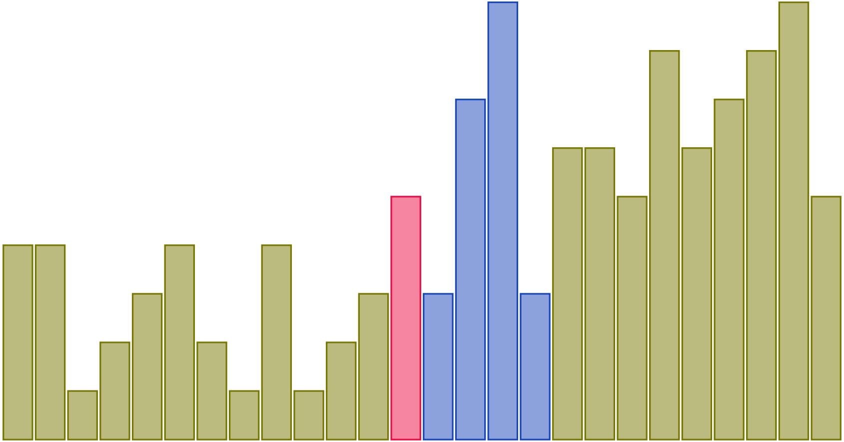


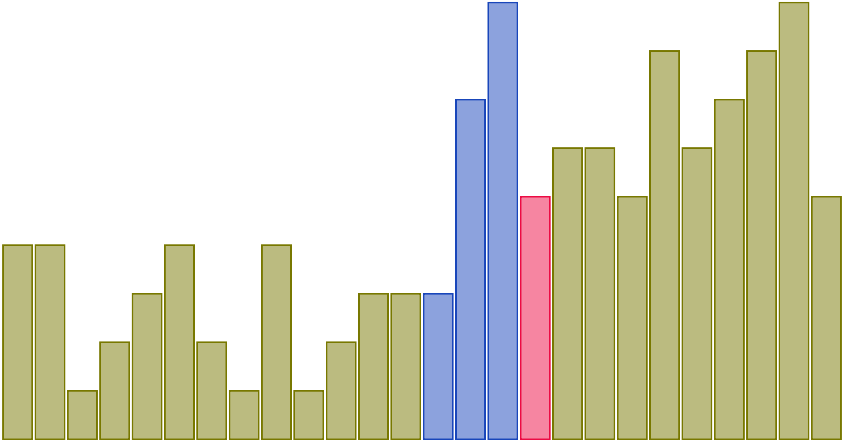


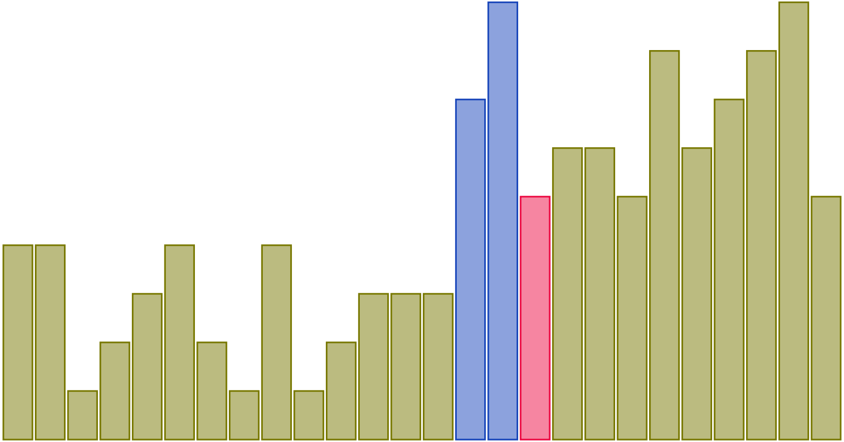


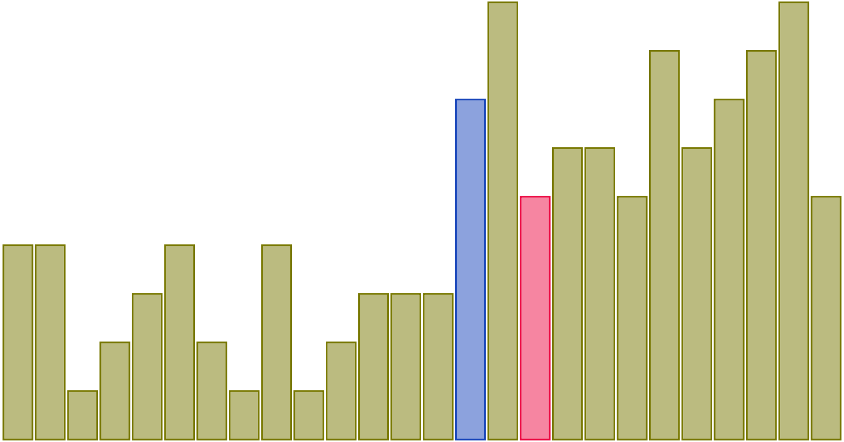




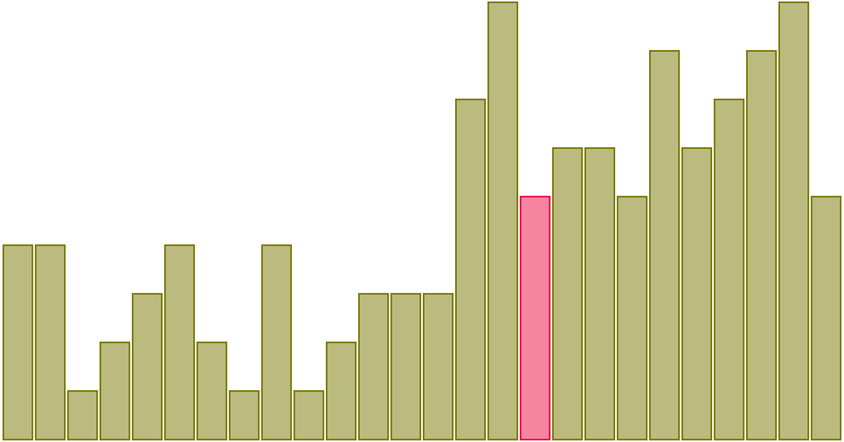


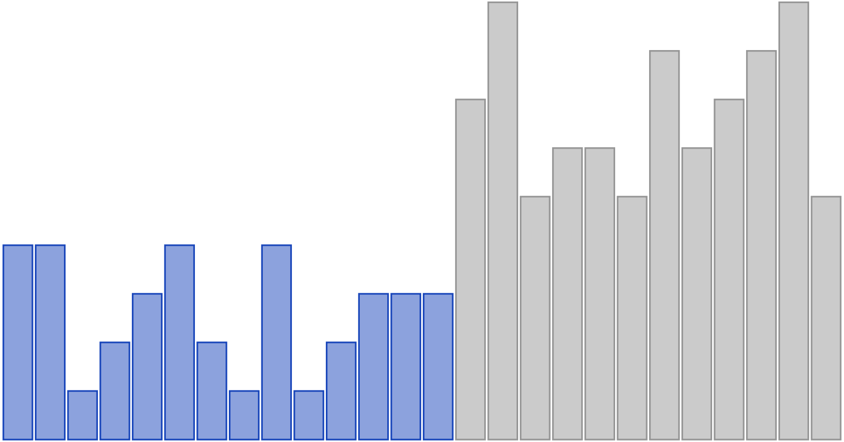


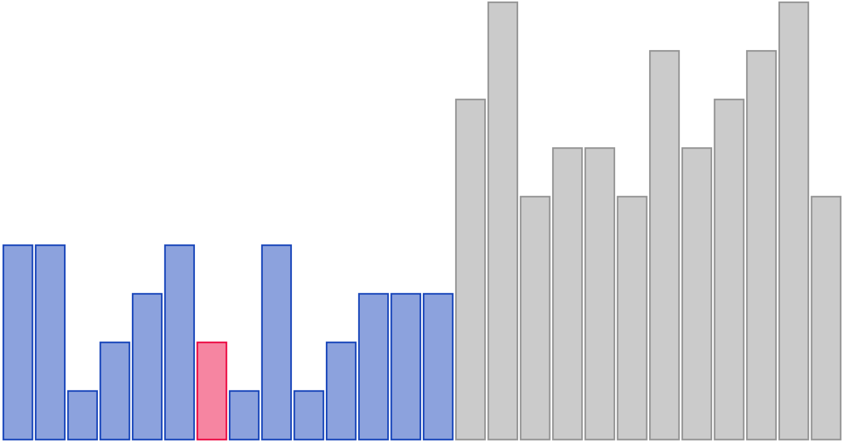


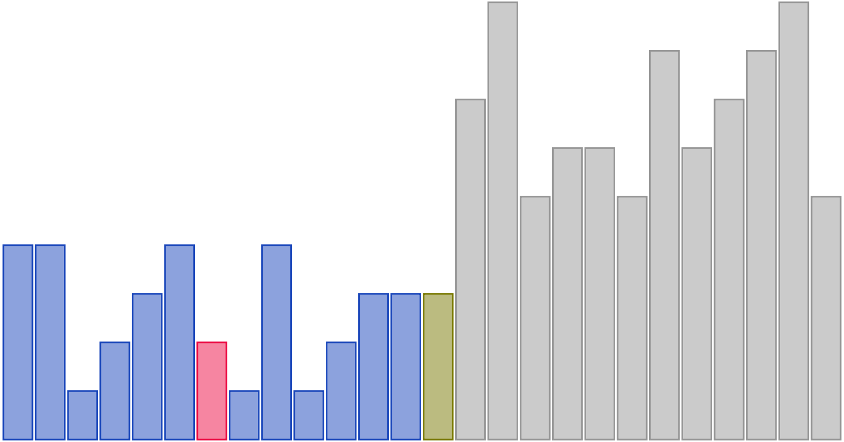


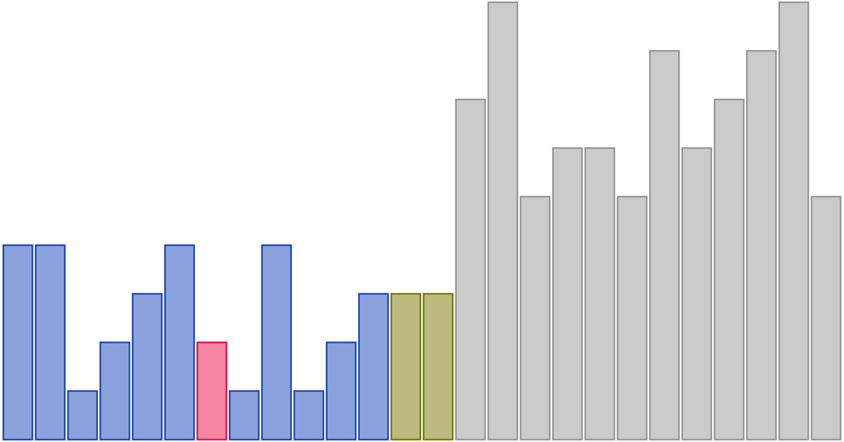
Quicksort: Example

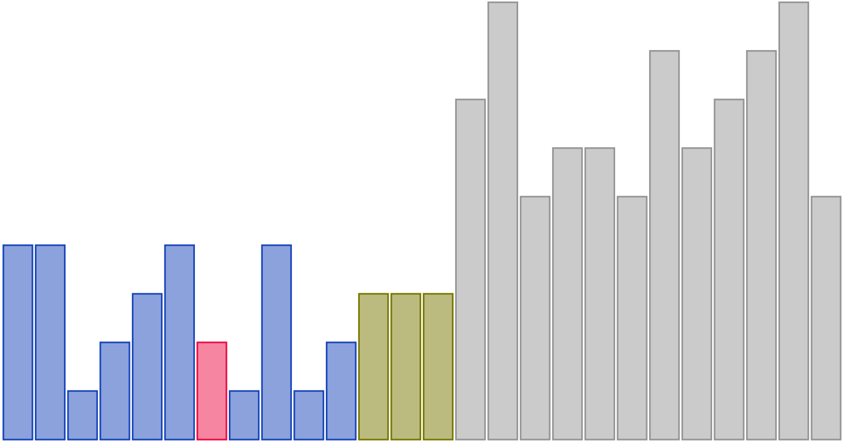


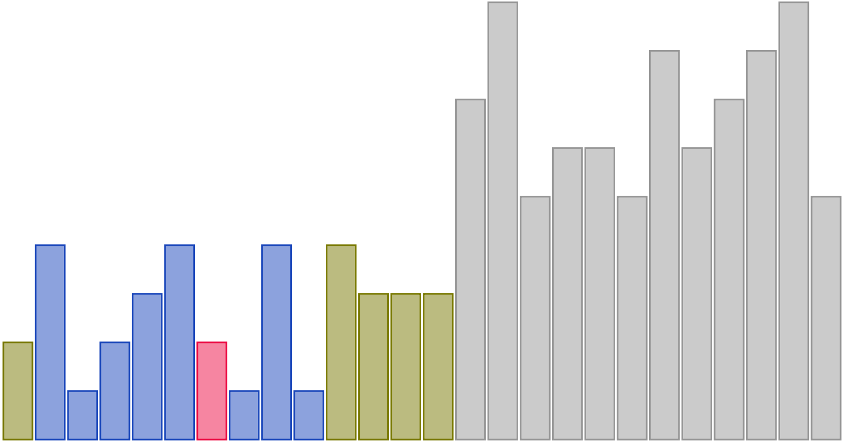


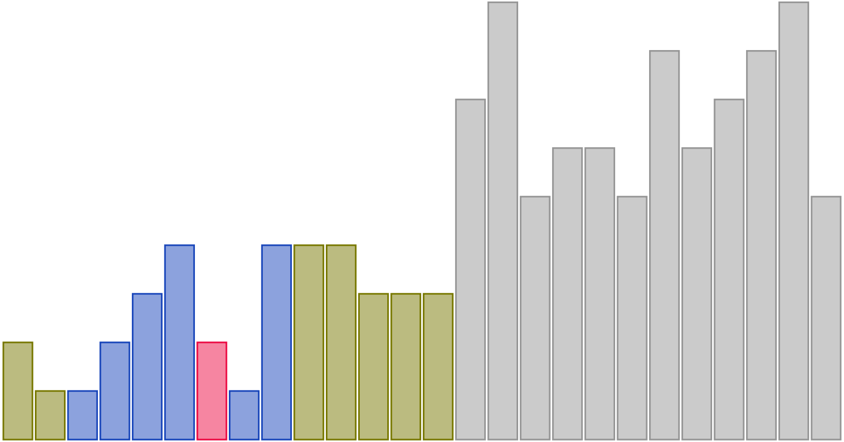


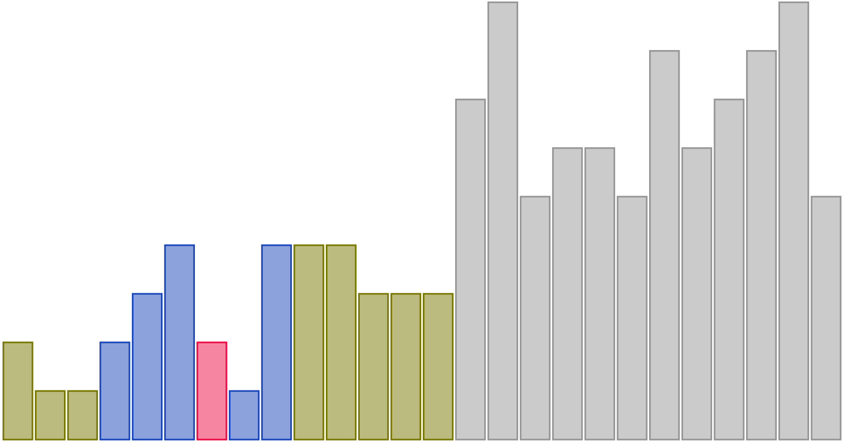


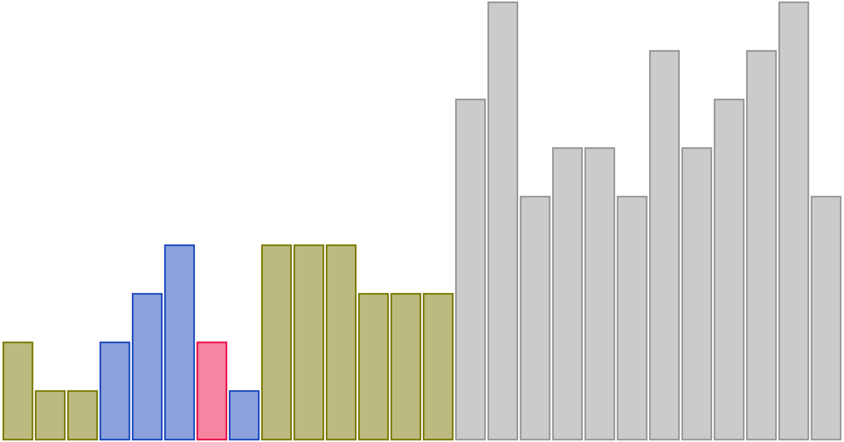


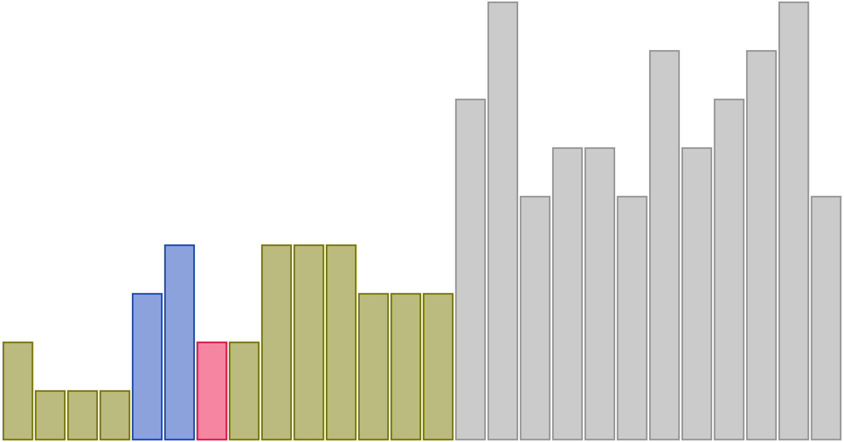


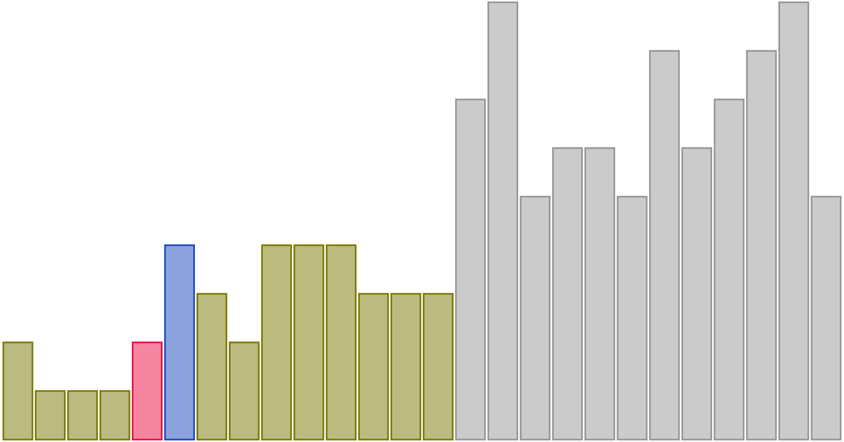


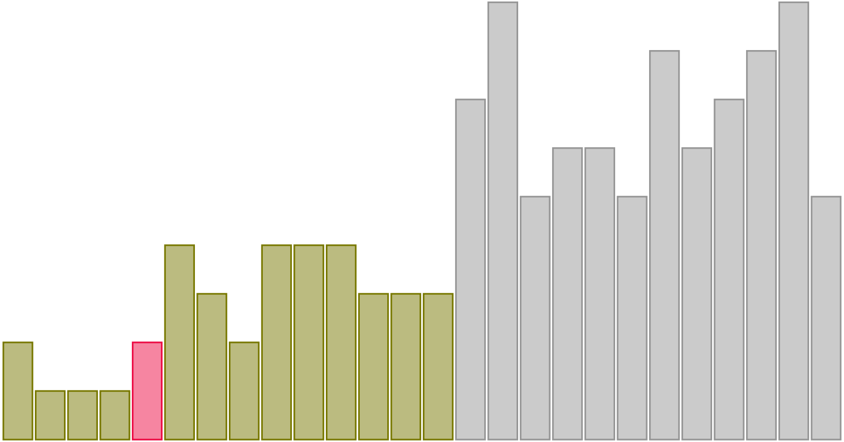


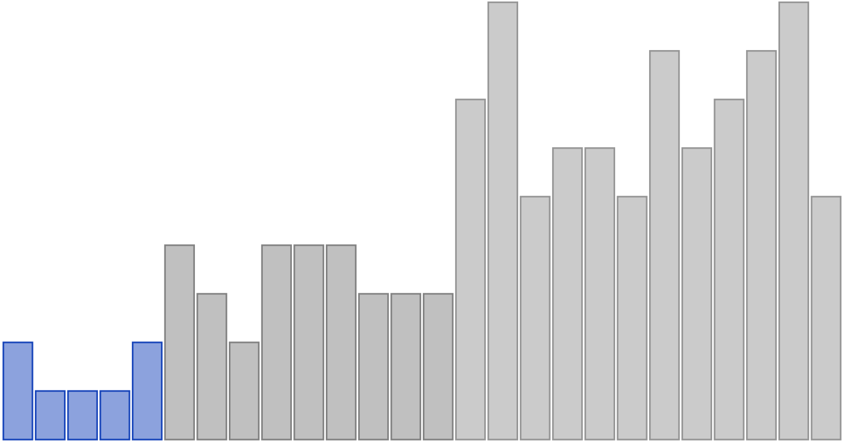


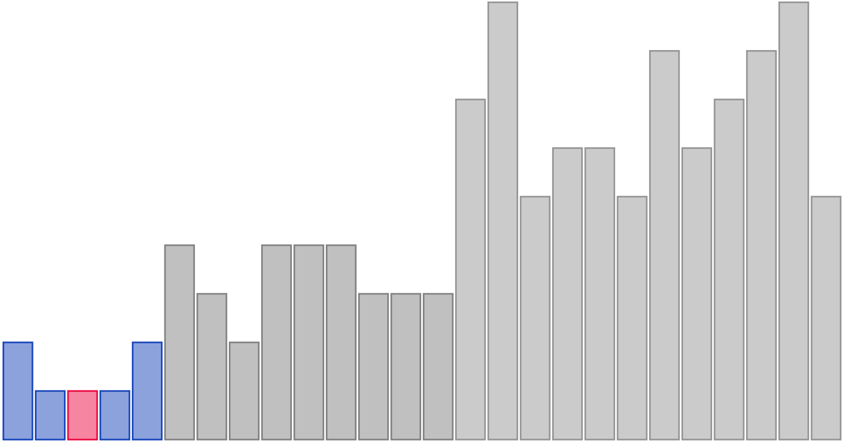


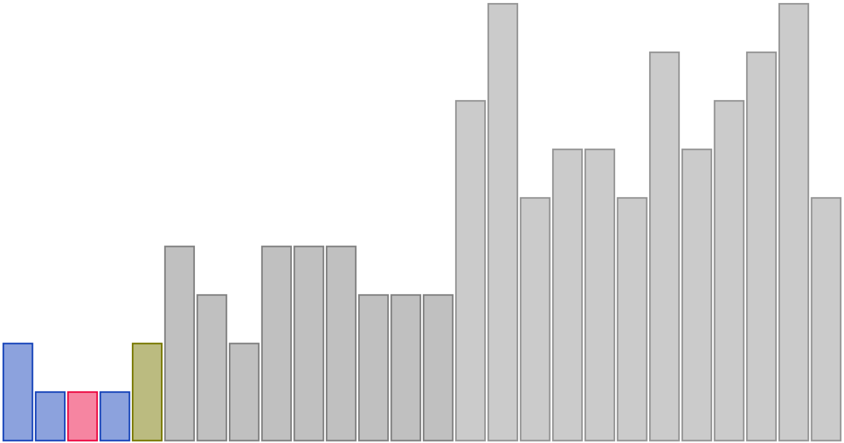


