15.415x Foundations of Modern Finance

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Lecture 7: Arbitrage Pricing Theory (APT)

Key Concepts

- The Main Idea of APT
- Factor Models
- Well Diversified Portfolios
- Expected Returns on Diversified Portfolios
- Factor Risk Prices / Risk Premia
- Factor-Mimicking Portfolios
- APT for Individual Securities
- Implementation of APT (Macro Factor Model)
- Implementation of APT (Portfolio Factor Model)

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Main steps of APT

- Factor model of returns in which risk can be decomposed into two components:
  - Systematic risks (common to many assets);
  - Non-systematic risks (specific to individual assets).
- Diversification eliminates risk → For diversified portfolios, $\bar{r}_p$ depends only on systematic factors (arbitrage otherwise).
- Portfolios come from risky assets → For “almost all” risky assets.
  - Expected return $\bar{r}_i$ depends only on systematic factors.
- End result: Model to price risky assets by their exposure to systematic risks.

Systematic vs. idiosyncratic risk

- Uncertainty in asset returns has two sources: Common factors and firm-specific shocks.
- Common factors:
  - Proxy for economic conditions or events that affect all firms and investors.
  - Such factors may include interest rates, price of oil, government policy shocks, etc.
  - Example: If return on an asset increases when inflation increases, it can be used to hedge uncertainty in future inflation rate → smaller risk premium as a result of investors' extra demand for this asset.
  - Represent systematic risks that cannot be diversified away.

Systematic vs. idiosyncratic risk

- Firm-specific events:
  - Such events may include new product innovations, lawsuits, changes in management, labor strikes, …
  - These firm-specific or idiosyncratic risks can be diversified away.

Example: a 2-factor model

- Suppose that the only two systematic sources of risk are:
  - Unanticipated changes in economic growth; and
  - Unanticipated changes in energy prices.
- The return on any stock respond to both sources of macro shocks and to firm-specific shocks:

expected return | factor loadings | firm-specific shocks

$$\tilde{r}_i = \bar{r}_i + b_{i,GR}\ \tilde{f}_{GR} + b_{i,EN}\ \tilde{f}_{EN} + \tilde{\epsilon}_i$$

systematic shocks

Example: a 2-factor model

$$\tilde{r}_i = \bar{r}_i + b_{i,GR}\,\tilde{f}_{GR} + b_{i,EN}\,\tilde{f}_{EN} + \tilde{\epsilon}_i$$

- A solar panel installer.
- Cash flows have moderate exposure to economic growth → b_{GR} is positive.
- Benefits from rising energy costs → b_{EN} is likely positive and large.

- A long-distance trucking firm.
- Cash flows are very sensitive to economic activity → b_{GR} is likely positive and large.
- Sensitive to energy costs → b_{EN} is negative and large.

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A single-factor model

- A large number of risk assets, $i = 1, 2, 3, \ldots$
  - $\tilde{r}_i$ is the (random) return.
  - $\bar{r}_i$ is the expected return.
- Returns are driven by a common, systematic factor, and idiosyncratic shocks.
- $\tilde{F}$ is a systematic factor that affects most asset returns (e.g., return on the market portfolio).
- $\tilde{f}$ is the news component of this common factor: $\tilde{f} = \tilde{F} - \overline{F}$.
- Idiosyncratic shock to asset i: $\tilde{\epsilon}_i$ with zero mean, $E[\tilde{\epsilon}_i] = 0$.
- A key assumption: $\tilde{\epsilon}_i$ are asset-specific, i.e., $\tilde{\epsilon}_i$ are uncorrelated across assets:

$$Cov(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0 \text{ for } i \neq j$$

A single-factor model

- Describe asset returns as

$$\tilde{r}_i = \underbrace{\bar{r}_i}_{\substack{\text{expected} \\ \text{return}}} + \underbrace{b_i \tilde{f} + \tilde{\epsilon}_i}_{\text{risk}}$$

- Return variance

$$\sigma_i^2 = \underbrace{b_i^2 \sigma_f^2}_{\substack{\text{systematic} \\ \text{risk}}} + \underbrace{\text{Var}(\tilde{\epsilon}_i)}_{\substack{\text{idiosyncratic} \\ \text{risk}}}$$

- Return covariance

$$\text{Cov}\left(\tilde{r}_i, \tilde{r}_j\right) = \text{Cov}(\bar{r}_i + b_i \tilde{f} + \tilde{\epsilon}_i, \bar{r}_j + b_j \tilde{f} + \tilde{\epsilon}_j) = b_i b_j \sigma_f^2$$

because $\text{Cov}\left(\tilde{f}, \tilde{\epsilon}_i\right) = \text{Cov}(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0$.

