ITMO UNIVERSITY

How to Win Coding Competitions: Secrets of Champions

Week 3: Sorting and Search Algorithms
Lecture 7: Lower bound. Stable sorting. Comparators

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Lower bound for comparison sorting

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$$\log_2(N!) \approx \log_2\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right) = n \log_2 n - \Theta(n) + O(\log n) = \Theta(n \log n)$$

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Mergesort is asymptotically optimal.
Quicksort is asymptotically optimal on average.

Stable sorting I

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- Mergesort: stable, as merge always picks the left element from two equal ones
- Quicksort: not stable. In an array of two equal items, it will swap them

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Cons:

- Additional memory needed for storing the indices
- More complicated ordering, may further decrease performance

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Requirements for a comparator:

- Total (every two elements should be reported as either “$<$”, “$=$”, or “$>$”)
- If it reports $a = b$, then a and b must be equal (in a problem-dependent sense)
- More specifically, if $a = b$, then $b = a$
- If $a < b$, then it should be that $b > a$ (and vice versa)

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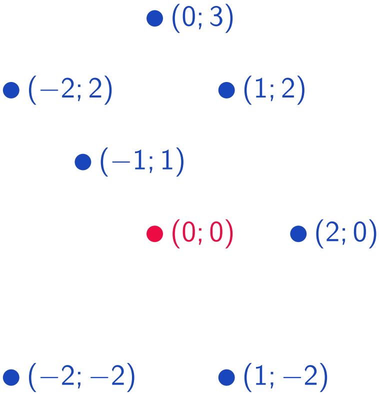
Implementation of a comparator:

- `Java`-way: return 0 if "equal", negative if "less", positive if "greater"
- `C++`-way: return `true` if "less", `false` otherwise

A non-trivial comparator



Sort the points counterclockwise around the red dot

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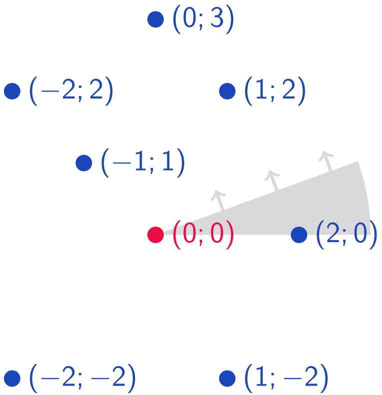

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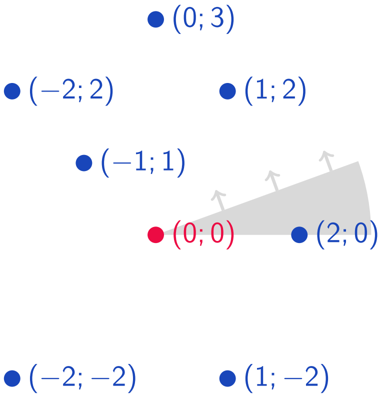

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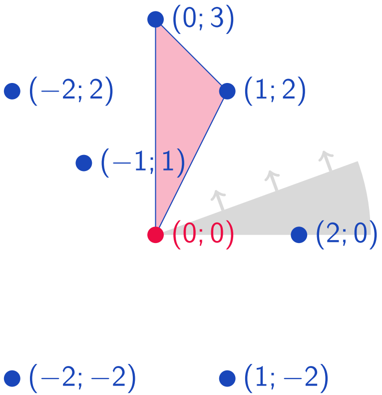

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```
function LessThan($(x_1, y_1)$, $(x_2, y_2)$)
    return $x_1 \cdot y_2 - x_2 \cdot y_1 > 0$
end function
```

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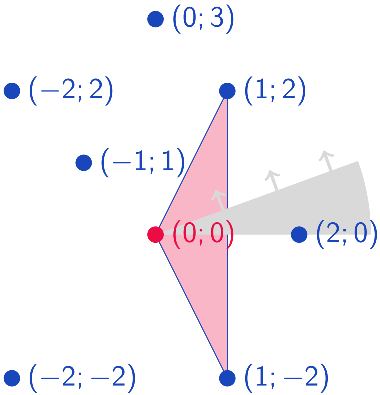

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Wait, what happens there?

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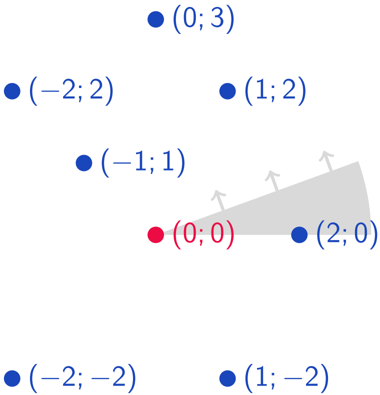

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…but first check the halfplanes

```
function LessThan($(x_1, y_1)$, $(x_2, y_2)$)
    $h_1 \leftarrow 0$, $h_2 \leftarrow 0$
    if $y_1 < 0$ or ($y_1 = 0$ and $x_1 < 0$) then $h_1 \leftarrow 1$ end if
    if $y_2 < 0$ or ($y_2 = 0$ and $x_2 < 0$) then $h_2 \leftarrow 1$ end if
    if $h_1 \neq h_2$ then return $h_1 < h_2$ end if
    return $x_1 \cdot y_2 - x_2 \cdot y_1 > 0$
end function
```

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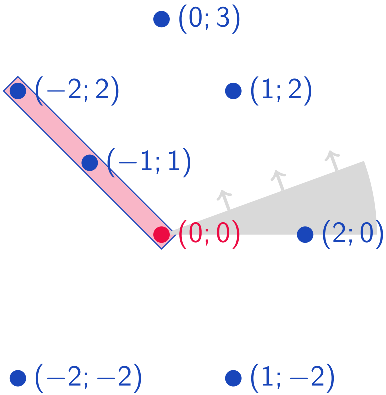

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Okay, but what happens for colinear points?

```
function LessThan($(x_1, y_1)$, $(x_2, y_2)$)
    $h_1 \leftarrow 0$, $h_2 \leftarrow 0$
    if $y_1 < 0$ or ($y_1 = 0$ and $x_1 < 0$) then $h_1 \leftarrow 1$ end if
    if $y_2 < 0$ or ($y_2 = 0$ and $x_2 < 0$) then $h_2 \leftarrow 1$ end if
    if $h_1 \neq h_2$ then return $h_1 < h_2$ end if
    return $x_1 \cdot y_2 - x_2 \cdot y_1 > 0$
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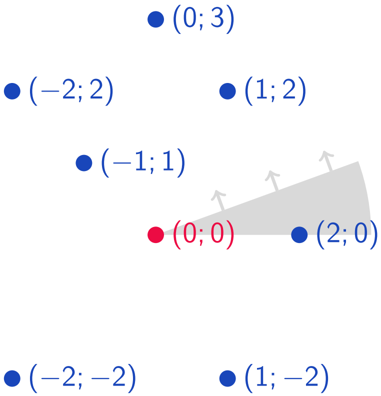

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. . . and, from colinear points, favor closer ones.

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    if $y_2 < 0$ or ($y_2 = 0$ and $x_2 < 0$) then $h_2 \leftarrow 1$ end if
    if $h_1 \neq h_2$ then return $h_1 < h_2$ end if
    $z \leftarrow x_1 \cdot y_2 - x_2 \cdot y_1$
    if $z = 0$ then return $x_1^2 + y_1^2 < x_2^2 + y_2^2$ end if
    return $z > 0$
end function
```