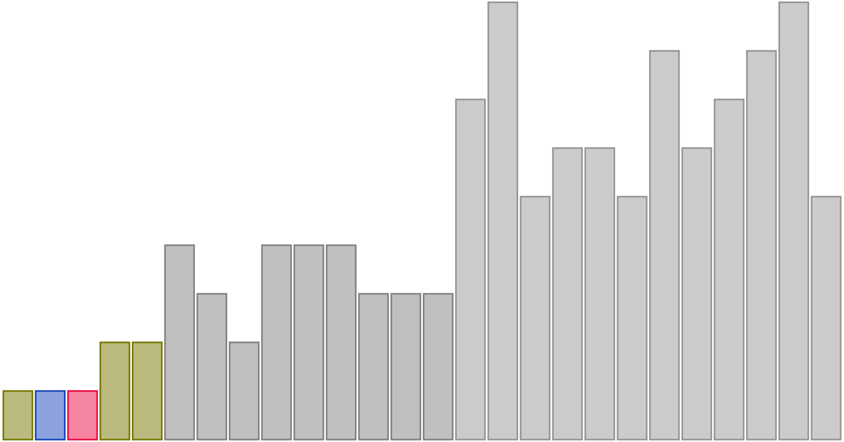


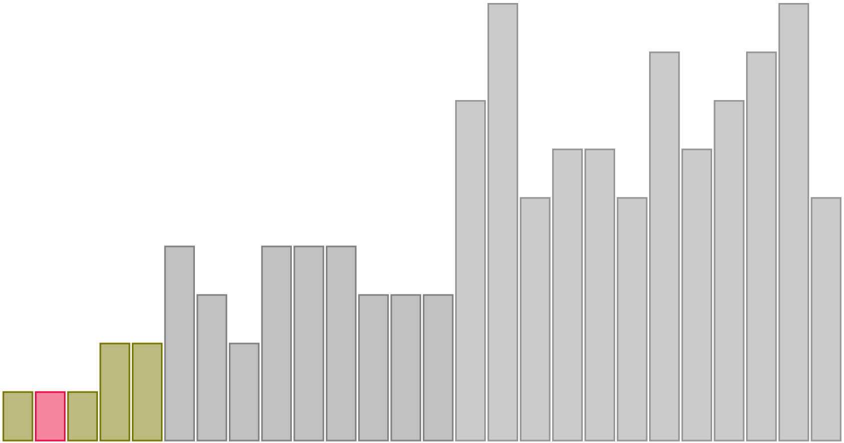


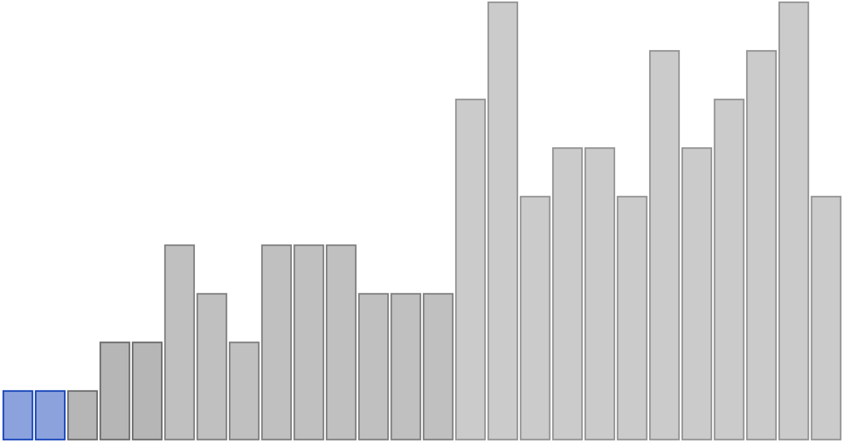


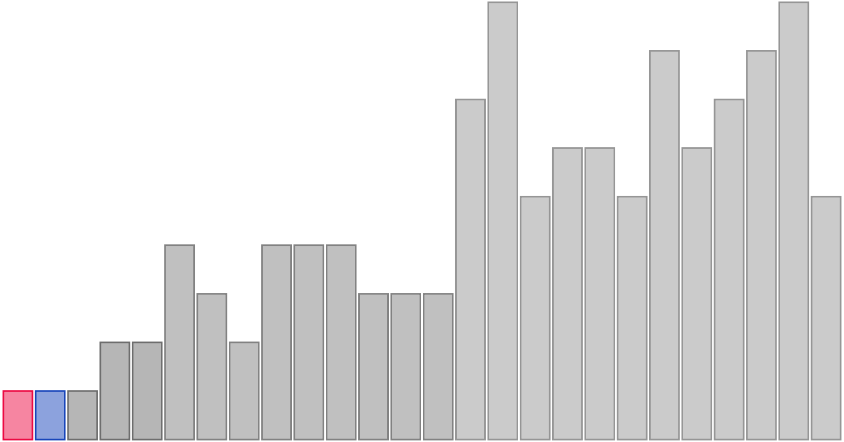


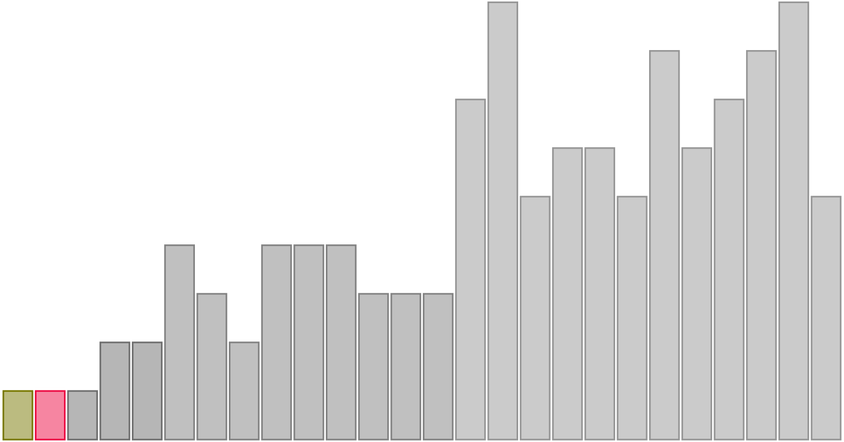


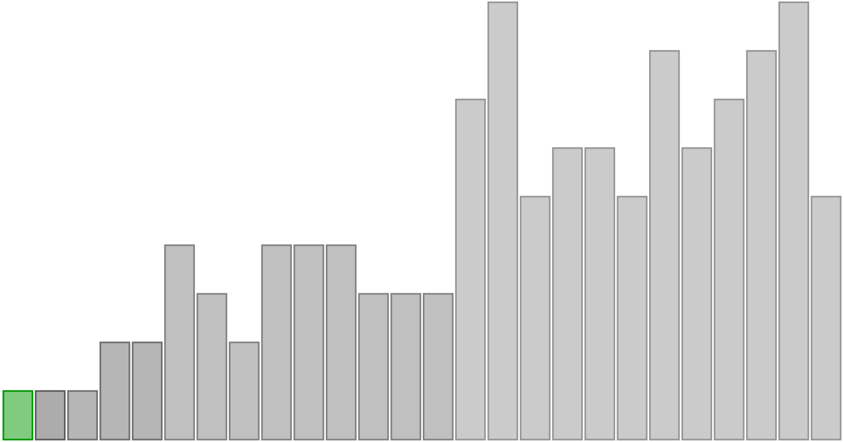


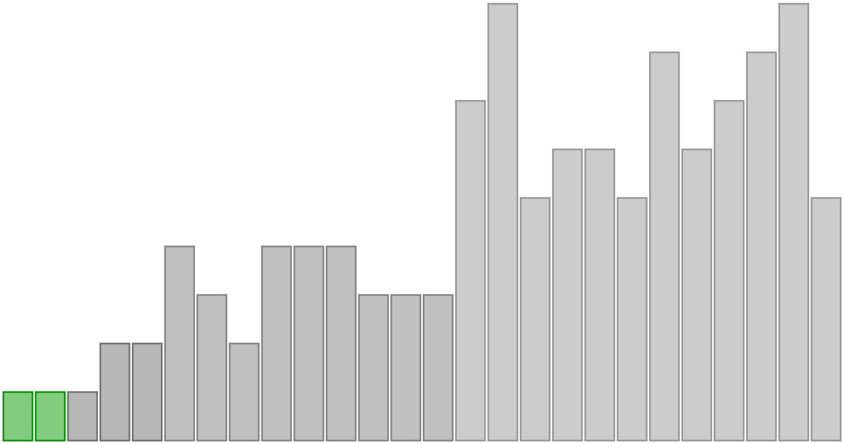


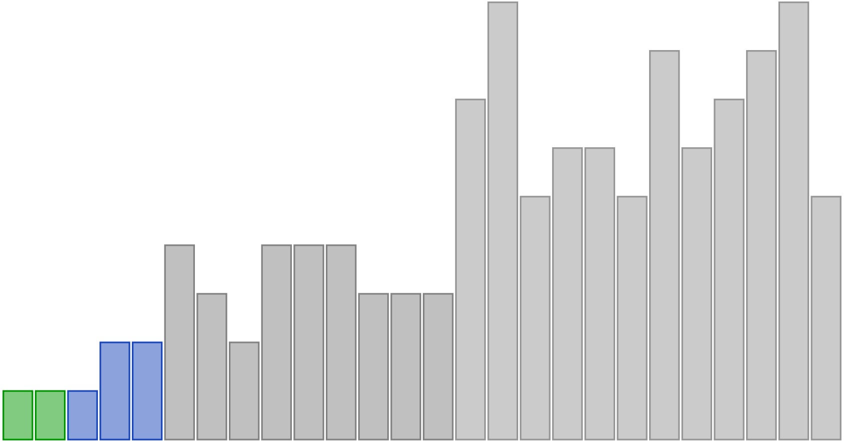


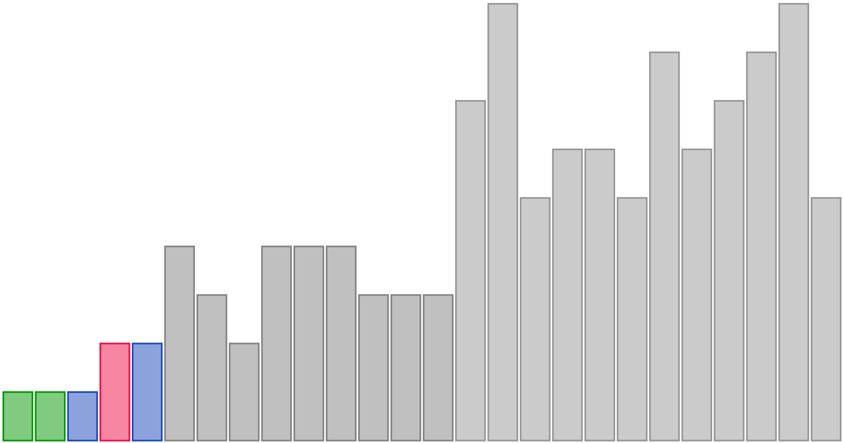


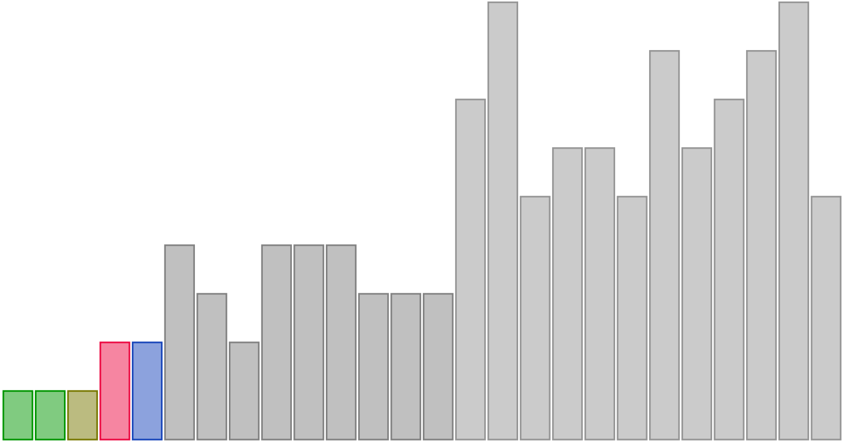


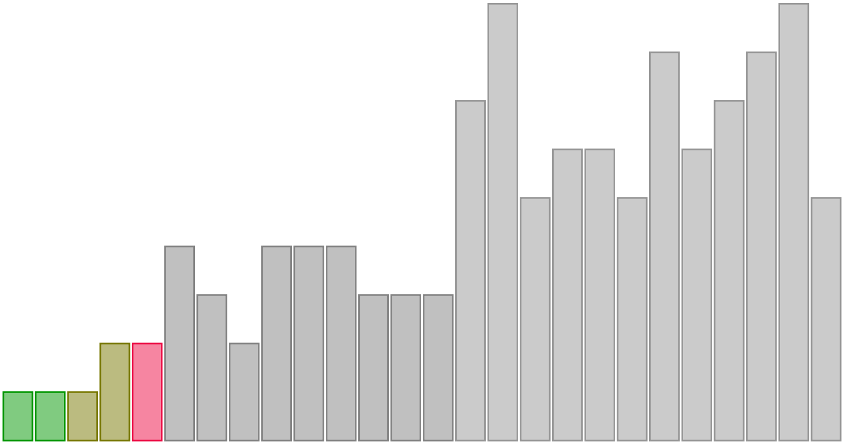


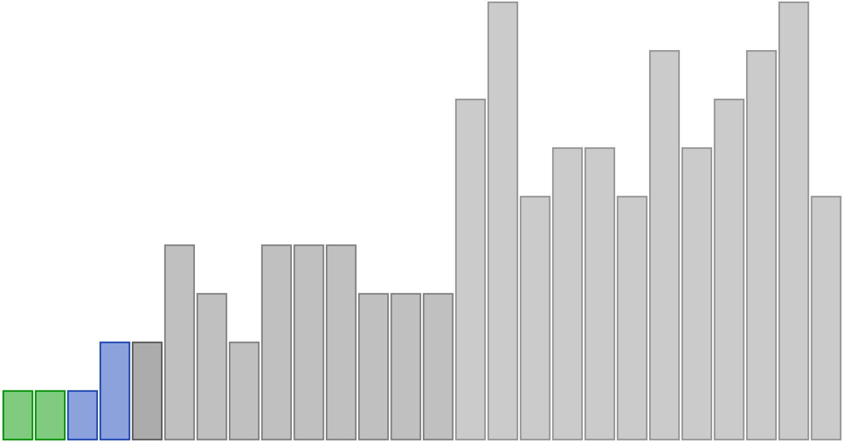


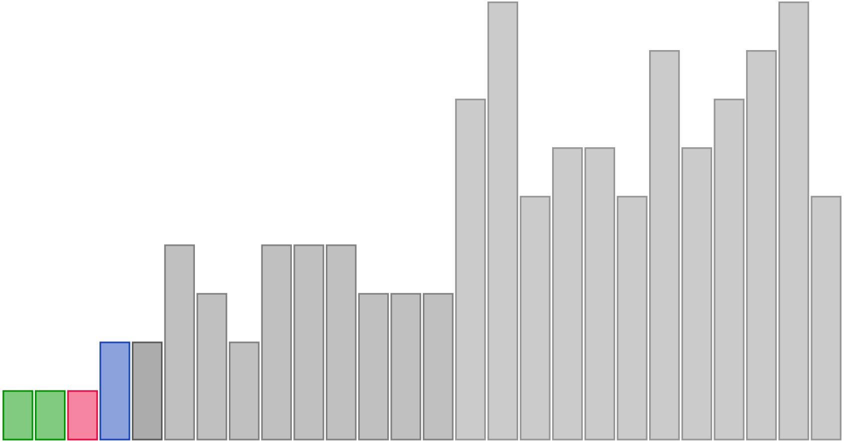


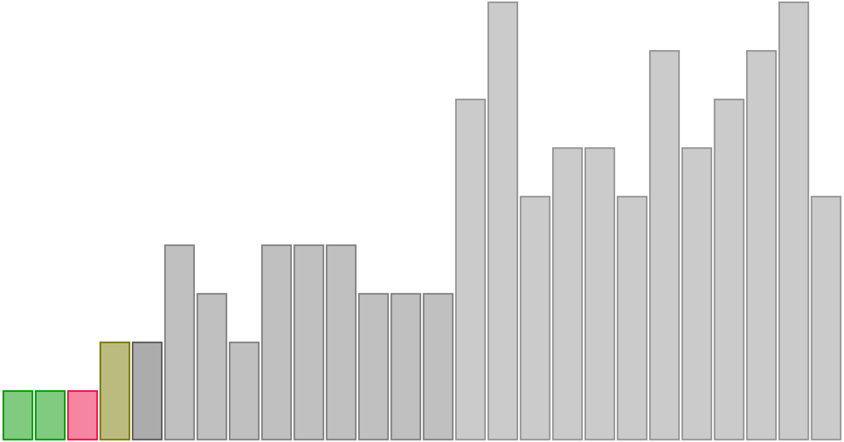


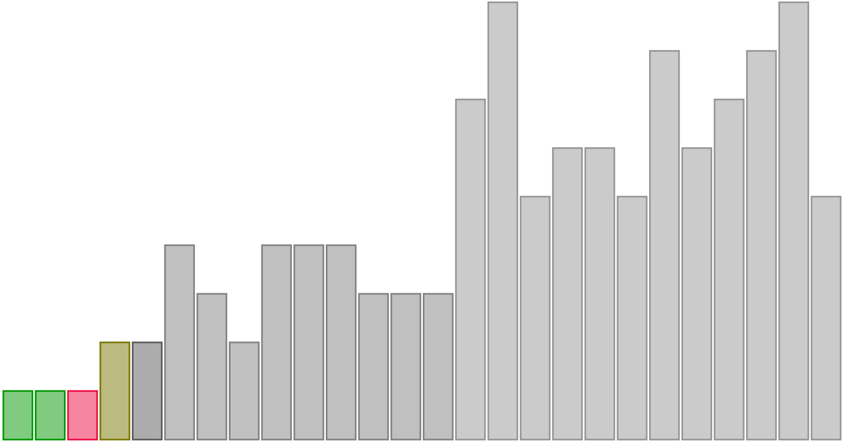


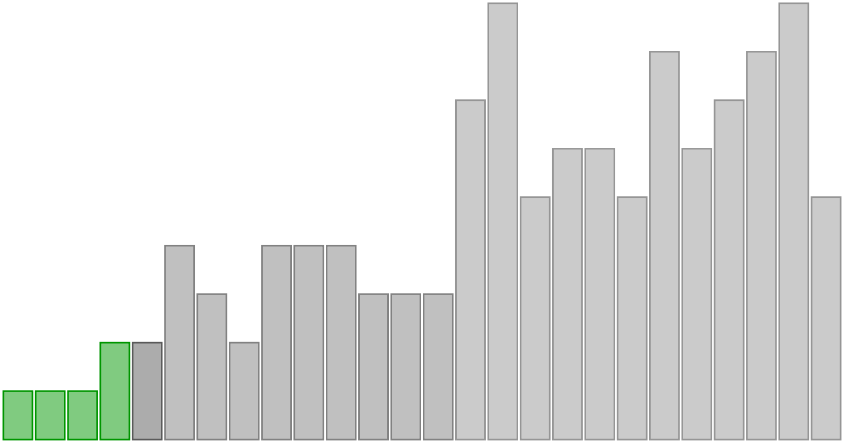


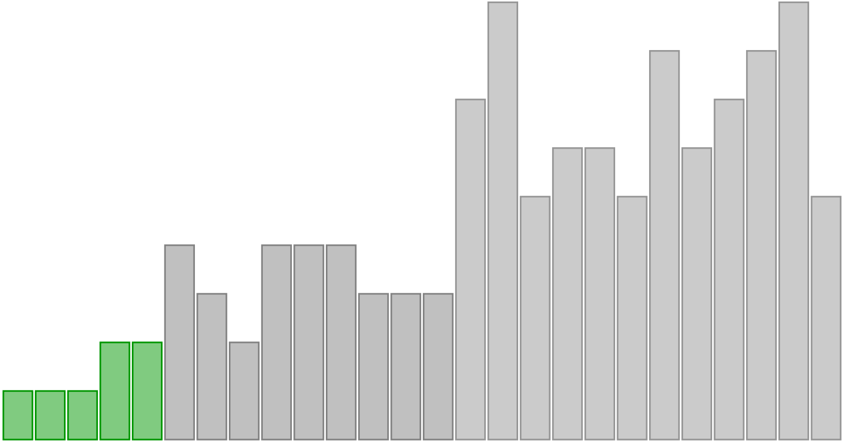


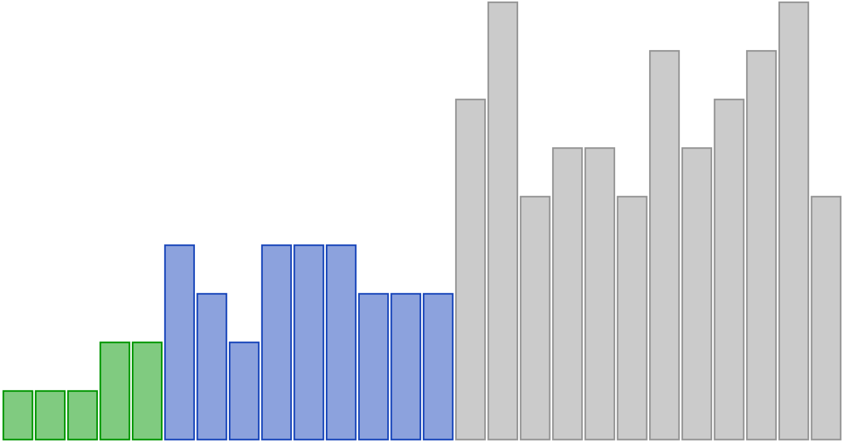


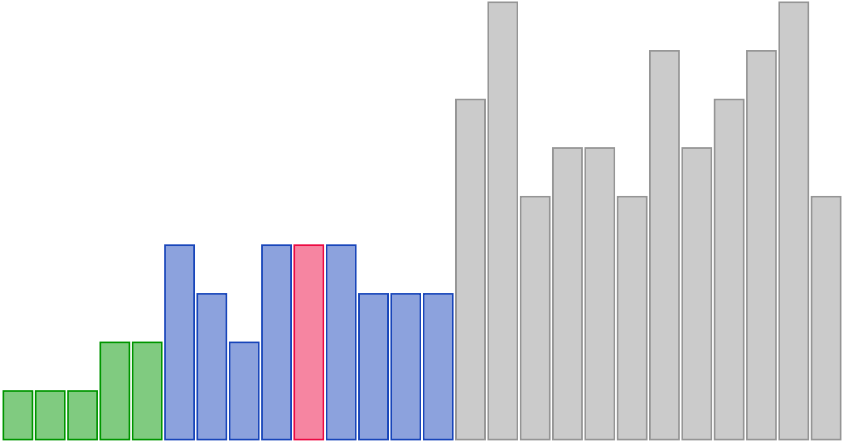


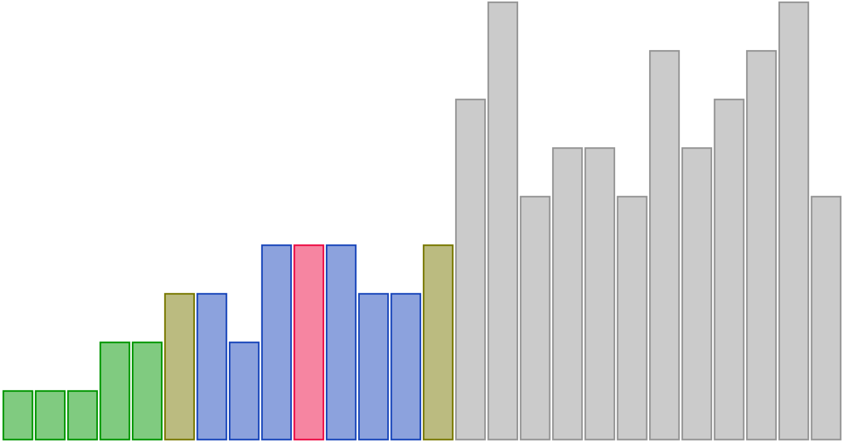