  - Factor exposure determines how much asset returns co-move.
  - Idiosyncratic risk affects individual return variance.

Multifactor models

- A multifactor model specifies

$$\tilde{r}_i = \bar{r}_i + \underbrace{b_{i,1}\,\tilde{f}_1 + b_{i,2}\,\tilde{f}_2 + \cdots + b_{i,K}\,\tilde{f}_K}_{\substack{\text{systematic} \\ \text{component}}} + \tilde{\epsilon}_i$$

- The $\tilde{f}_1$, $\tilde{f}_2$, …, $\tilde{f}_K$ are the common factors.
  - Common factors may be correlated with each other.
  - The $b_{i,1}$, $b_{i,2}$,…, $b_{i,K}$, are the asset's factor sensitivities (or factor loadings or factor betas).
- The residuals are firm-specific:

$$\mathrm{Cov}(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0 \quad \text{for all } i \neq j$$

- We assume that all factor shocks have zero mean, $\mathrm{E}[\tilde{f}_k] = 0$, $k = 1,2, \dots, K$.

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Portfolio return

- The return process of a portfolio is

$$\tilde{r}_p = \bar{r}_p + b_{p,1}\,\tilde{f}_1 + b_{p,2}\,\tilde{f}_2 + \cdots + b_{p,K}\,\tilde{f}_K + \tilde{\epsilon}_p$$

where

$$\bar{r}_p = \sum_{i=1}^{N} w_i\,\bar{r}_i\,, \quad b_{p,k} = \sum_{i=1}^{N} w_i\,b_{i,k}\,, \quad \tilde{\epsilon}_p = \sum_{i=1}^{N} w_i\,\tilde{\epsilon}_i$$

- Because $\tilde{\epsilon}_i$'s are uncorrelated, the non-systematic variance of a portfolio is

$$\mathrm{Var}(\tilde{\epsilon}_p) = \mathrm{Var}\left(\sum_{i=1}^{N} w_i\tilde{\epsilon}_i\right) = \sum_{i=1}^{N} w_i^2\mathrm{Var}(\tilde{\epsilon}_i)$$

Well diversified portfolios

- Consider an equally-weighted portfolio with $w_i = 1/N$.
- Let $\overline{\sigma_i^2}$ denote the average non-systematic variance:

$$\overline{\sigma_i^2} = \frac{1}{N}\sum_{i=1}^{N} \mathrm{Var}(\tilde{\epsilon}_i)$$

- Then, idiosyncratic portfolio variance is

$$\mathrm{Var}(\tilde{\epsilon}_p) = \frac{1}{N}\overline{\sigma_i^2}$$

- When N goes to infinity → non-systematic variance goes to zero!
  - This result does not require that portfolios have equal weights. The conclusion holds as long as portfolio weights are relatively evenly distributed across the assets.

Well diversified portfolios

- Asset-specific risk is uncorrelated across assets, it can be diversified away by holding large diversified portfolios.
- A well-diversified portfolio is a portfolio that distributes holdings over a large number of securities so that the non-systematic variance $\mathrm{Var}(\tilde{\epsilon}_p)$ is negligible.
- In a well-diversified portfolio, firm-specific effects average out:

$$\tilde{\epsilon}_p \approx 0$$

- For a well-diversified portfolio, only systematic (factor) risk is present:

$$\tilde{r}_p = \bar{r}_p + b_{p,1}\,\tilde{f}_1 + b_{p,2}\,\tilde{f}_2 + \cdots + b_{p,K}\,\tilde{f}_K$$

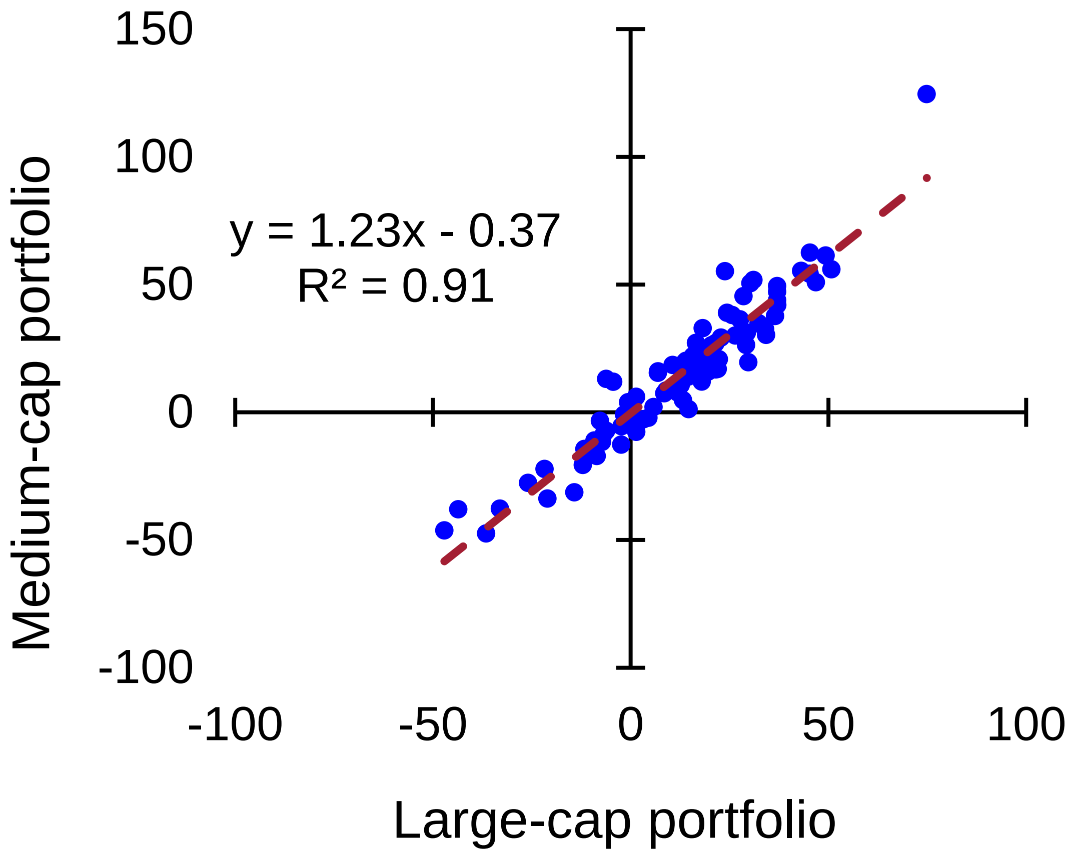
Example

- Consider two portfolios:
  - Annual returns, starting in 1926. Source: Kenneth R. French's Data Library.
  - P_1 contains the largest US stocks, top 30% relative to NYSE stocks by size (~500 securities in recent years).
  - P_2 contains mid-cap US stocks, next 40% relative to NYSE stocks by size (~1,000 recently).
- These portfolios are well-diversified, and do not overlap in holdings.
- If return distribution was described by a single-factor model, we would observe an approximate linear relation between the two portfolios

$$\tilde{r}_{P_i} = \bar{r}_{P_i} + b_{P_i}\tilde{f}, \qquad i = 1,2$$

Example: returns of size-sorted portfolios

- Both portfolios are exposed to the market-wide shocks, which account for most of return variation for each portfolio.



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An arbitrage argument

- Example: single systematic factor, two well-diversified portfolios.