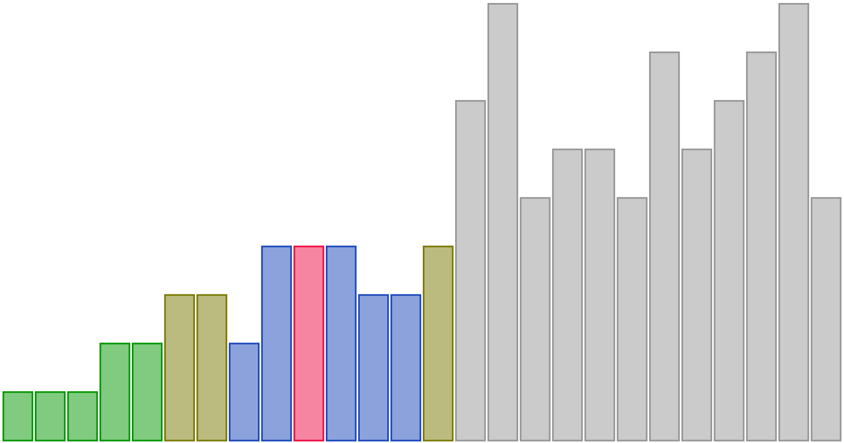


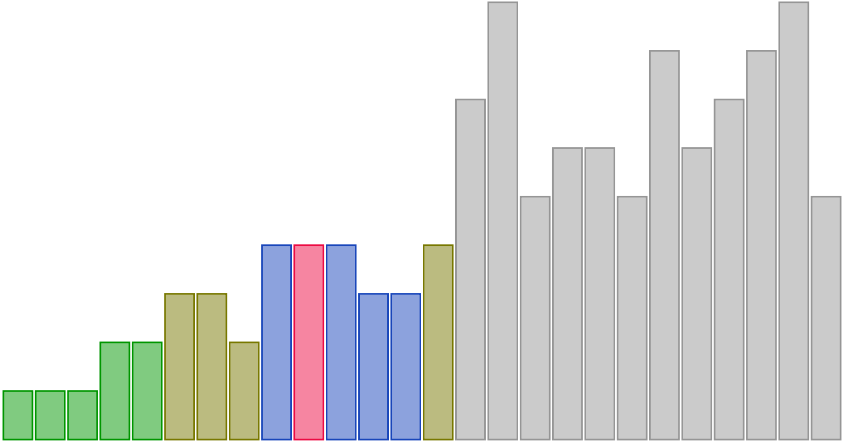


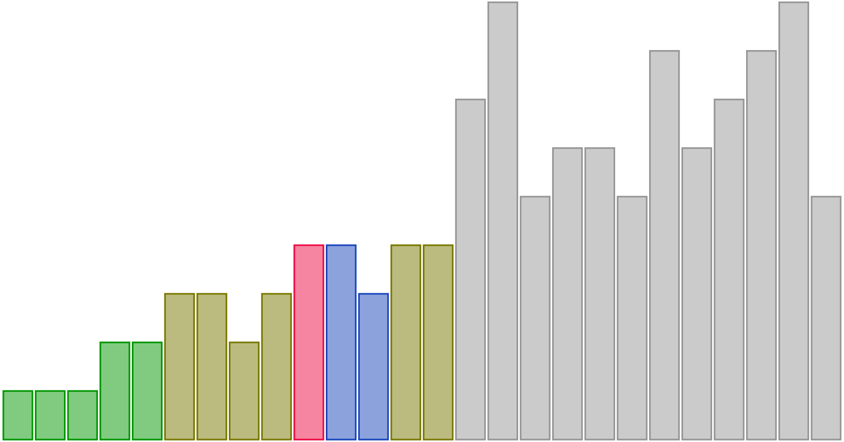


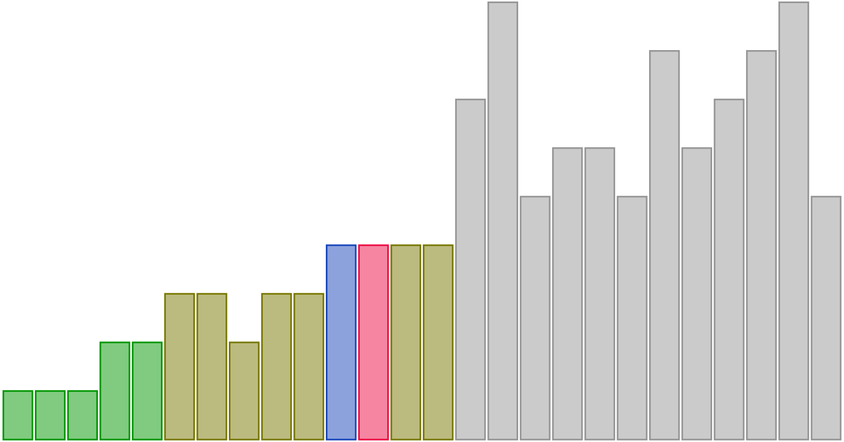


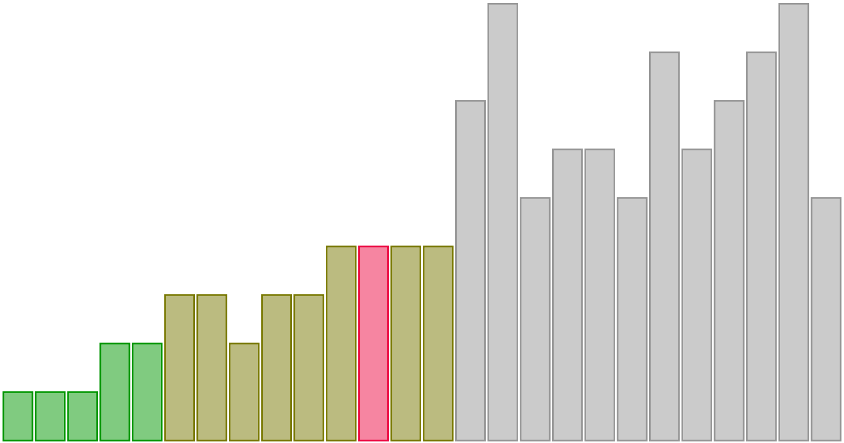


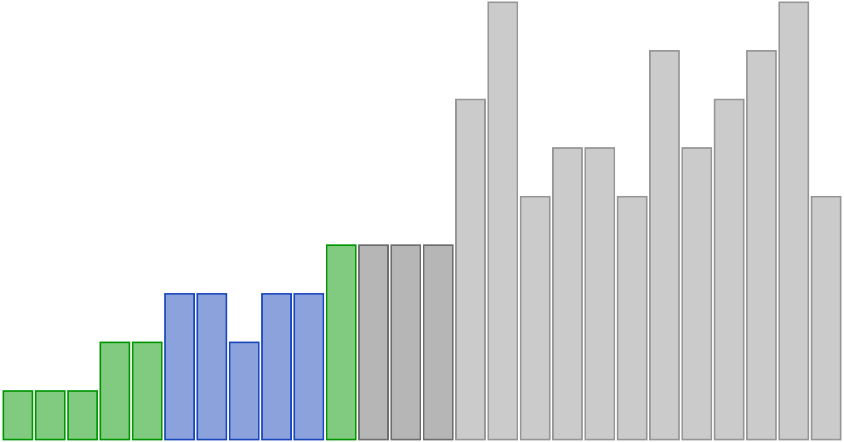


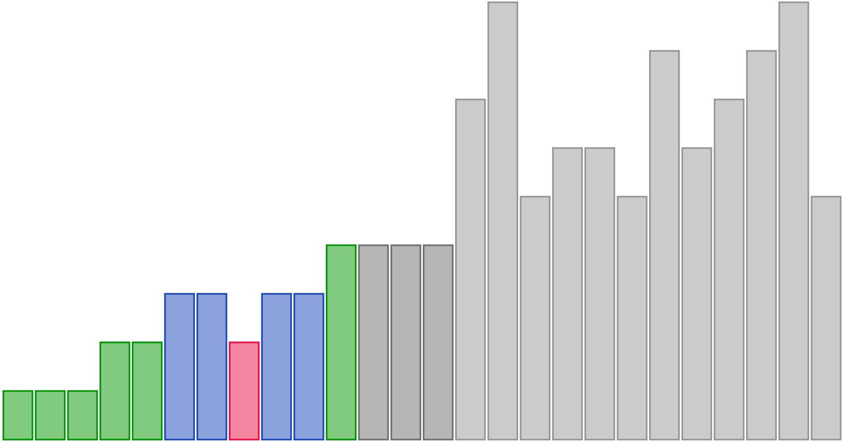


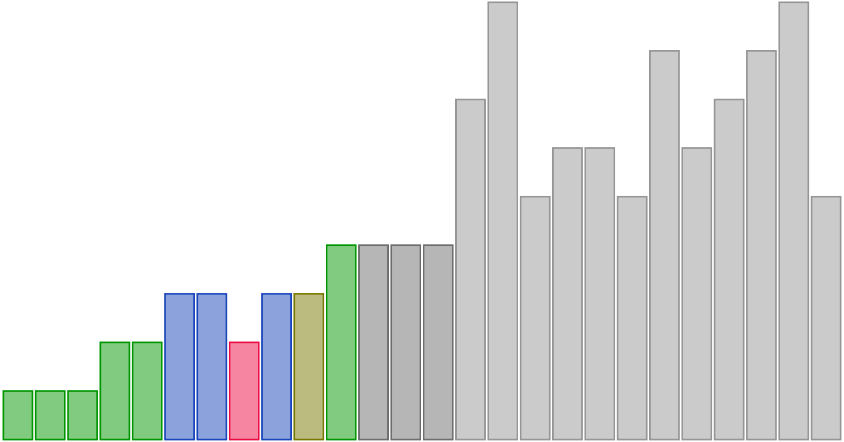


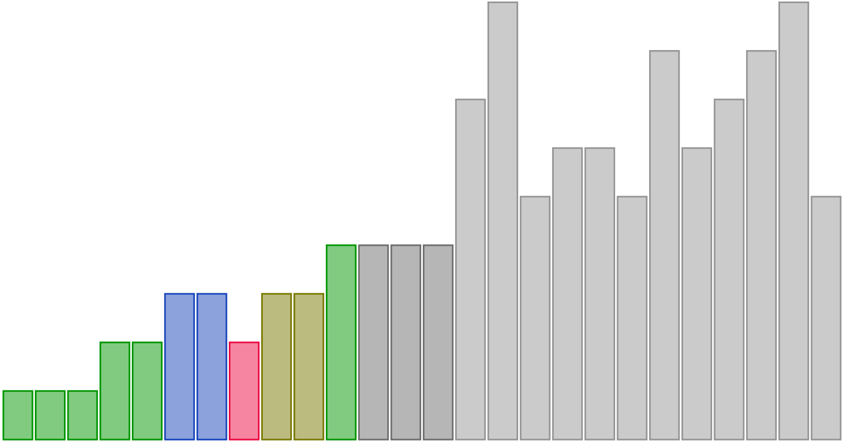


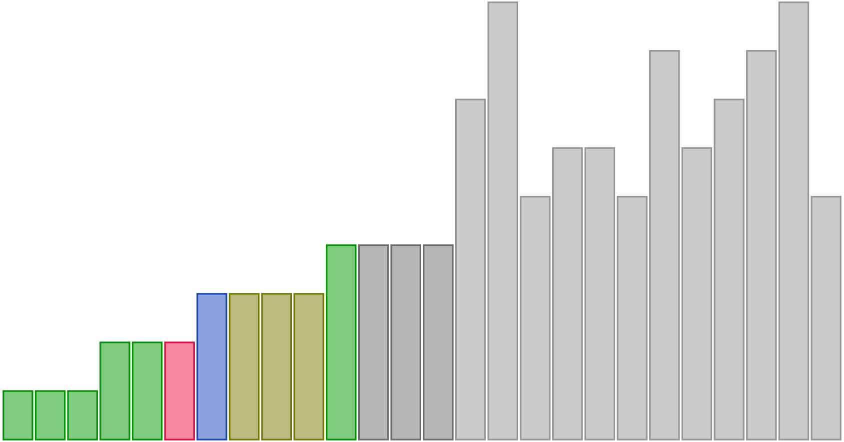


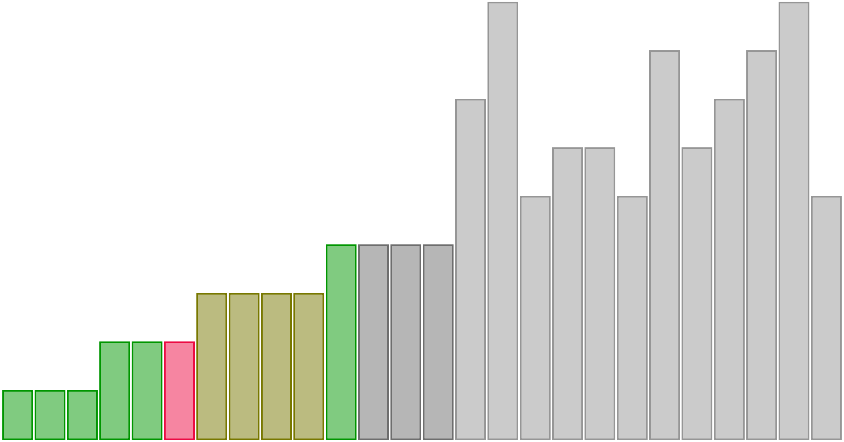


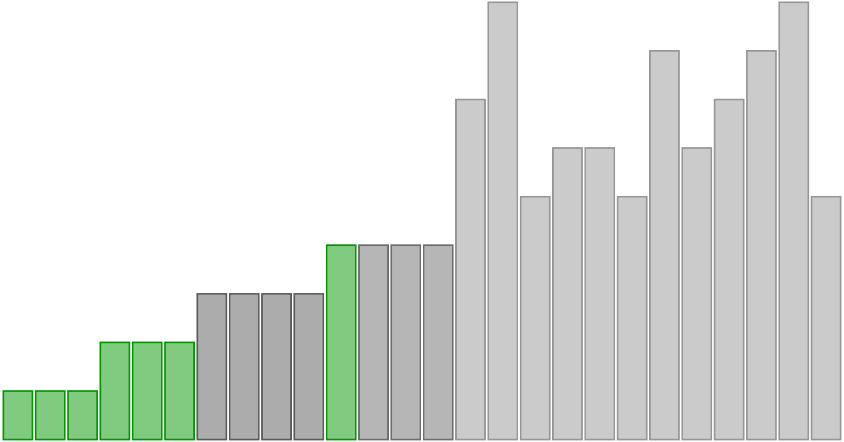


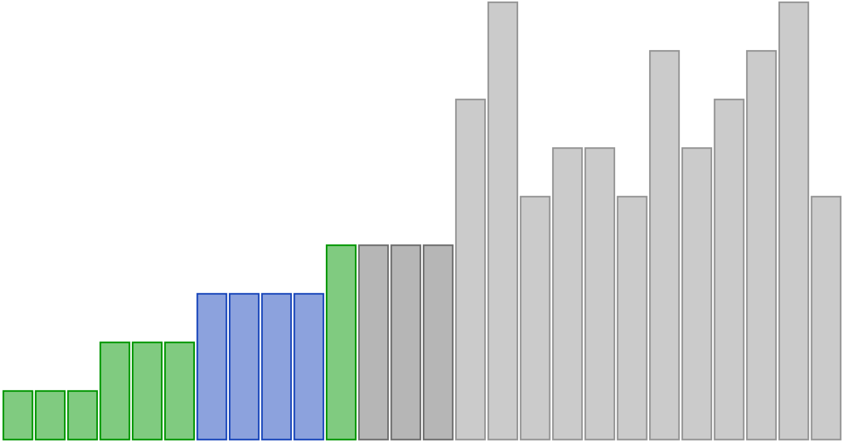


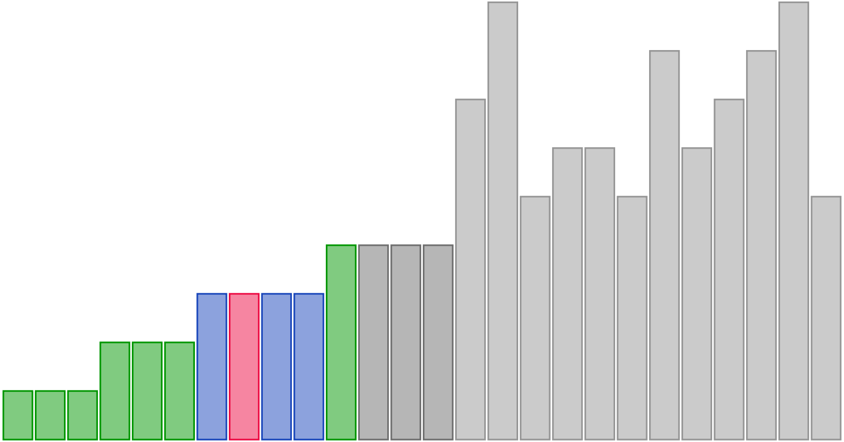


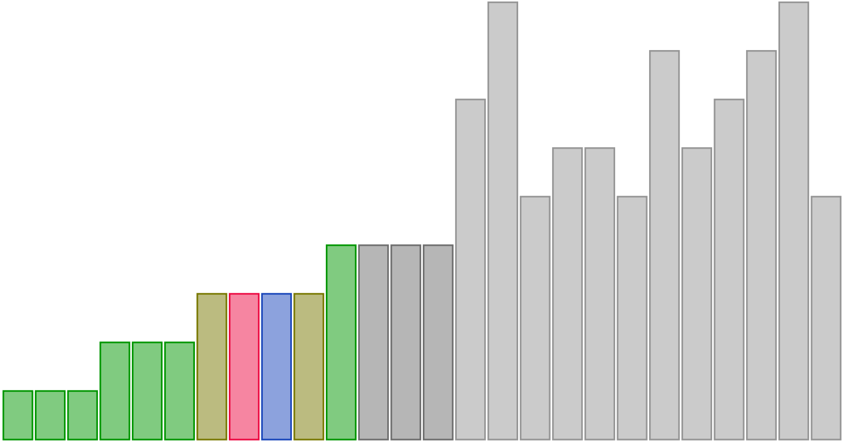


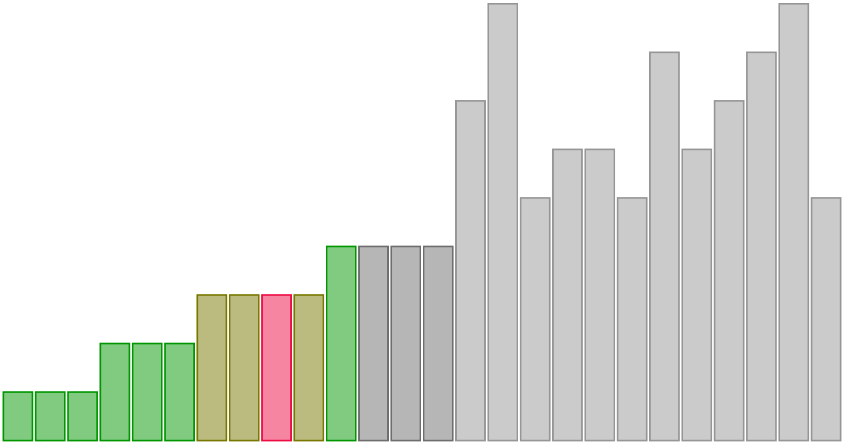


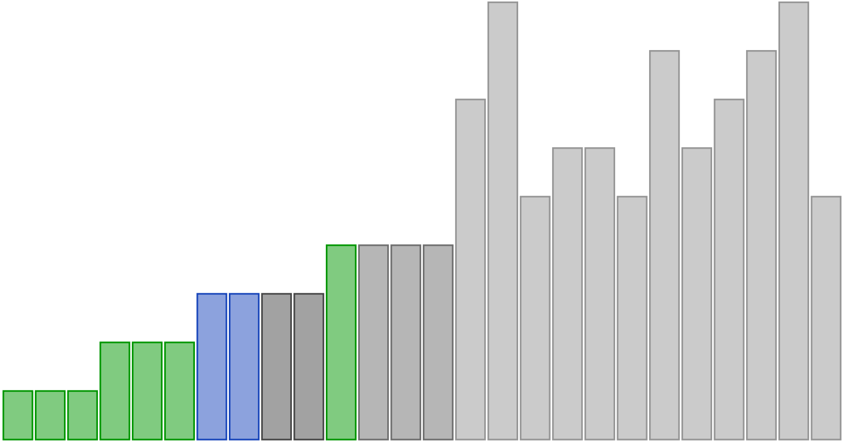


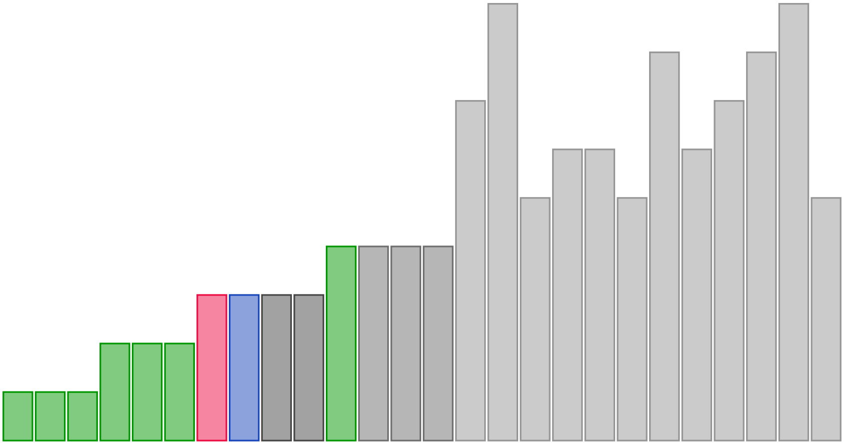


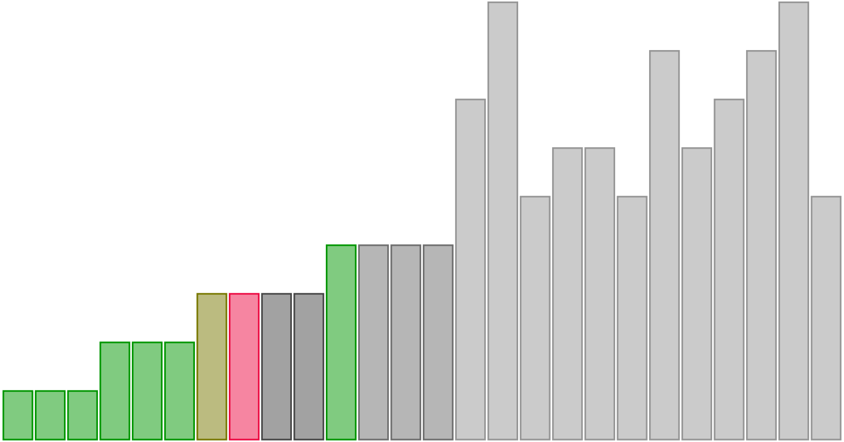


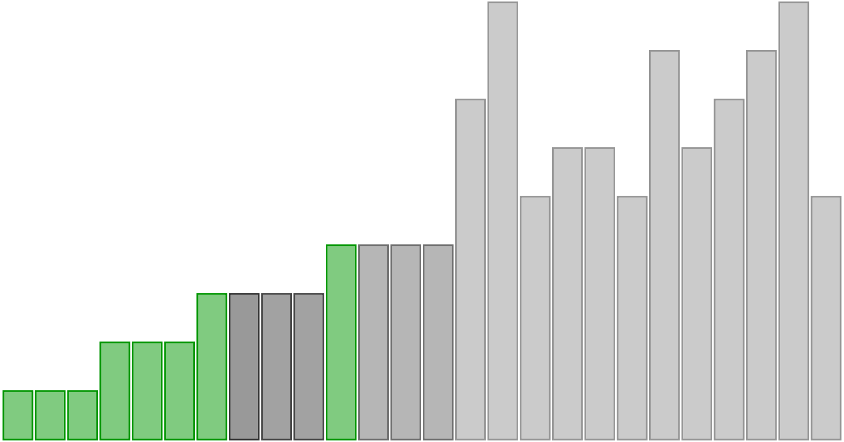