	Expected excess return	Factor loading
Portfolio A	5%	1.0
Portfolio B	8%	2.0

- There is arbitrage in this market!
- Arbitrage strategy:
  - Borrow \$1;
  - Short \$1 of Portfolio B;
  - Invest \$2 of Portfolio A.
  - No risk (zero factor loading), zero investment, and positive payoff.

$$\begin{aligned}\text{Payoff} = & -(1 + r_f) \times \$1 \\ & -(1 + r_f + 8\% + 2.0\tilde{f}) \times \$1 \\ & +(1 + r_f + 5\% + 1.0\tilde{f}) \times \$2 \\ = & \$0.02\end{aligned}$$

APT pricing relation

- Expected excess returns and factor loading must be linearly related.
- For a single-factor model, expected excess returns on diversified portfolios must be proportional to the factor loading:

$$\bar{r}_p - r_f = \lambda\, b_p$$

- Suppose this is not the case:

$$\bar{r}_q - r_f = \lambda' b_q, \quad \lambda' \neq \lambda, \; b_q \neq 0.$$

- Create an arbitrage trade:
  - Short \$1 of portfolio p;
  - Buy $\$(b_p/b_q)$ of portfolio q;
  - Borrow $\$(b_p/b_q - 1)$.

$$\begin{aligned}\text{Payoff} = & -\left(1 + r_f + \lambda\, b_p + b_p \tilde{f}\right) \times 1 \\ & +\left(1 + r_f + \lambda' b_q + b_q \tilde{f}\right) \times (b_p/b_q) \\ & -\left(1 + r_f\right) \times (b_p/b_q - 1) \\ = & \; (\lambda' - \lambda) b_p \neq 0 \Rightarrow \text{arbitrage}\end{aligned}$$

APT pricing relation

- Arbitrage opportunities cannot exist in a frictionless market.
- To avoid arbitrage, expected excess returns (risk premia) on all well-diversified portfolios must satisfy

$$\bar{r}_p - r_f \quad = \quad \lambda \quad \times \quad b_p$$

Risk premium = Price of risk × Quantity of risk

- λ tells us how much compensation one earns in the market for a unit of factor risk exposure.
- λ is called the market price of risk of the factor, or the factor risk premium.

APT relation for multi-factor models

- APT pricing relation generalizes to multi-factor models

$$\bar{r}_p - r_f = \lambda_1 b_{p,1} + \lambda_2 b_{p,2} + \cdots + \lambda_K b_{p,K}$$

- Expected excess return on a diversified portfolio is determined by its loadings on the common factors:
  - Factor exposures measure portfolio risk;
  - Multi-dimensional nature of risk: each factor exposure carries its own risk premium.
- Intuition: can construct multiple portfolios with the same factor loadings – these all must have the same risk premium to avoid arbitrage.
- Therefore, portfolio risk premium is determined by its factor loadings.

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Factor risk premia

- We can use the APT relation

$$\bar{r}_p - r_f = \lambda_1 b_{p,1} + \lambda_2 b_{p,2} + \cdots + \lambda_K b_{p,K}$$

to recover prices of risk for each factor as implied by expected returns on other assets.

Recovering risk prices from portfolio returns

- Consider an example with $K = 2$ factors: economic growth shock (GR) and energy price shock (EN).

- Start with the general APT relation

$$\bar{r}_p - r_f = \lambda_1 b_{p,1} + \lambda_2 b_{p,2} + \cdots + \lambda_K b_{p,K}$$

- Observe risk premia on two well-diversified portfolios, A and B:

	Expected Return	Factor Loadings	
		GR	EN
Portfolio A	12%	1.0	1.25
Portfolio B	10%	2.0	-0.50
Risk-free asset	2%		

- Want to recover factor risk premia for GR and EN.

Recovering risk prices from portfolio returns

- APT relation implies two equations for expected excess returns on portfolios A and B:

$$\underbrace{12\% - 2\%}_{\text{risk premium}} = \underbrace{1.0}_{\text{factor loading}} \times \underbrace{\lambda_{GR}}_{\text{price of risk}} + \underbrace{1.25}_{\text{factor loading}} \times \underbrace{\lambda_{EN}}_{\text{price of risk}} \quad (A)$$

$$10\% - 2\% = 2.0\,\lambda_{GR} - 0.50\,\lambda_{EN} \quad (B)$$

- Solving these equations, we find

$$\lambda_{GR} = 5\%,$$

$$\lambda_{EN} = 4\%.$$

- All other portfolios must have expected returns consistent with these factor premia, e.g., if portfolio C has factor loadings $b_{GR} = 1.0$, and $b_{EN} = 0.5$, then

$$\bar{r}_C - r_f = 1.0\,\lambda_{GR} + 0.5\,\lambda_{EN} = 7\%$$

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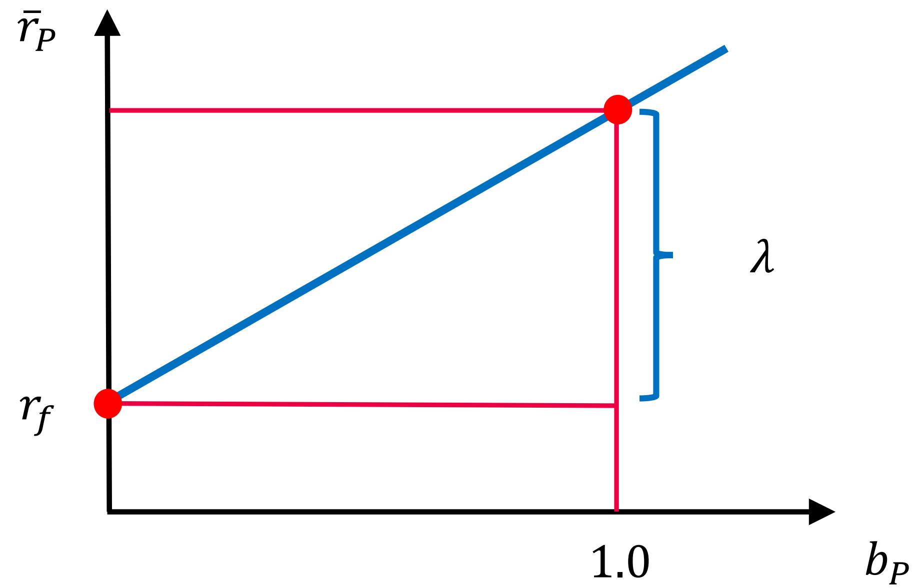
Factor mimicking portfolios

- Consider a special case of a single-factor model.
- Factor mimicking portfolios are portfolios with unit factor exposure, $b_P = 1$.
- Risk premium on the factor-mimicking portfolio equals the factor risk premium.
- This portfolio is perfectly correlated with the factor – can use it instead of the factor in the APT relation.



Multiple factors

- A model with K factors and linearly independent portfolios P_1, P_2, …, P_K.
- Construct factor-mimicking portfolios: risk premium of each factor equals the expected excess return on the factor-mimicking portfolio.
- A factor-mimicking portfolio for factor j is a well-diversified portfolio with a beta of 1 on factor j and a beta of 0 on any other factor.
- A factor-mimicking portfolio for factor k with weights $(w_0, w_1, w_2, \dots, w_K)$, w_0 in the risk-free asset, satisfies

$$\underbrace{w_1}_{\substack{\text{portfolio weight} \\ \text{of } P_1}} \times \underbrace{b_{P_1,1}}_{\substack{\text{factor loading} \\ \text{of } P_1 \text{ on factor 1}}} + w_2 b_{P_2,1} + \cdots + w_K b_{P_K,1} = 0 \qquad (1)$$

…

$$w_1 b_{P_1,j} + w_2 b_{P_2,j} + \cdots + w_K b_{P_K,j} = 1 \qquad (j)$$

…

$$w_1 b_{P_1,K} + w_2 b_{P_2,K} + \cdots + w_K b_{P_K,K} = 0 \qquad (K)$$

Example

- Mimic the Energy shock (EN) using portfolios A and B: weights w_A, w_B.