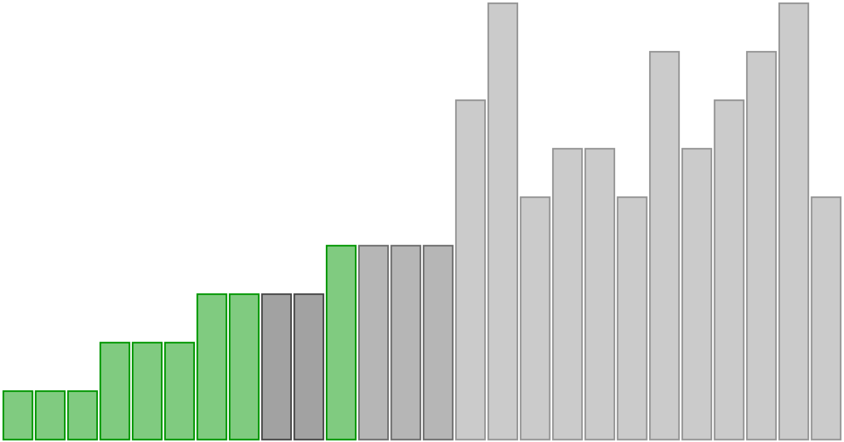


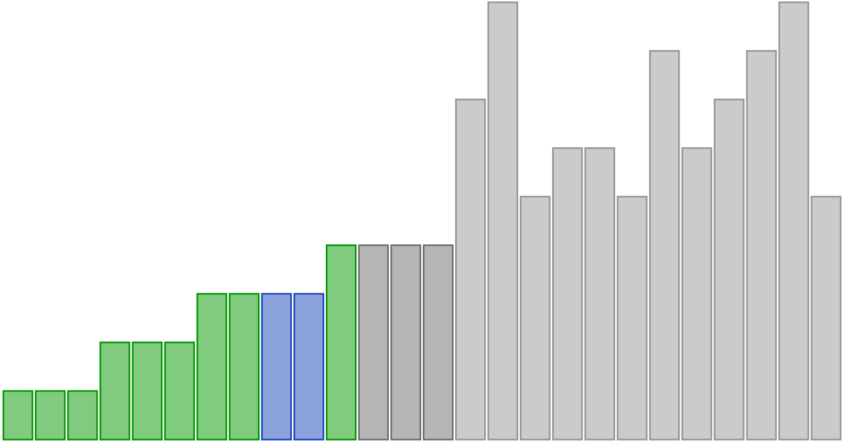


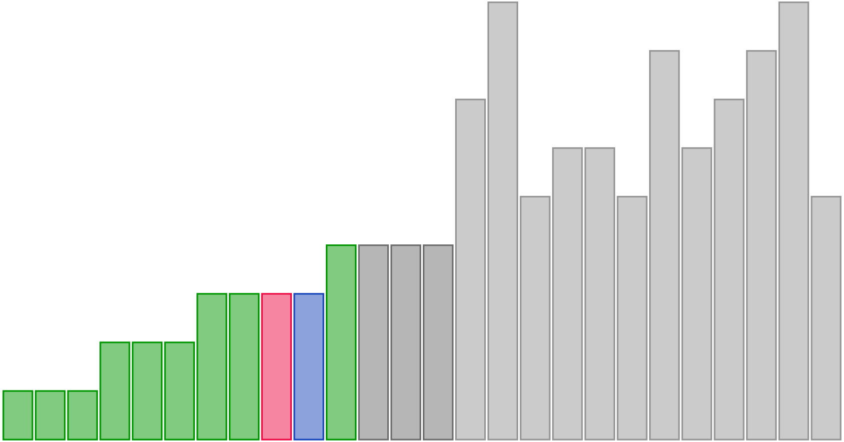


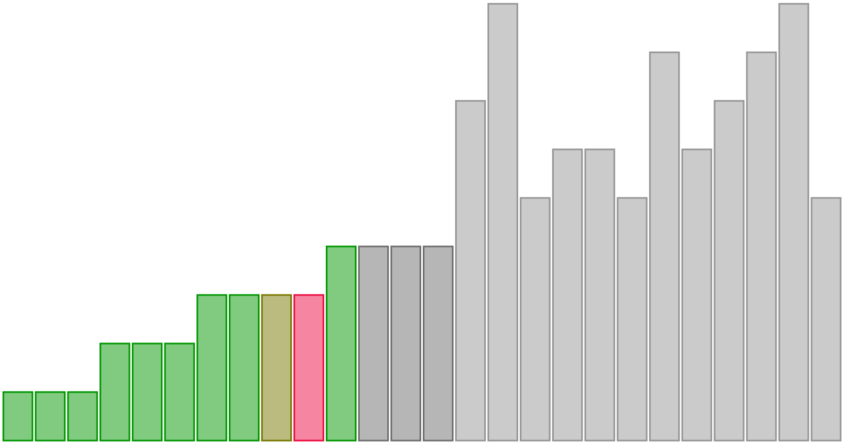


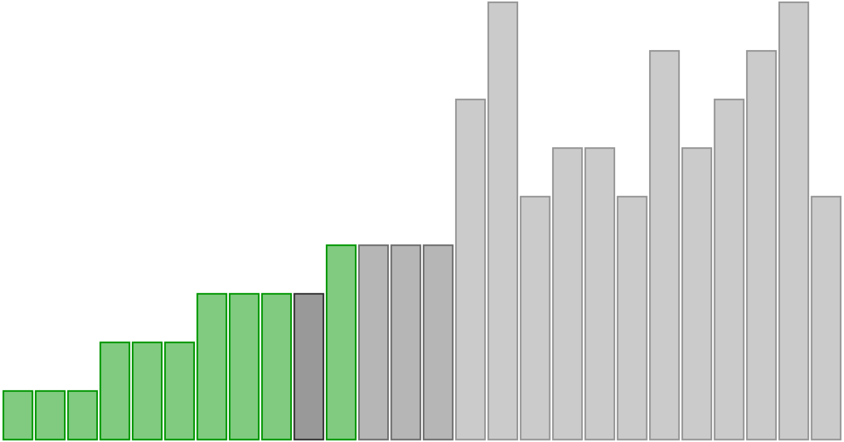


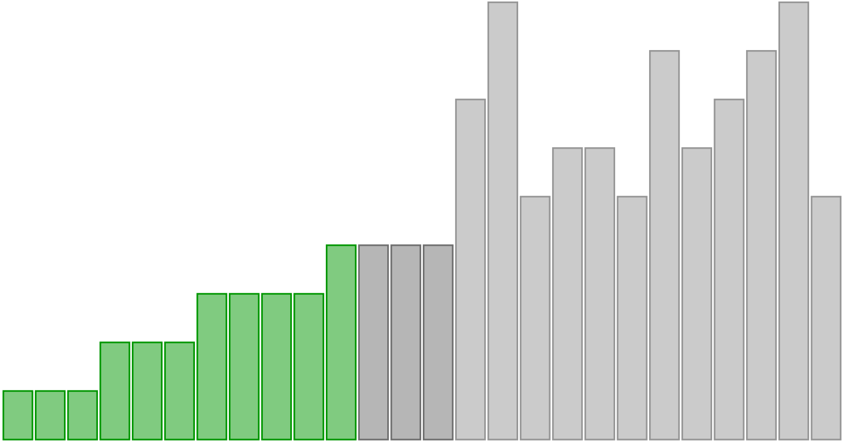


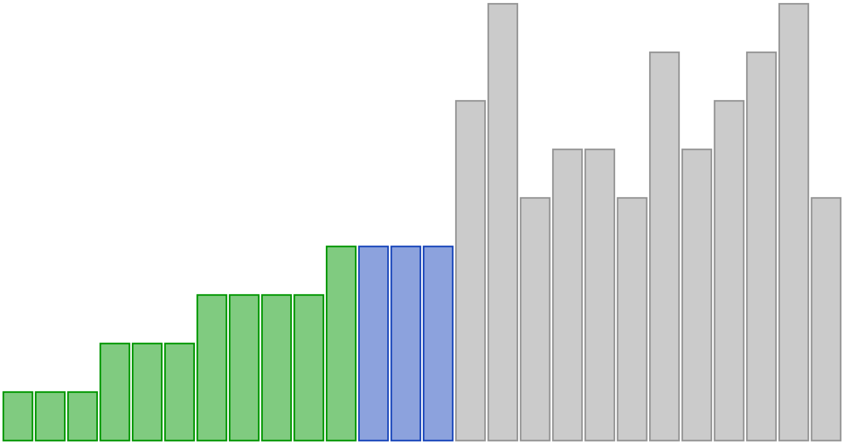


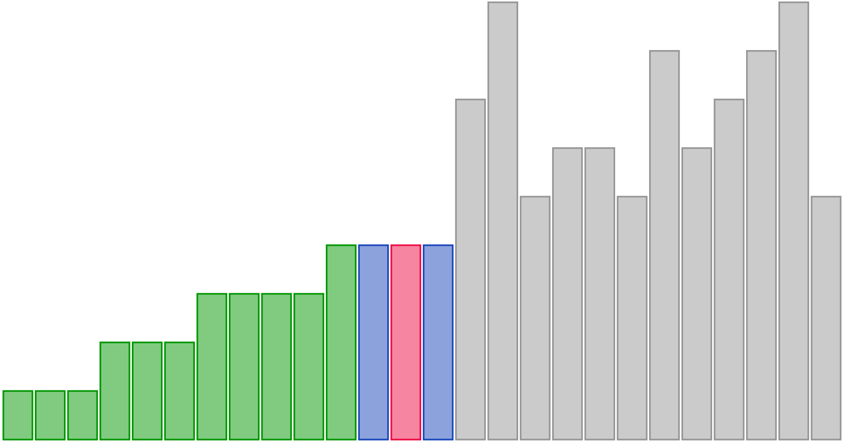


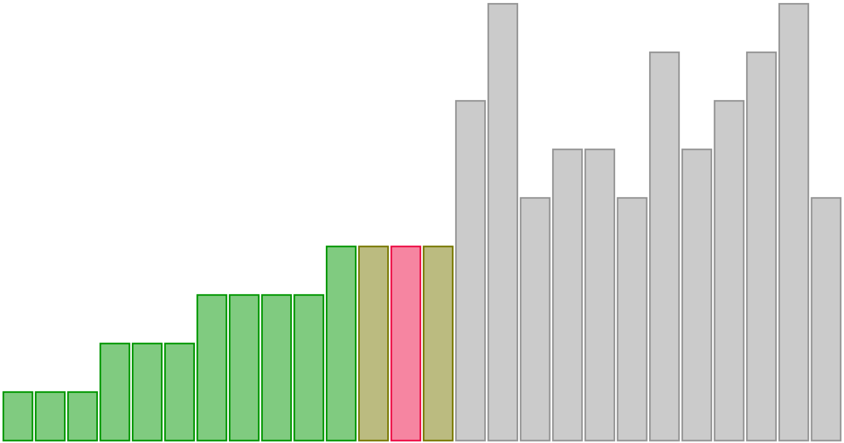


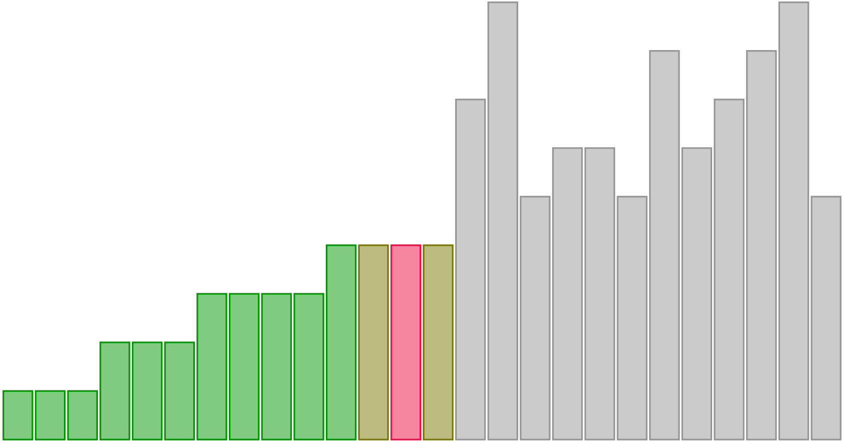


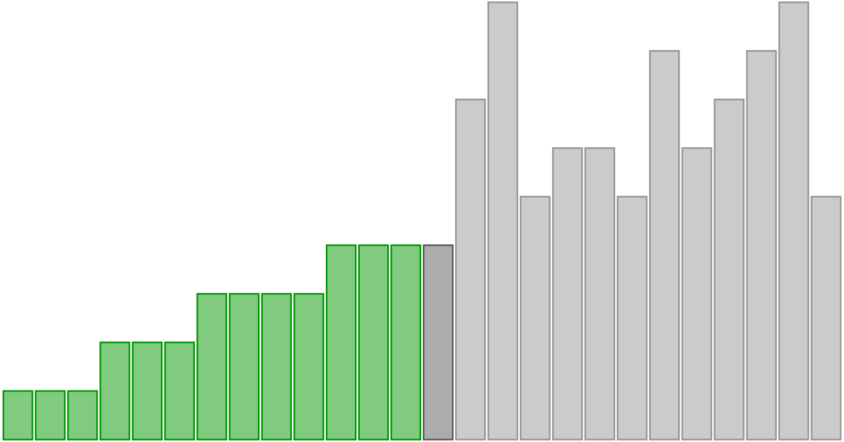


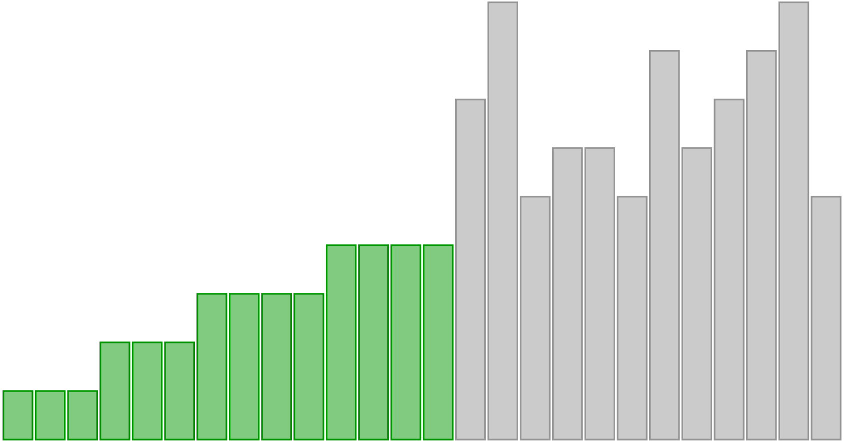


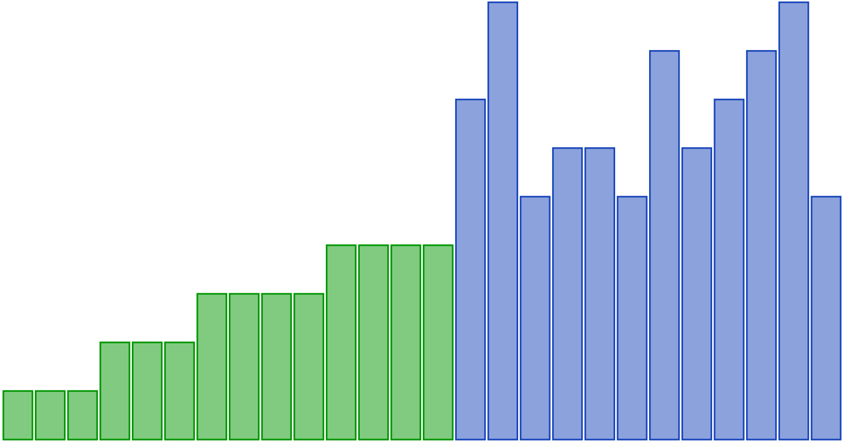


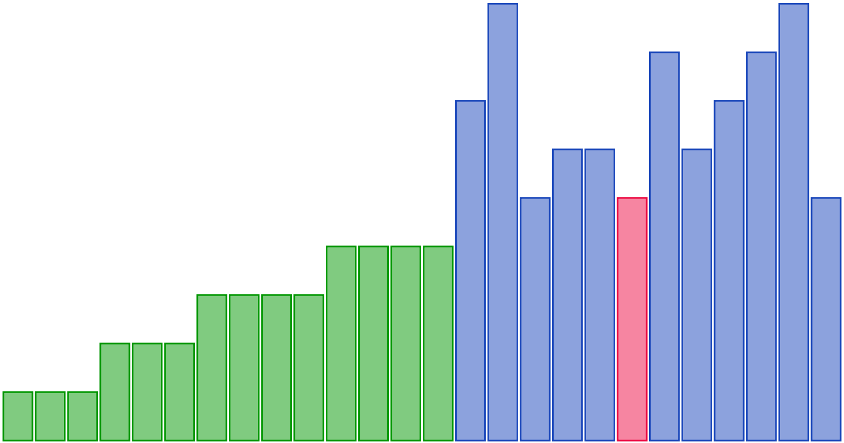


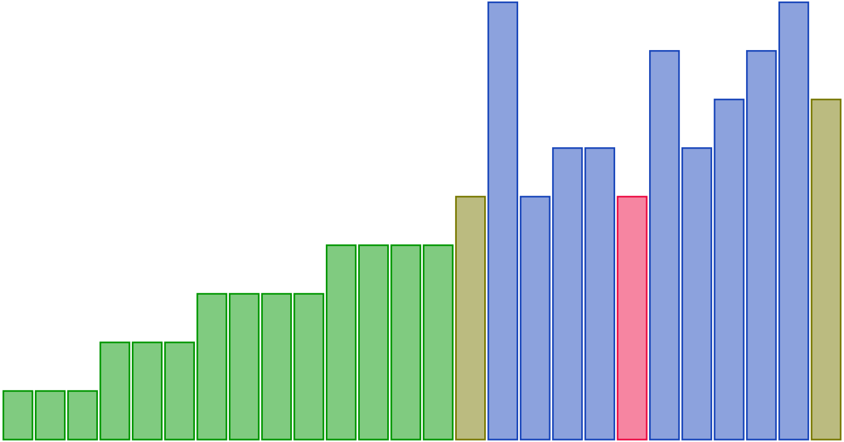


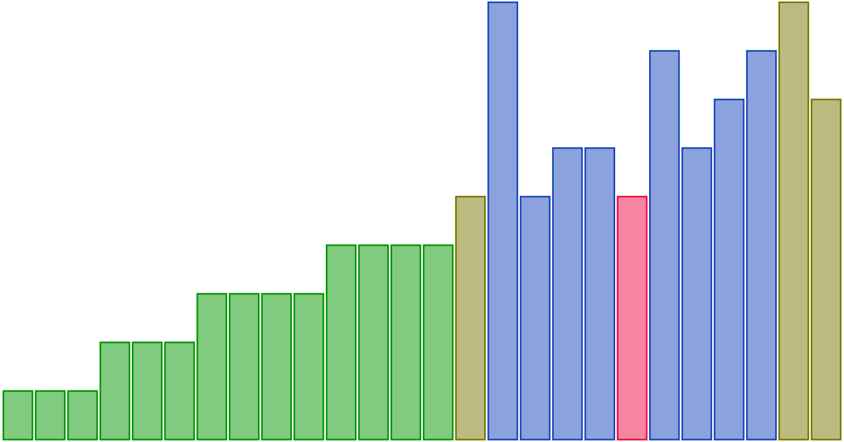


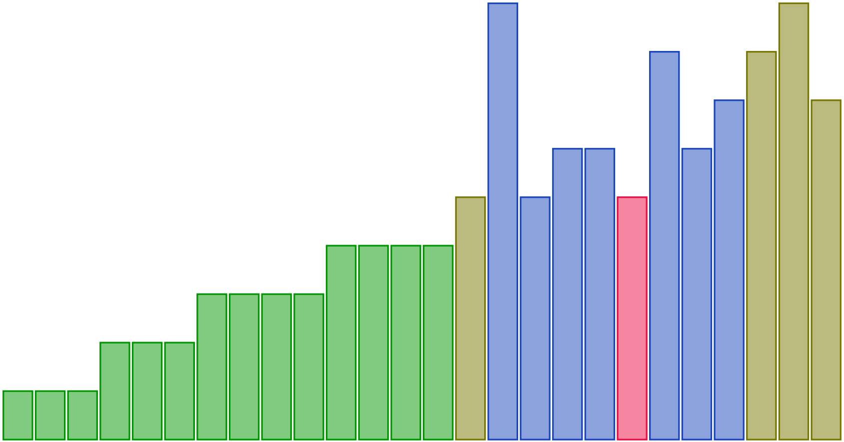


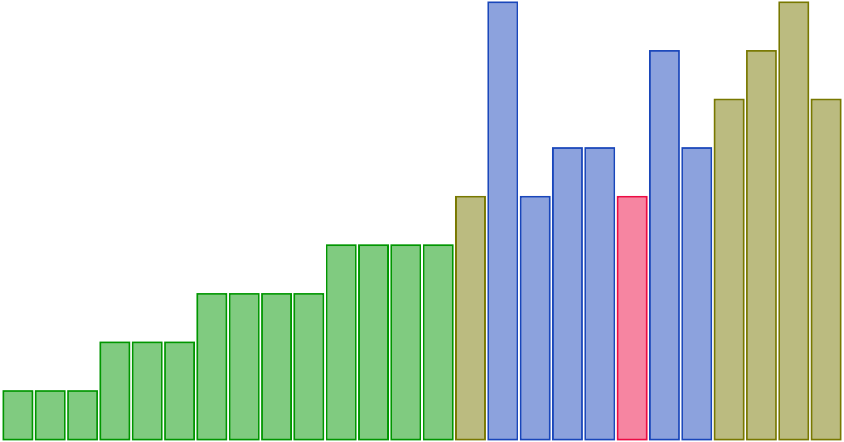


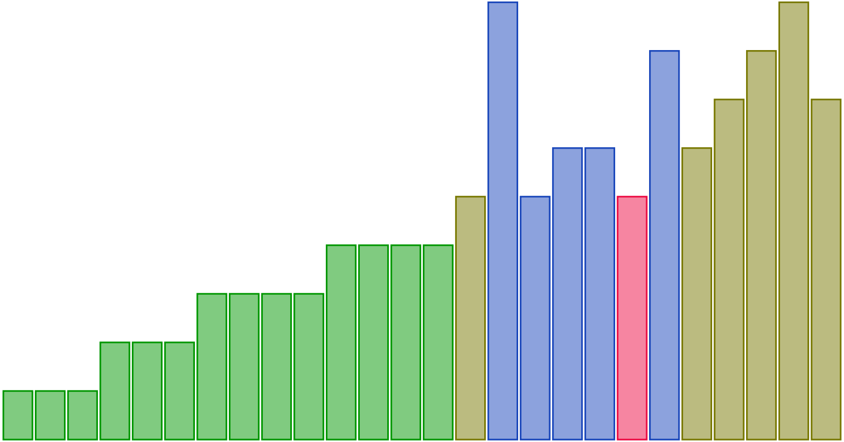


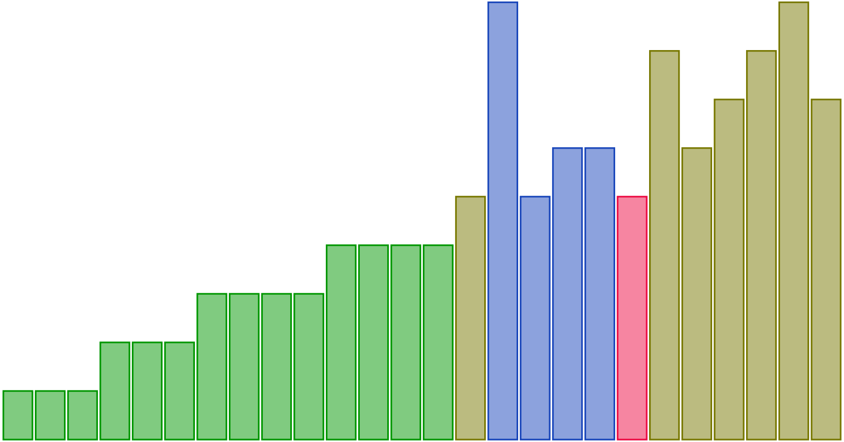