	Expected Return	Factor Loadings	
		GR	EN
Portfolio A	12%	1.0	1.25
Portfolio B	10%	2.0	-0.50
Risk-free asset	2%		

$$1.0\, w_A + 2.0\, w_B = 0 \quad (GR)$$
$$1.25\, w_A - 0.50\, w_B = 1 \quad (EN)$$

- Result: $w_A = 0.67, \;\; w_B = -0.33.$

- The risk premium on the Energy factor is then

$$\lambda_{EN} = 10\%\, w_A + 8\%\, w_B = 4\%$$

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APT for individual securities

- For any well-diversified portfolio p with factor sensitivities $b_{p,1}, \ldots, b_{p,K}$, the risk premium equals

$$\bar{r}_p - r_f = \lambda_1 b_{p,1} + \lambda_2 b_{p,2} + \cdots + \lambda_K b_{p,K}$$

where λ_n is the risk premium on the n^{th} factor.

- This result also applies to almost all individual securities.

- This is the Arbitrage Pricing Theory (APT), developed by Stephen Ross in 1976.

APT for individual securities: intuition

- Suppose that many assets violate the APT relation.
- Then can find many assets for which $\alpha_i \neq 0$ in

$$\bar{r}_i - r_f = \alpha_i + \lambda_1 b_{i,1} + \lambda_2 b_{i,2} + \cdots + \lambda_K b_{i,K}$$

- Suppose many assets have a positive alpha (negative values work analogously).
- Combine them in a well-diversified, equally-weighted portfolio p^*:

$$\bar{r}_{p^*} - r_f = \overline{\alpha} + \lambda_1 b_{p^*,1} + \lambda_2 b_{p^*,2} + \cdots + \lambda_K b_{p^*,K}$$

  where $\overline{\alpha}$ is the average alpha across assets in p^*.

- This contradicts the APT results for diversified portfolios, so we cannot find many assets that violate APT.

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Implementation of APT

- Three steps:
  - Identify/choose the factors.
    - Economic variables that are thought to affect asset returns.
    - How many and which?
  - Estimate factor loadings of assets.
    - Usually by a time-series regression of diversified portfolio returns on factors.
  - Estimate factor premia.
    - Usually by a cross-sectional regression of excess returns on factor loadings.
- End up with an assessment of which factors matter and how much.

A Macro-factor model

- Chen, Roll, and Ross (1986, *Journal of Business*).
- In addition to the market factor, use economic variables to represent systematic factors explaining the returns of financial assets.
- Monthly growth rate of industrial production (MP).
- Changes in expected inflation (DEI).
  - Measured by changes in T-Bill rates.
- Unexpected inflation (UI).

A Macro-factor model

- Unexpected changes in risk premium (UPR), measured as the difference between returns on bonds rated Baa (or lower), and long-term government bonds.
- Unexpected changes in the term premium (UTS), measured as the difference between returns on long-term government bonds and T-Bills.
- Data: Monthly observations from 1953 to 1984.

Estimation: betas

- Group stocks into 20 size portfolios (5% smallest to 5% largest).
- Run time-series regressions (5 years of monthly data) to obtain factor sensitivities.
  - For each size portfolio i, estimate β's in

$$R_{i,t} = a_i + \beta_{i,RM} RM_t + \beta_{i,MP} MP_t + \beta_{i,DEI} DEI_t$$
$$+\beta_{i,UI} UI_t + \beta_{i,UPR} UPR_t + \beta_{i,UTS} UTS_t + \epsilon_{i,t}$$

  where RM is for the return on the market index (e.g., value-weighted stock index).

- Result: estimates $\beta_{i,RM}, \beta_{i,MP}, \beta_{i,DEI}, \beta_{i,UI}, \beta_{i,UPR}, \beta_{i,UTS}$ for each portfolio i.

Estimation: risk premia

- Run cross-sectional regressions to get factor risk premia and determine if they are statistically significant.
  - Using the β's of the 20 portfolios, estimate λ's in a cross-sectional regression of monthly returns on the betas

$$R_i = a_i + \lambda_{RM}\beta_{i,RM} + \lambda_{MP}\beta_{i,MP} + \lambda_{DEI}\beta_{i,DEI} + \lambda_{UI}\beta_{i,UI} + \lambda_{UPR}\beta_{i,UPR} + \lambda_{UTS}\beta_{i,UTS} + \epsilon_i$$

  - Average $\hat{\lambda}_k$ over time to estimate the risk premium for factor k.
  - Result: estimates of the risk premium (λ) for each of the factors.

Results

- Factors are not very highly correlated.
- All economic factors are priced.
- Market factor is not priced separately from other factors.

	VWNY	MP	DEI	UI	UPR	UTS	Constant
1958–84	−2.403	11.756	−.123	−.795	8.274	−5.905	10.713
	(−.633)	(3.054)	(−1.600)	(−2.376)	(2.972)	(−1.879)	(2.755)
1958–67	1.359	12.394	.005	−.209	5.204	−.086	9.527
	(.277)	(1.789)	(.064)	(−.415)	(1.815)	(−.040)	(1.984)
1968–77	−5.269	13.466	−.255	−1.421	12.897	−11.708	8.582
	(−.717)	(2.038)	(−3.237)	(−3.106)	(2.955)	(−2.299)	(1.167)
1978–84	−3.683	8.402	−.116	−.739	6.056	−5.928	15.452
	(−.491)	(1.432)	(−.458)	(−.869)	(.782)	(−.644)	(1.867)

NOTE.—VWNY = return on the value-weighted NYSE index; EWNY = return on the equally weighted NYSE index; MP = monthly growth rate in industrial production; DEI = change in expected inflation; UI = unanticipated inflation; UPR = unanticipated change in the risk premium (Baa and under return − long-term government bond return); UTS = unanticipated change in the term structure (long-term government bond return − Treasury-bill rate); and YP = yearly growth rate in industrial production. *t*-statistics are in parentheses.

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The Fama-French factor model

- Fama and French (1993 Journal of Financial Economics, 1996 Journal of Finance).
- Factors do not necessarily have to be macroeconomic variables.
- Sufficient that they correlate with changes in the macroeconomy.
- Generate factor-mimicking portfolios by sorting firms by size and book-to-market ratio.
    - Intuition: small firms and high B/M firms are exposed differently to macroeconomic factors.

The Fama-French factor model

- Factor portfolios:
  - $R_M - R_f$: Return on the value-weighted market minus the T-Bill rate.
  - SMB ("Small minus Big"): Return on small-cap stocks minus return on large-cap stocks.
  - HML ("High minus Low"): Return on stocks with high B/M ratio minus return on stocks with low B/M ratio.
- Factors carry significant risk premia: $\lambda_{R_M - R_f} = 0.43\%$, $\lambda_{SMB} = 0.27\%$, $\lambda_{HML} = 0.40\%$ (per month) (Fama and French 1993).

The Fama-French factor model

- Data: Monthly returns on all NYSE, AMEX, and NASDAQ stocks from 1963 to 1991.
- Methodology:
  - Form 25 stock portfolios based on size and book-to-market equity.
  - Run time-series regressions of monthly excess returns on the returns to the market portfolio and mimicking portfolios for size and book-to-market equity (see next page).

$$R_i - R_f = \alpha_i + b_i(R_M - R_f) + s_i SMB + h_i HML + \epsilon_i$$

  - Evaluate factor loadings and the intercepts (APT alphas).

Results

- B/M and Size portfolios exhibit a large spread in average returns.
- Small stocks (“Small” row) outperform large stocks (“Big” row) on average.
- Value stocks (“High” column) outperform growth stocks (“Low” column) on average.

	Book-to-Market Equity (BE/ME) Quintiles									
Size	Low	2	3	4	High	Low	2	3	4	High
Panel A: Summary Statistics										
	Means					Standard Deviations				
Small	0.31	0.70	0.82	0.95	1.08	7.67	6.74	6.14	5.85	6.14
2	0.48	0.71	0.91	0.93	1.09	7.13	6.25	5.71	5.23	5.94
3	0.44	0.68	0.75	0.86	1.05	6.52	5.53	5.11	4.79	5.48
4	0.51	0.39	0.64	0.80	1.04	5.86	5.28	4.97	4.81	5.67
Big	0.37	0.39	0.36	0.58	0.71	4.84	4.61	4.28	4.18	4.89

Results

- Factors do a good job explaining the cross-section of returns.