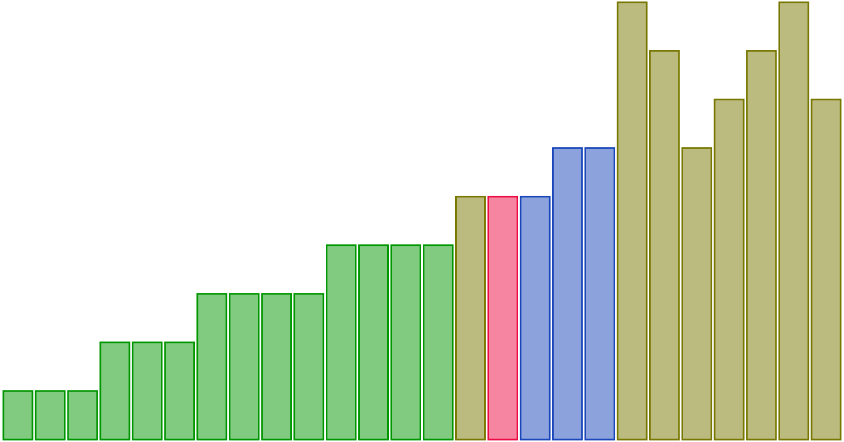


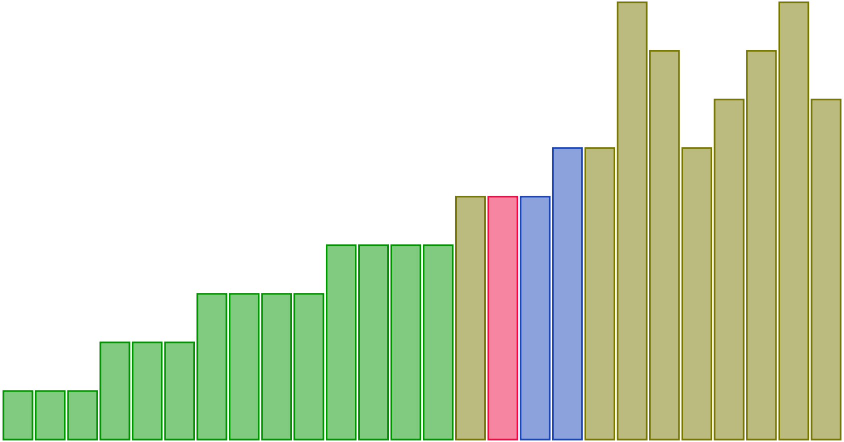


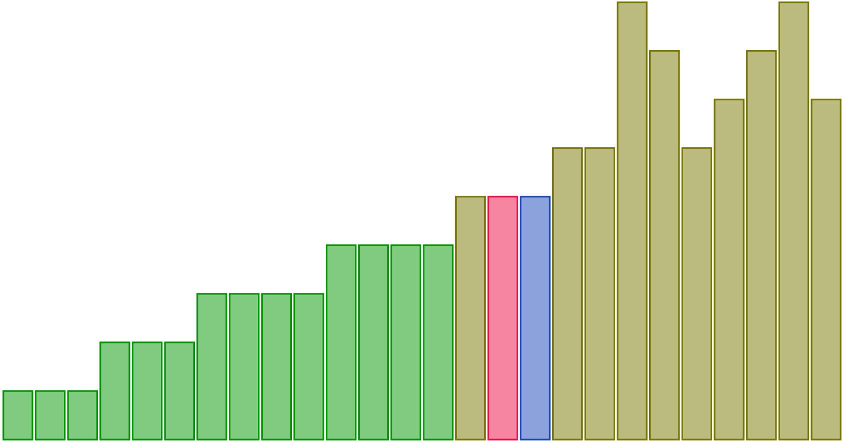


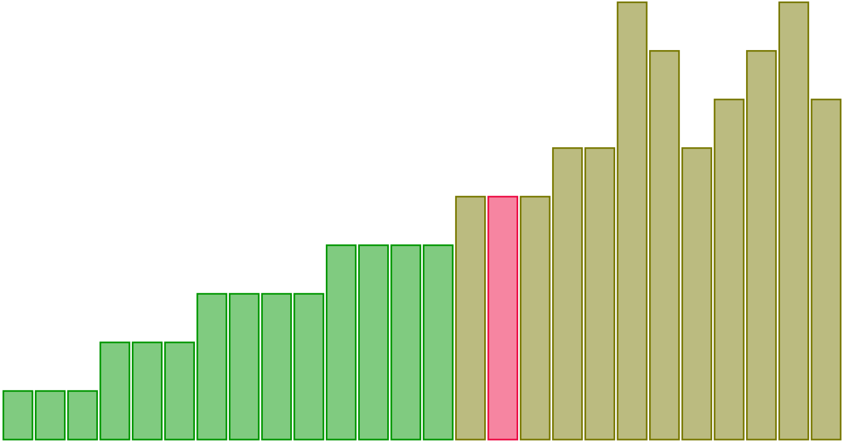


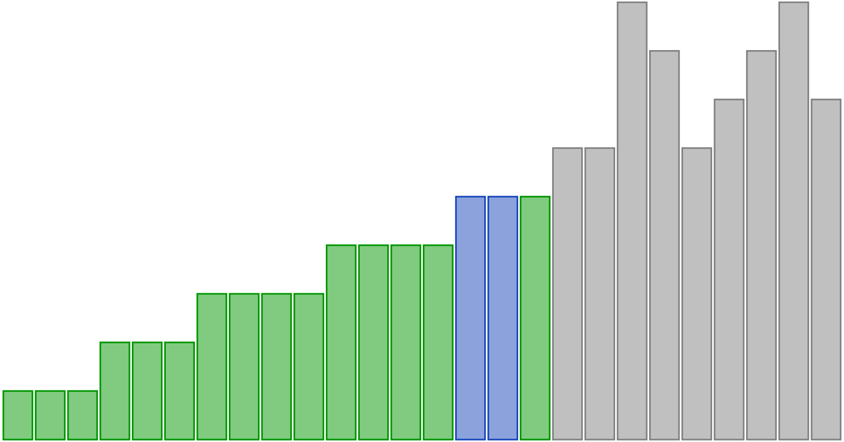


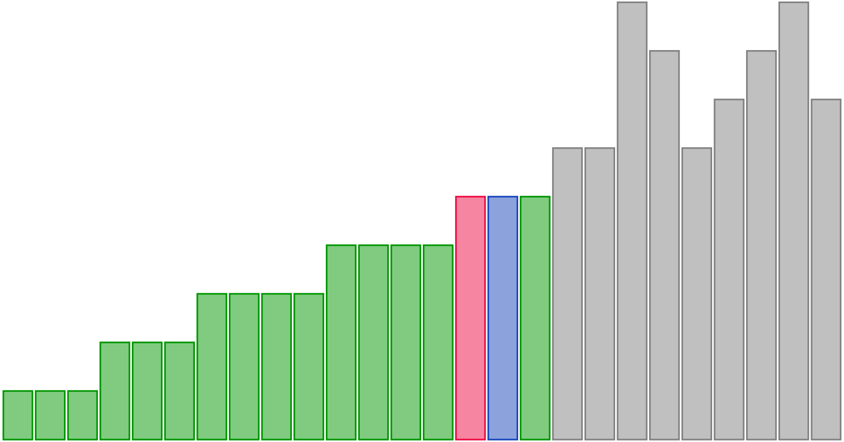


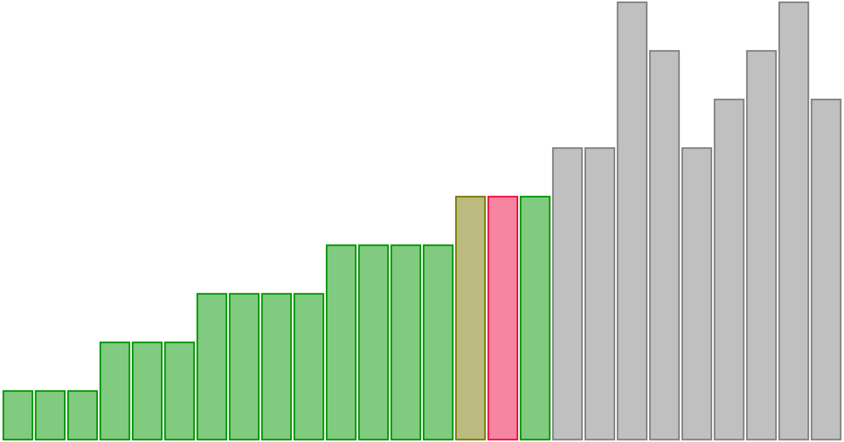


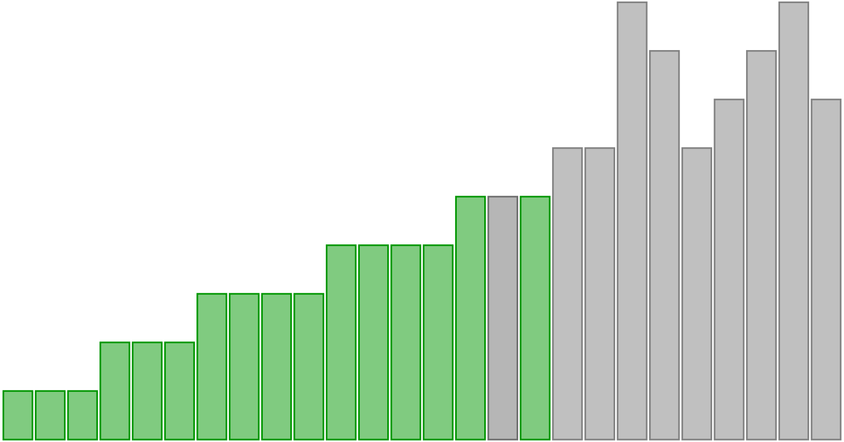


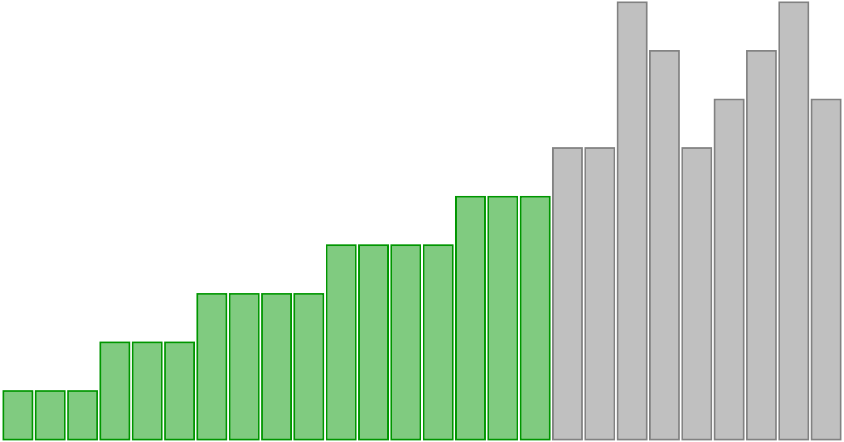


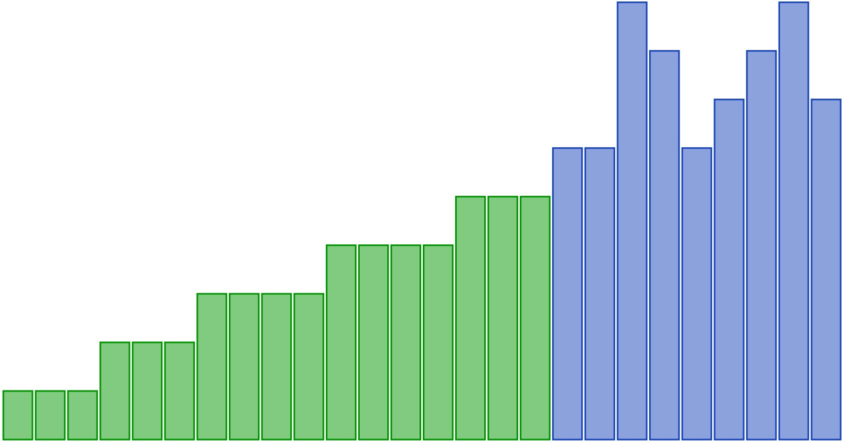


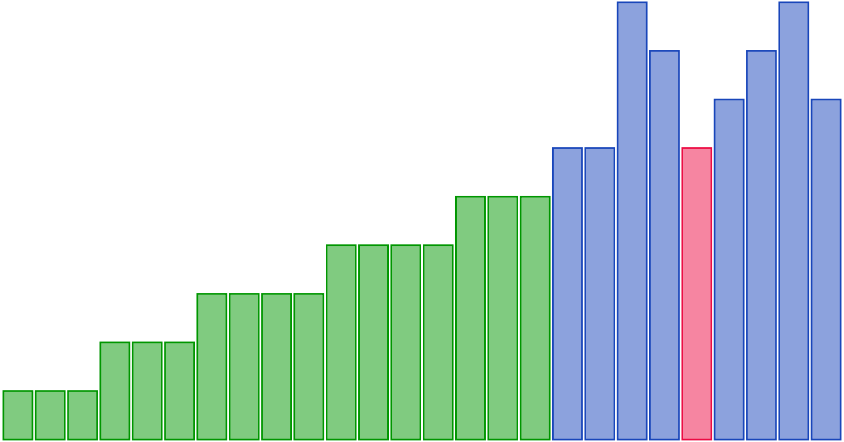


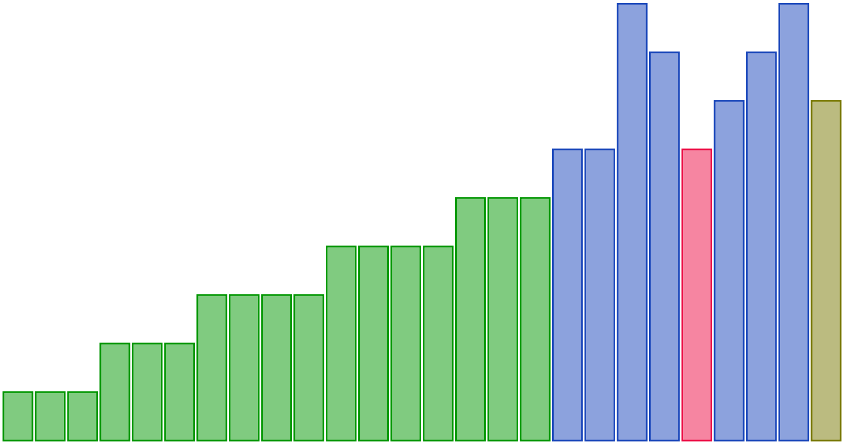


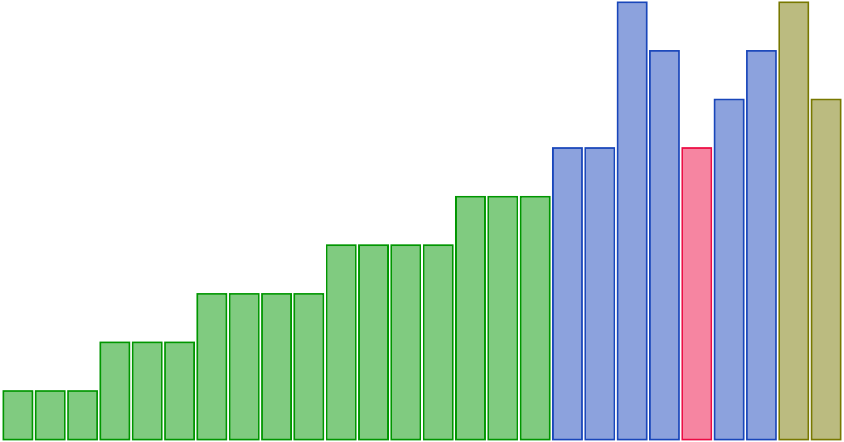


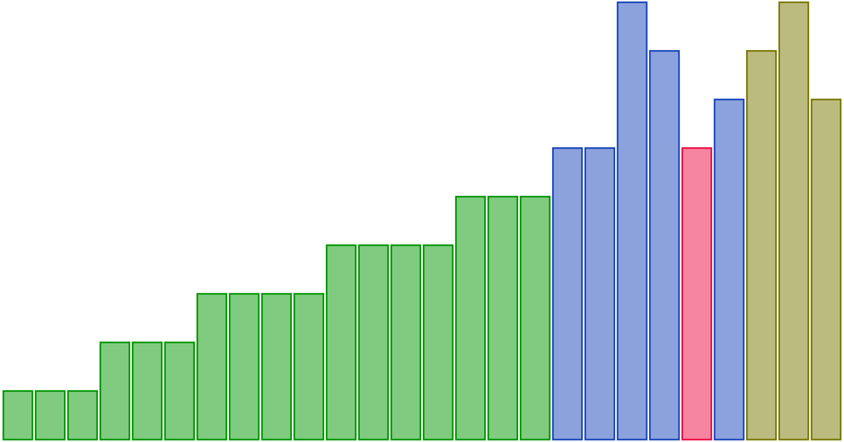


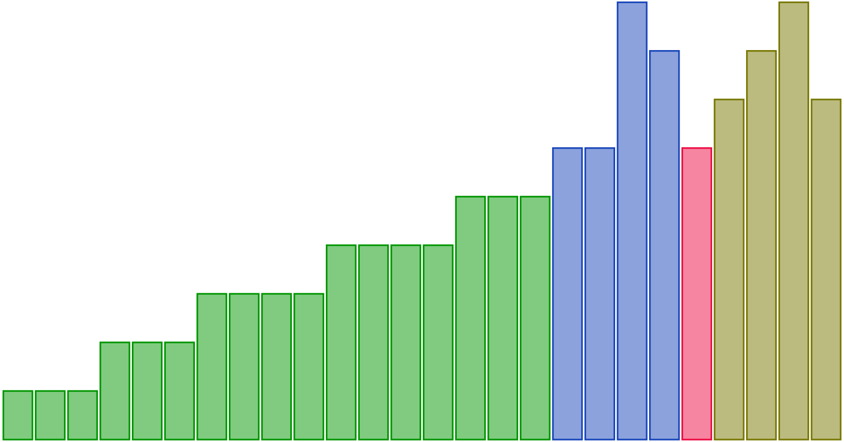


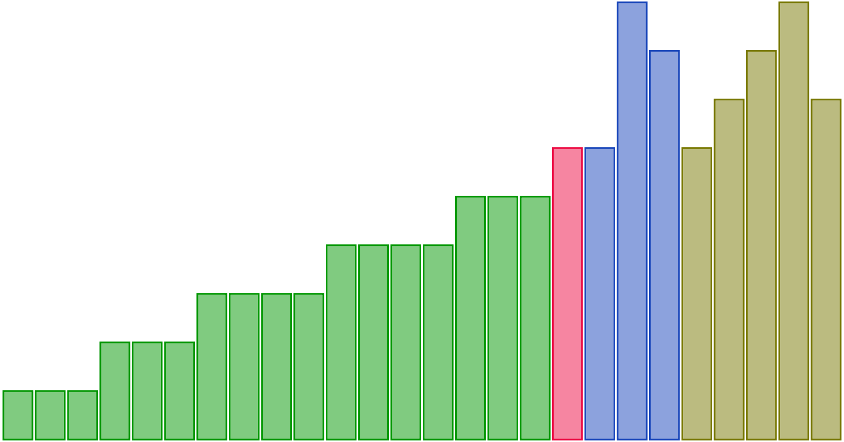


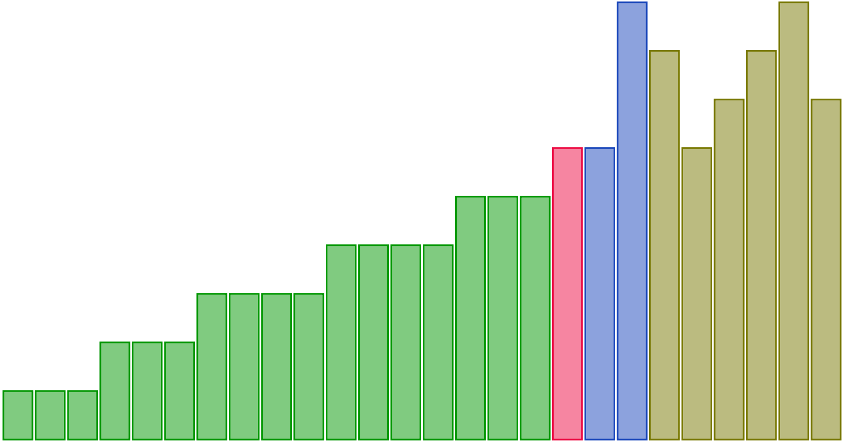


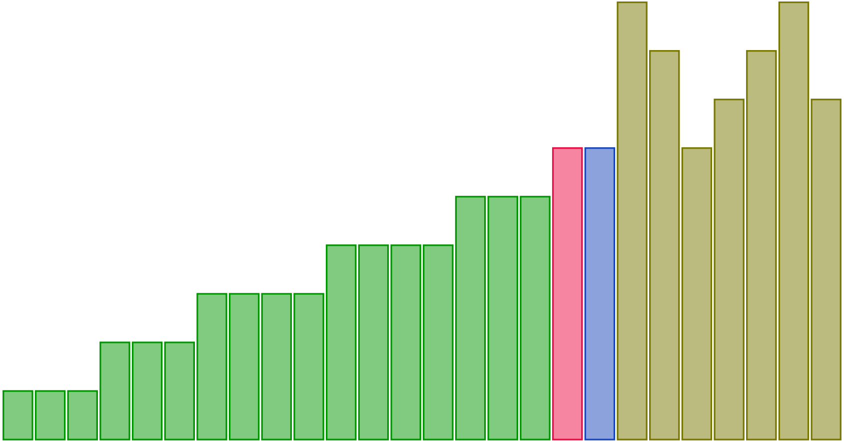


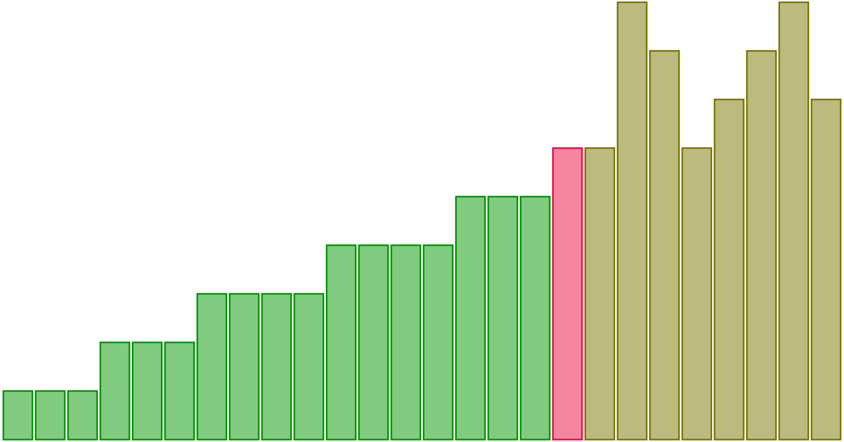


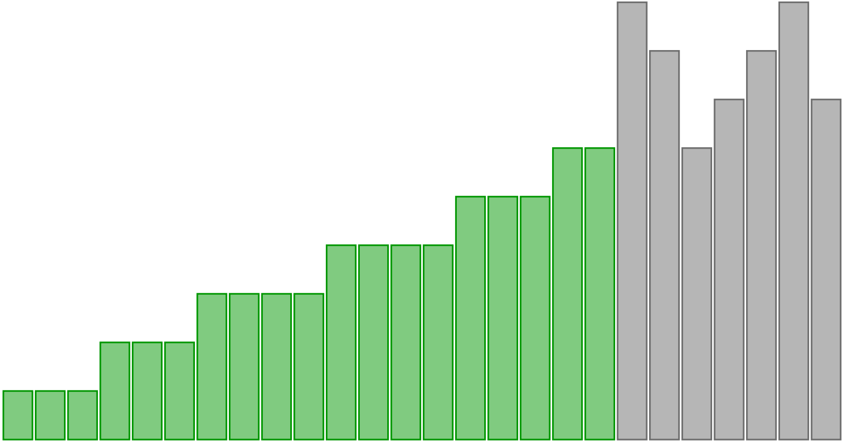


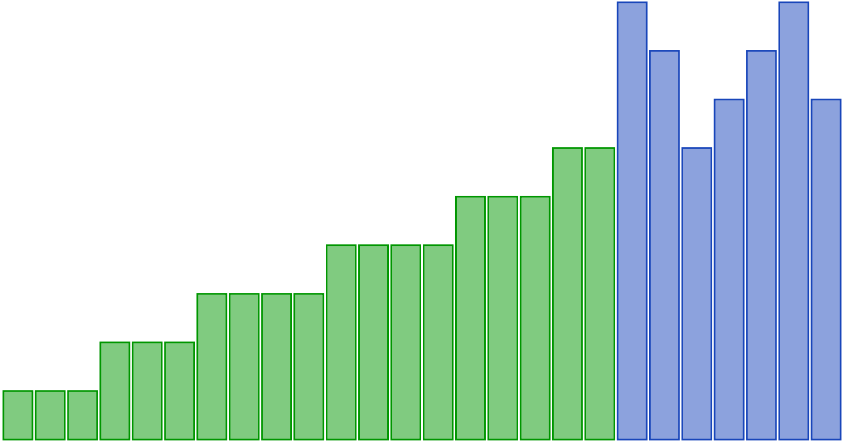


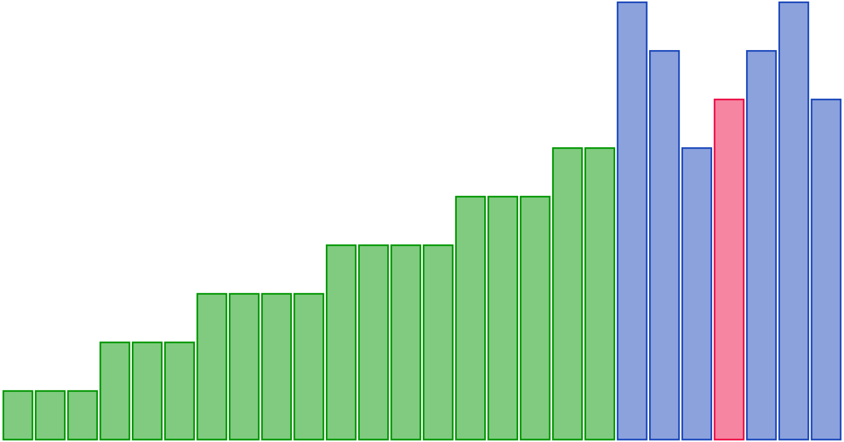


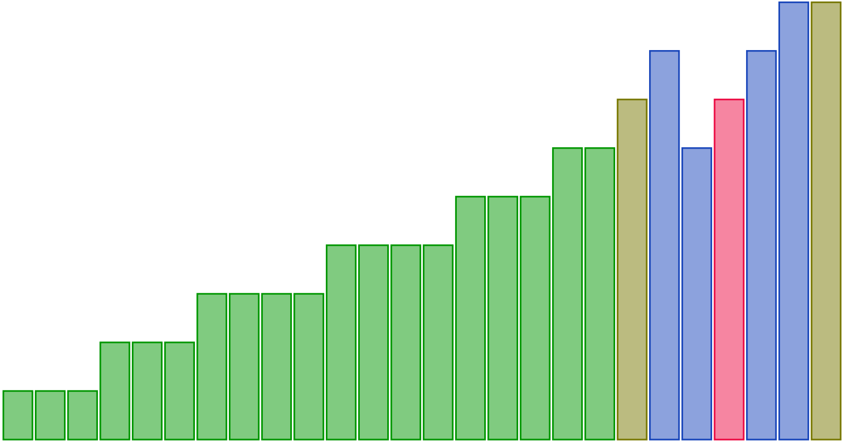


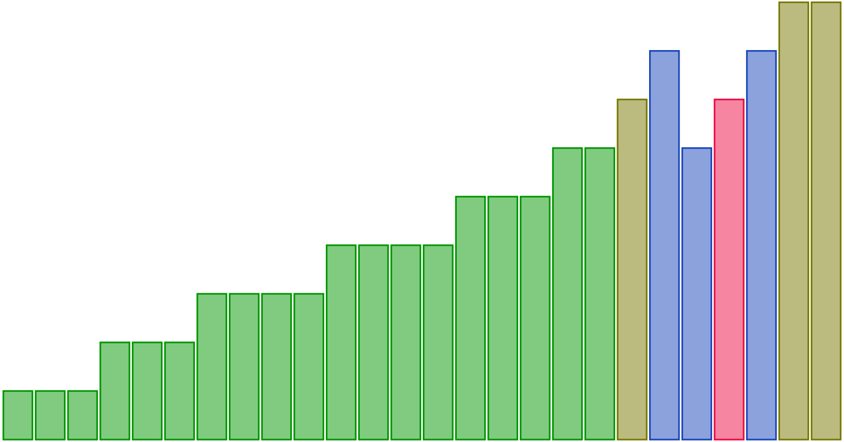


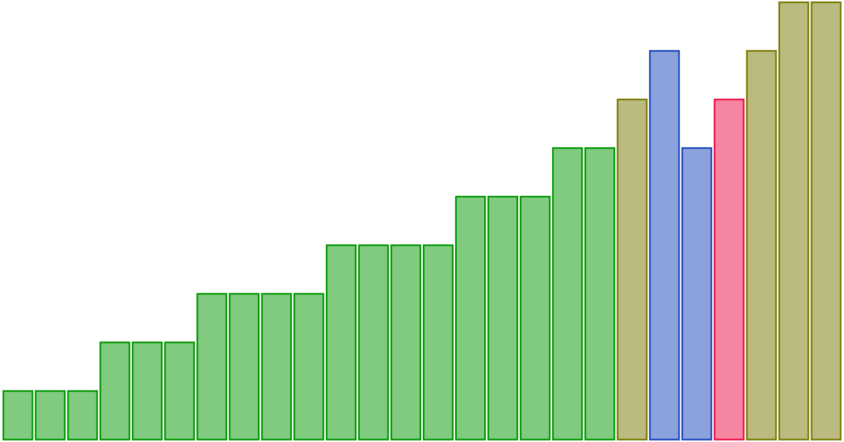


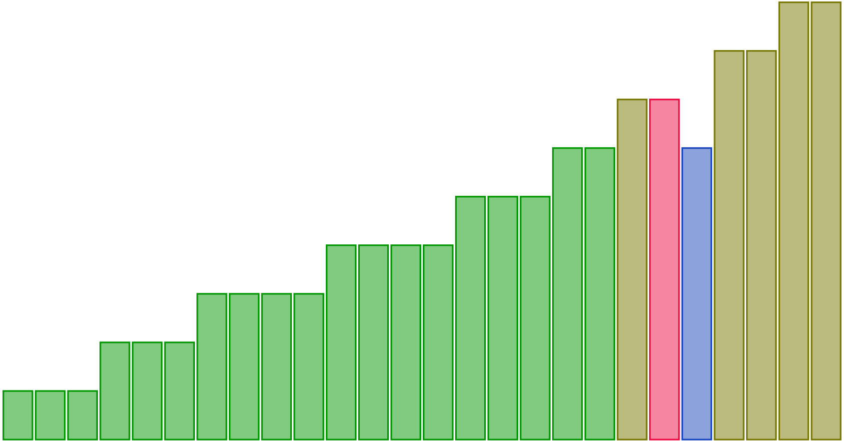


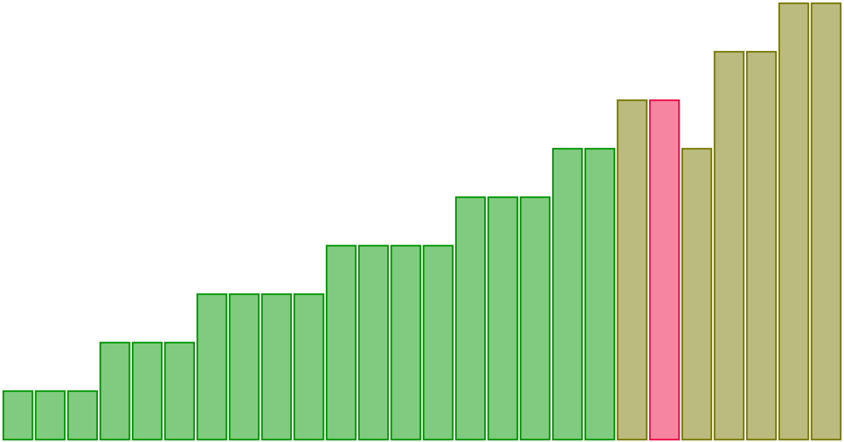


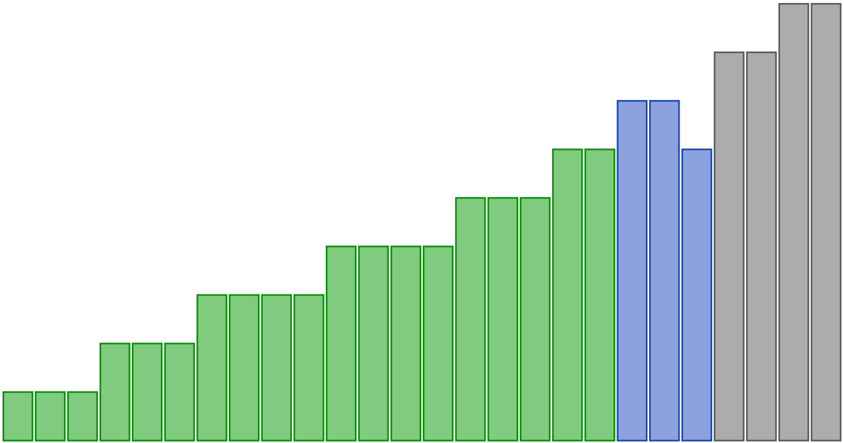


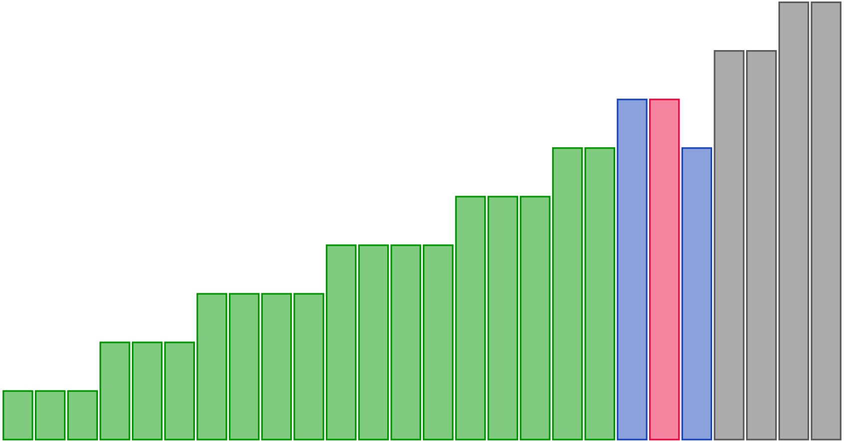


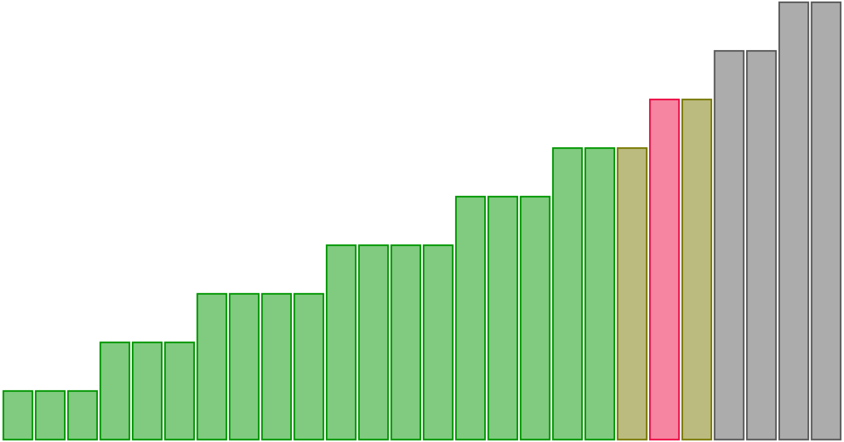


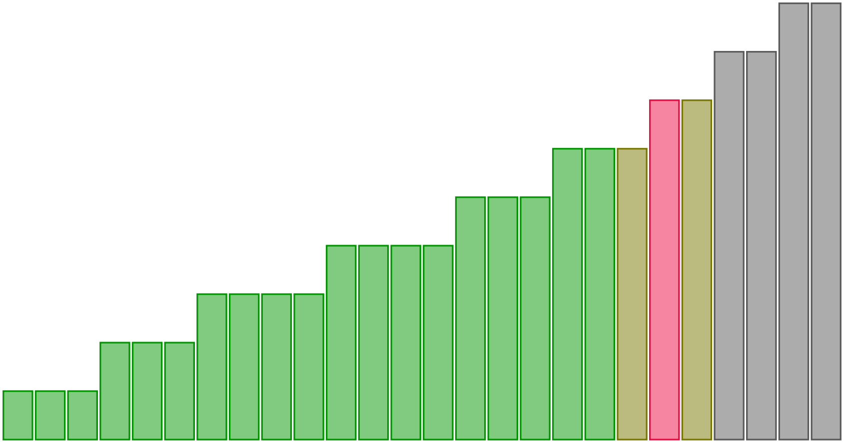


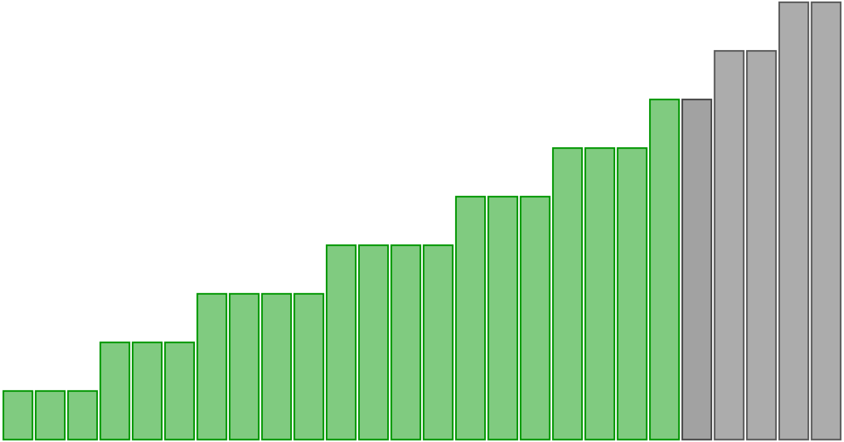


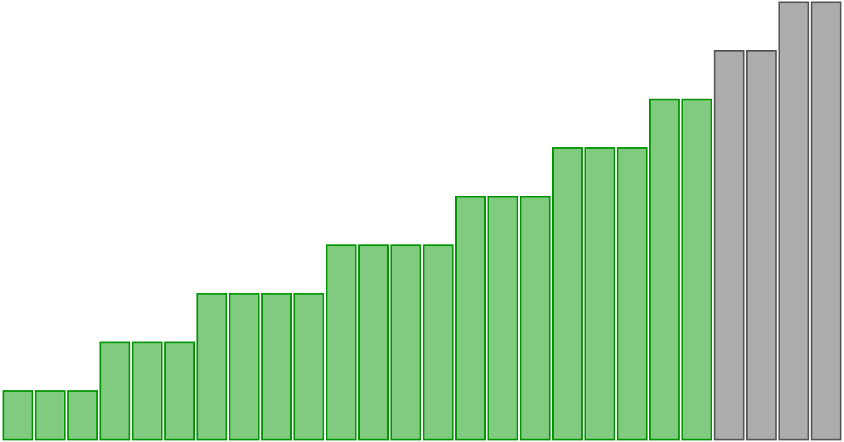


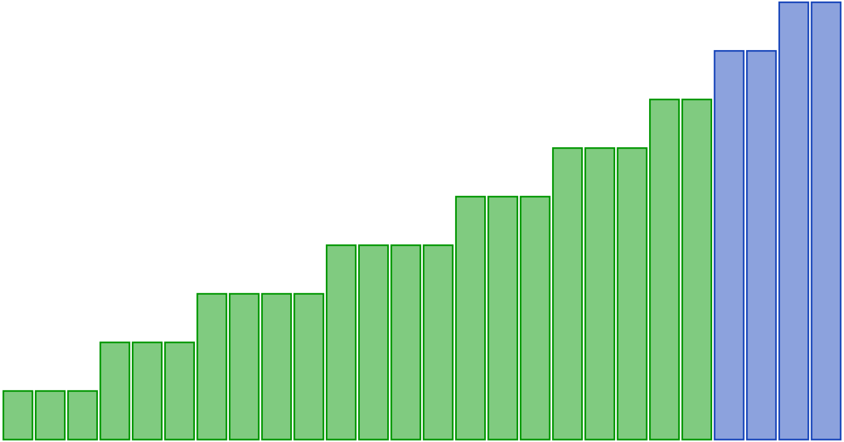


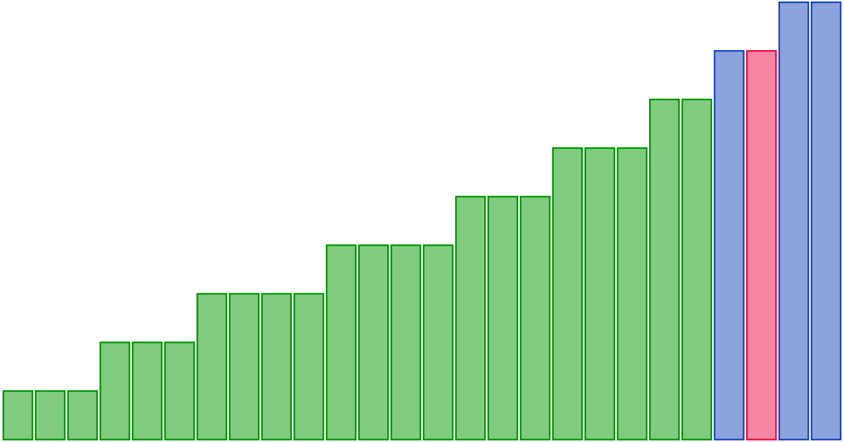


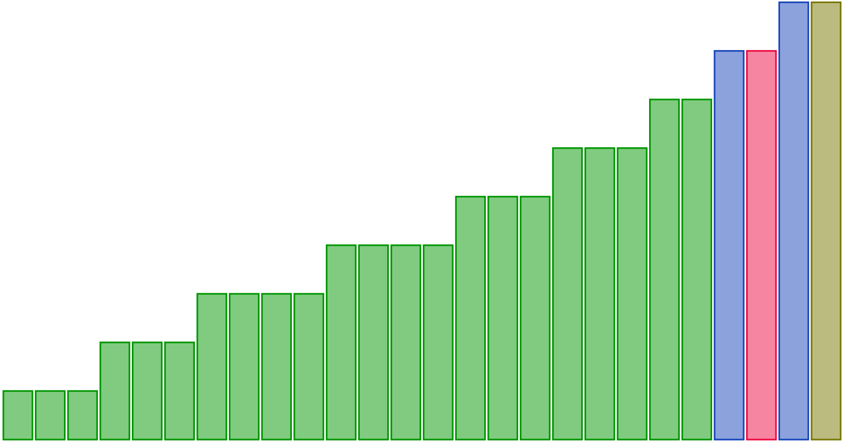


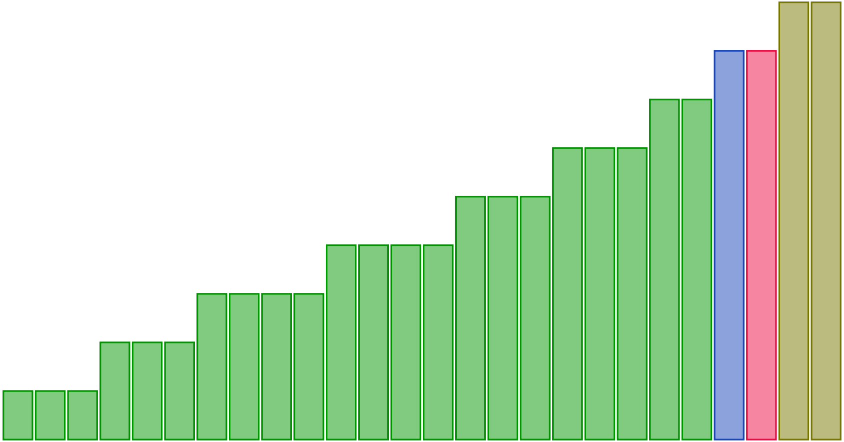


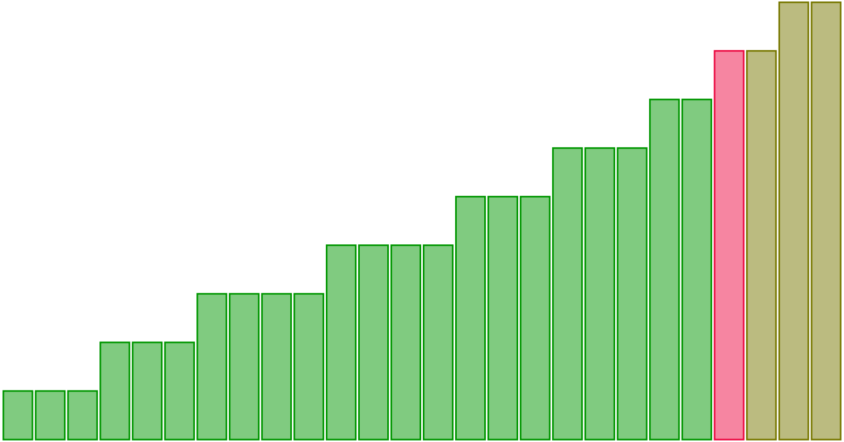


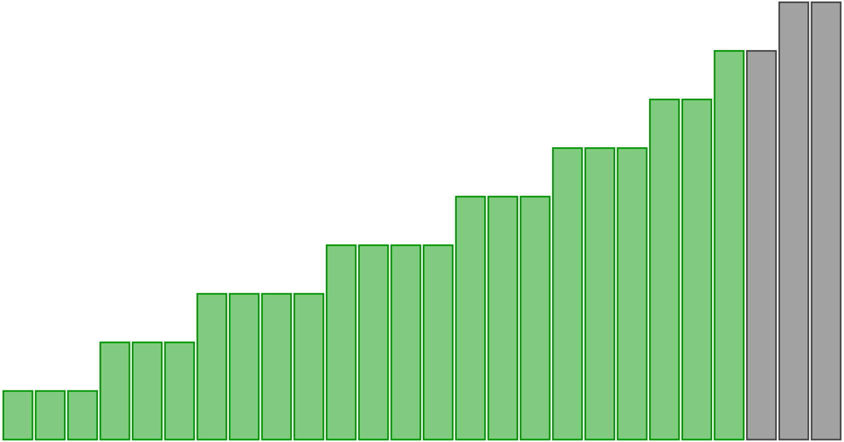


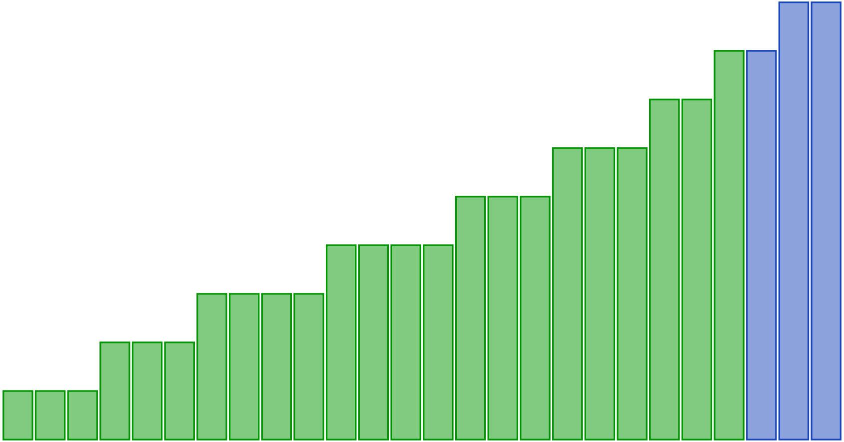