	b					t(b)				
Small	1.03	1.01	0.94	0.89	0.94	39.10	50.89	59.93	58.47	57.71
2	1.10	1.04	0.99	0.97	1.08	52.94	61.14	58.17	62.97	65.58
3	1.10	1.02	0.98	0.97	1.07	57.08	55.49	53.11	55.96	52.37
4	1.07	1.07	1.05	1.03	1.18	54.77	54.48	51.79	45.76	46.27
Big	0.96	1.02	0.98	0.99	1.07	60.25	57.77	47.03	53.25	37.18

	s					t(s)				
Small	1.47	1.27	1.18	1.17	1.23	39.01	44.48	52.26	53.82	52.65
2	1.01	0.97	0.88	0.73	0.90	34.10	39.94	36.19	32.92	38.17
3	0.75	0.63	0.59	0.47	0.64	27.09	24.13	22.37	18.97	22.01
4	0.36	0.30	0.29	0.22	0.41	12.87	10.64	10.17	6.82	11.26
Big	−0.16	−0.13	−0.25	−0.16	−0.03	−6.97	−5.12	−8.45	−6.21	−0.77

	h					t(h)				
Small	−0.27	0.10	0.25	0.37	0.63	−6.28	3.03	9.74	15.16	23.62
2	−0.49	0.00	0.26	0.46	0.69	−14.66	0.34	9.21	18.14	25.59
3	−0.39	0.03	0.32	0.49	0.68	−12.56	0.89	10.73	17.45	20.43
4	−0.44	0.03	0.31	0.54	0.72	−13.98	0.97	9.45	14.70	17.34
Big	−0.47	0.00	0.20	0.56	0.82	−18.23	0.18	6.04	18.71	17.57

Results

- Factors do a good job explaining the cross-section of returns: high R^2
- Portfolios are well diversified, returns are well explained by the common factors.

	R^2					s(e)				
Small	0.93	0.95	0.96	0.96	0.96	1.97	1.49	1.18	1.13	1.22
2	0.95	0.96	0.95	0.95	0.96	1.55	1.27	1.28	1.16	1.23
3	0.95	0.94	0.93	0.93	0.92	1.44	1.37	1.38	1.30	1.52
4	0.94	0.92	0.91	0.88	0.89	1.46	1.47	1.51	1.69	1.91
Big	0.94	0.92	0.87	0.89	0.81	1.19	1.32	1.55	1.39	2.15

Results

- Intercepts (α_i) from 3-factor regressions are close to 0.
  - Estimates of alphas are statistically indistinguishable from 0 for most portfolios.
- Some violations for small stocks (extreme B/M quintiles).

	Book-to-Market Equity (BE/ME) Quintiles									
Size	Low	2	3	4	High	Low	2	3	4	High
Panel B: Regressions: $R_i - R_f = a_i + b_i(R_M - R_f) + s_i SMB + h_i HML + e_i$										
	a					t(a)				
Small	−0.45	−0.16	−0.05	0.04	0.02	−4.19	−2.04	−0.82	0.69	0.29
2	−0.07	−0.04	0.09	0.07	0.03	−0.80	−0.59	1.33	1.13	0.51
3	−0.08	0.04	−0.00	0.06	0.07	−1.07	0.47	−0.06	0.88	0.89
4	0.14	−0.19	−0.06	0.02	0.06	1.74	−2.43	−0.73	0.27	0.59
Big	0.20	−0.04	−0.10	−0.08	−0.14	3.14	−0.52	−1.23	−1.07	−1.17

Conclusion

- We showed how risk premia on common risk factors can be inferred from historical returns on financial assets.
- Multiple techniques available, including cross-sectional and time-series regression methods.
- The main weakness of APT: the theory does not tell us what the common factors are, this is an empirical question.
- APT model is a flexible and general valuation framework.
- Absence of arbitrage imposes internal consistency, APT connects expected returns to measures of risk – loadings on the common factors.

Summary

- The Main Idea of APT
- Factor Models
- Well Diversified Portfolios
- Expected Returns on Diversified Portfolios
- Factor Risk Prices / Risk Premia
- Factor-Mimicking Portfolios
- APT for Individual Securities
- Implementation of APT (Macro Factor Model)
- Implementation of APT (Portfolio Factor Model)