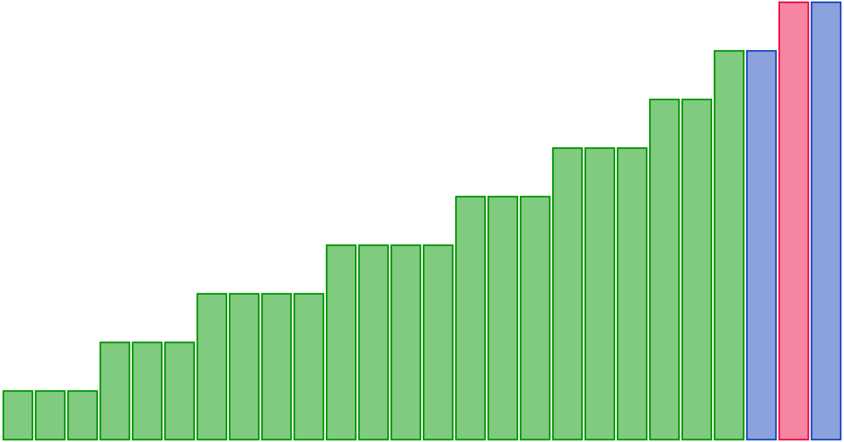


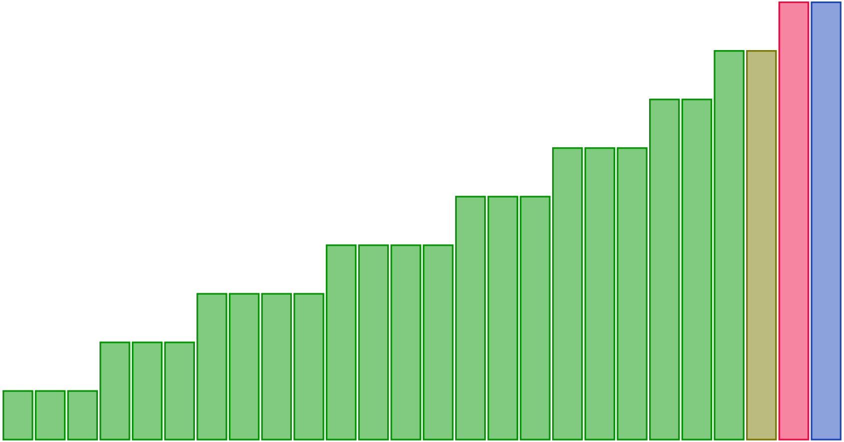


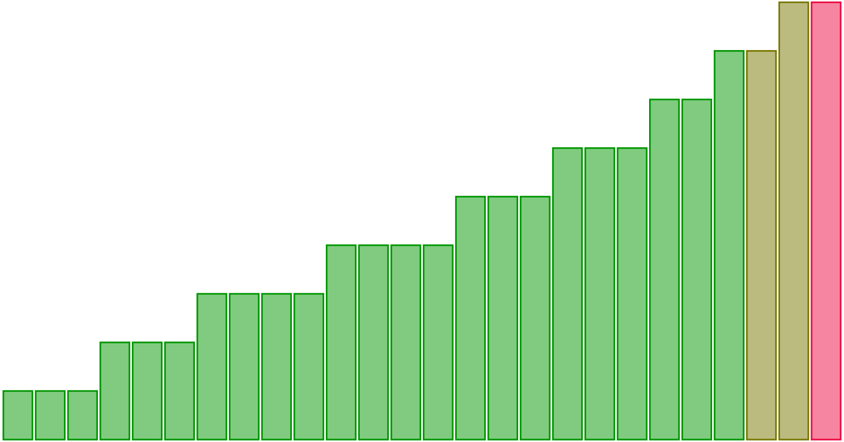


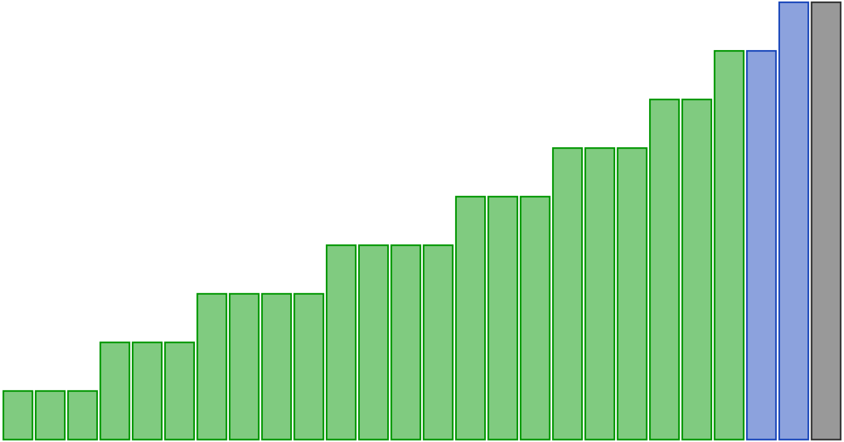


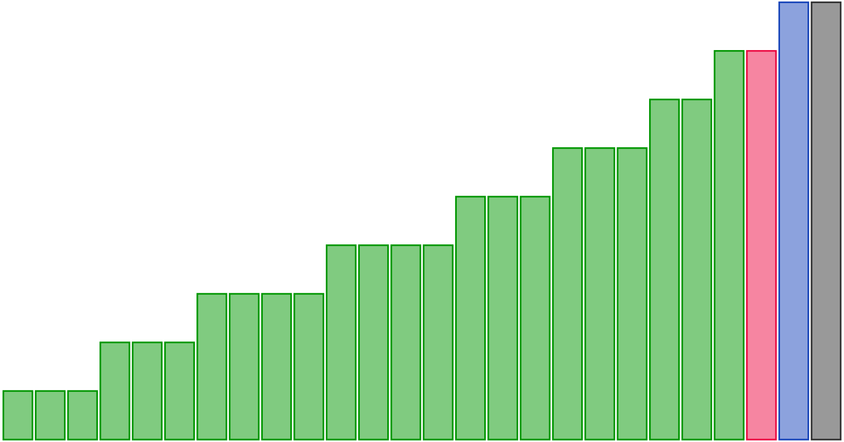


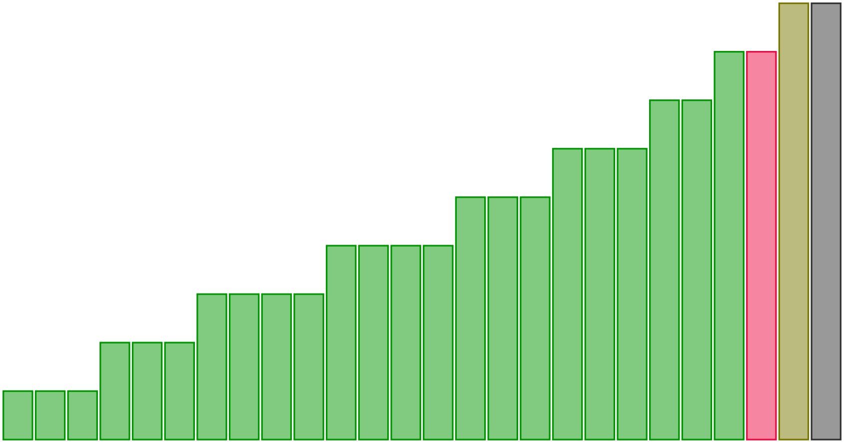


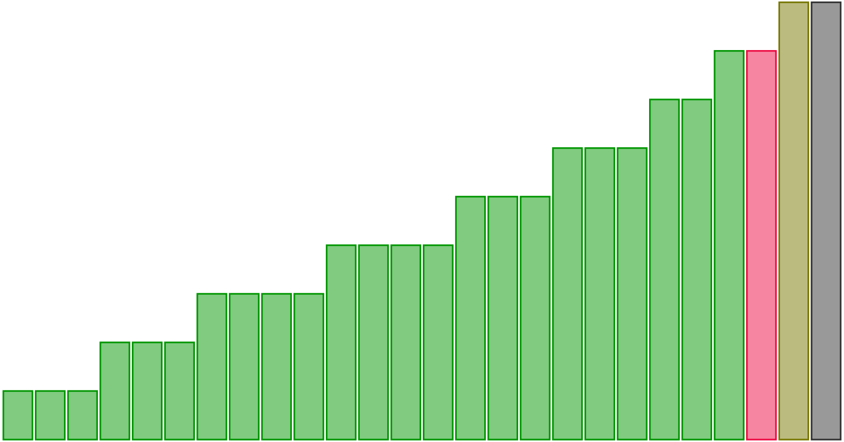


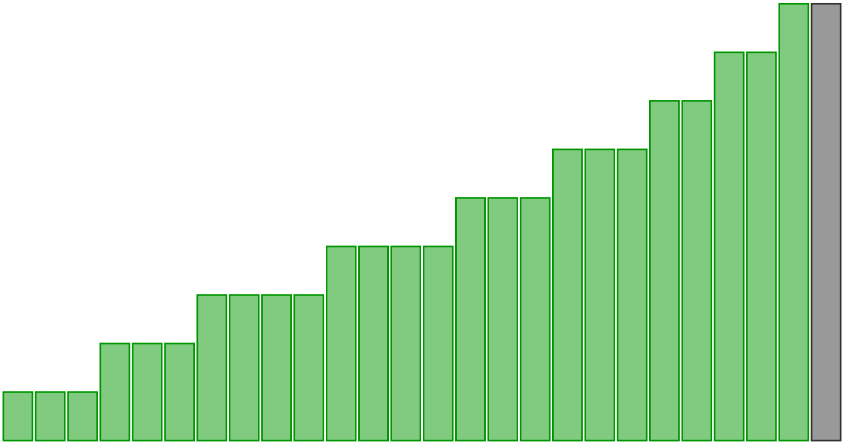


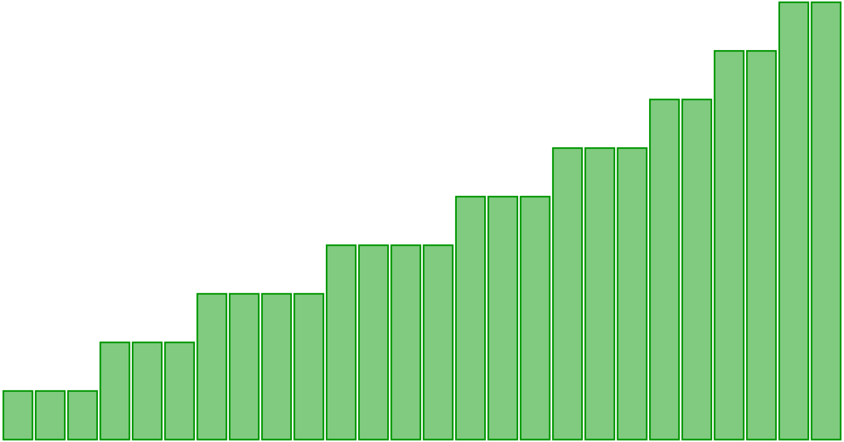












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  - Calls itself recursively on $[s; e']$ and $[s'; e]$
- Proof:
  - Quicksort terminates, because recursive calls work with strictly smaller array parts
  - Any single-element subarray is sorted by definition
  - After recursive calls are done, the subarrays $[s; e']$ and $[s'; e]$ are sorted, and the subarray $(e'; s')$ consists of equal elements, thus also sorted
  - Left part $\preceq$ middle part $\preceq$ right part $\rightarrow$ result is sorted

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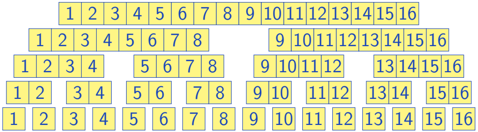

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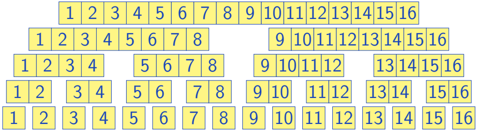

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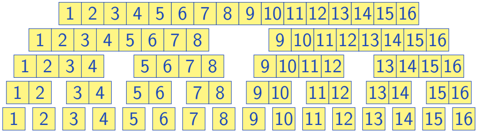

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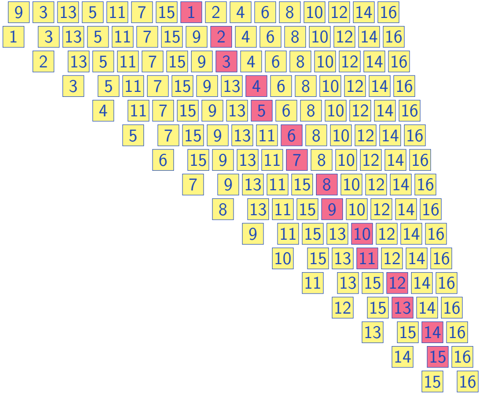

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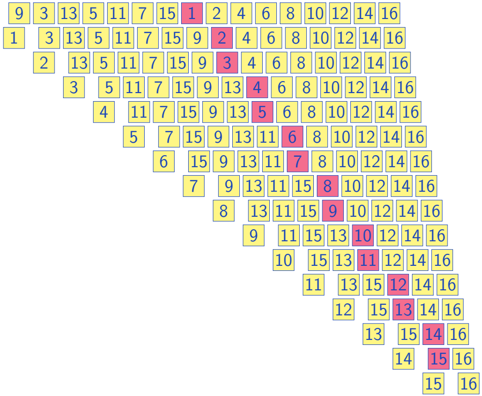

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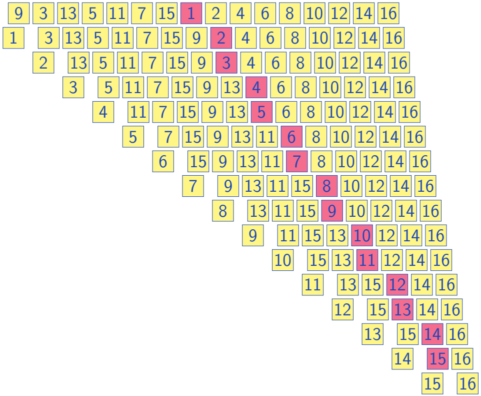

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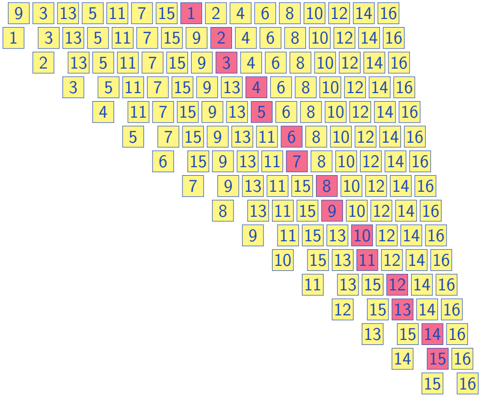
- But why is it “quick”?



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- Total work at each depth: $O(N) \to$ average runtime is $O(N \log N)$