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CHAPTER **7**

# Functions as Models

## Chapter Outline

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**7.1**    **REVIEW OF FUNCTIONS**

**7.2**    **CHOOSING A FUNCTION MODEL**

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## 7.1 Review of Functions

### Learning Objectives

- Review the definition and notation for a function.
- Understand how functions are a special class of relations.
- Recognize that a function can be represented as an ordered pair, an equation or a graph.
- Review function families that we will use to model relationships in our data.

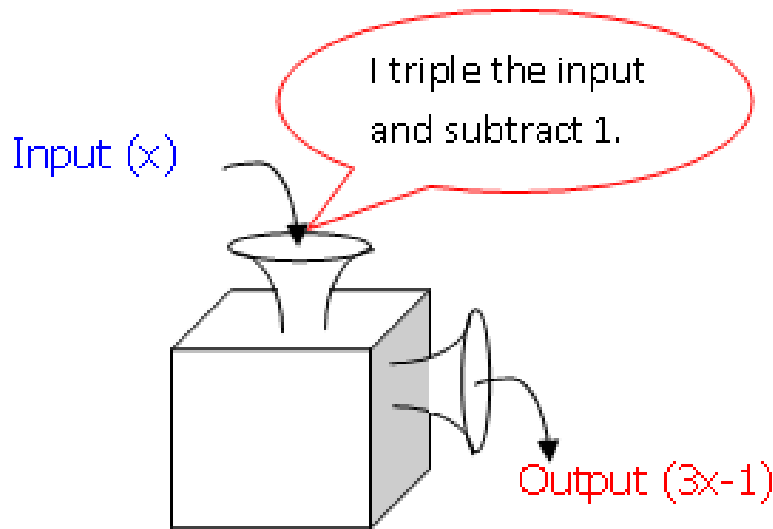
### Definition

We start with remembering the definition of a function. A function is a set of ordered pairs in which the first coordinate, usually  $x$ , matches with exactly one second coordinate,  $y$ . Equations that follow this definition can be written in function notation. The  $y$  coordinate represents the **dependent variable**, meaning the values of this variable depend upon what is substituted for the other variable.

A function can be expressed as an equation, as shown below. In the equation,  $f$  represents the function name and  $(x)$  represents the variable. In this case the parentheses do not mean multiplication; rather, they separate the function name from the **independent variable**.

$$\begin{array}{c}
 \textit{input} \\
 \downarrow \\
 \underbrace{f(x)} = y \leftarrow \textit{output} \\
 \textit{function} \\
 \textit{box}
 \end{array}$$

You may have seen functions represented by a function machine. These emphasize the fact that functions are rules that explain how the input and output are related. For example, the function below triples the value of the input ( $x$ ) and subtracts 1 from it. If 3 is fed into the machine,  $3(3) - 1 = 8$  comes out.



When naming a function the symbol  $f(x)$  is often used. The symbol  $f(x)$  is pronounced as “ $f$  of  $x$ .” This means that the equation is a function that is written in terms of the variable  $x$ . So, for example, if the function above is named  $f$ , then we write the function as  $f(x) = 3x - 1$ .

### Functions as Relations

Consider two situations shown below:

*Situation 1: You are selling candy bars for a school fundraiser. Each candy bar costs \$3.00*

*Situation 2: You collect data from several students in your class on their ages and their heights:*

(18, 65"), (17, 64"), (18, 67"), (18, 68"), (17, 66")

In Situation 1, let the variable  $x$  represent the number of candy bars that you sell, and let  $y$  represent the amount of money you make. If you sell  $x$  candy bars, you will make  $y = 3x$  dollars. For example, if you sell 25 candy bars, you will make  $3(25) = \$75.00$ . Notice that you can use the number of candy bars you sell to predict how much money you will make.

Now consider Situation 2. Can you similarly use the data to predict specific height, based on age? No, this is not the case in the second situation. For example, if a student is 18 years old, there are several heights that the student could be.

Situation 1 is an example of a *function*, and Situation 2 is *not* a function.

**A function is a relationship where each input number corresponds to one and only one output number.**

In Situation 1, for each different number of candy bar sales you *input*, there is one and only one *output* number representing your profit.

In Situation 2, if you *input* "18 years", there are multiple *outputs*, so you can't identify a specific relationship between age and height.

See the difference?

It is important to note that both situations above are relations. A **relation** is simply a relationship between two sets of numbers or data. For example, in Situation 2, we created a relationship between students' ages and heights, just by writing each student's information as an ordered pair. In Situation 1, there is a relationship between the number of candy bars you sell and the amount of money you make. The first example is different from the second because it

represents a **function**: *every  $x$  is paired with only one  $y$ .*

## Representing a Function

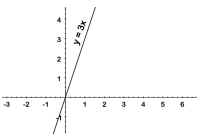
Functions may be presented in many ways. Some of the most common ways to represent functions include:

- a graph
- ordered pairs
- an equation
- a table of values
- an arrow or mapping diagram

### Example A

Determine if each relation is a function:

**TABLE 7.1:**

| Representation       | Example  |
|----------------------|--|
| Set of ordered pairs | (1,3), (2,6), (3,9), (4,12) (a subset of the ordered pairs for this function)        |
| Equation             | $y = 3x$   |
| Graph                |  |

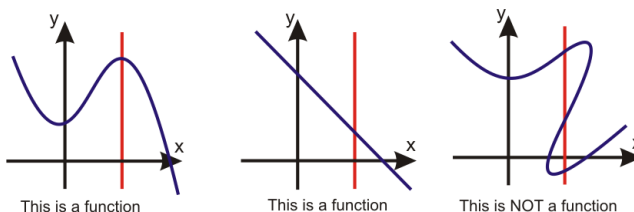
### Solution

The figure above actually shows the same function depicted in three different ways!

In the first representation, we are given a set of ordered pairs. To verify that this is a function, we must ensure that each  $x$ -value is associated with a single  $y$ -value. In this example, the first number in each pair (the  $x$ -value) is different, so we can be certain that there are no cases where a particular  $x$  is associated with more than one  $y$ .

In the second representation, the equation of a line, it is apparent that any number put in place of  $x$  will result in a different  $y$ , since the  $x$  number is simply being multiplied by 3.

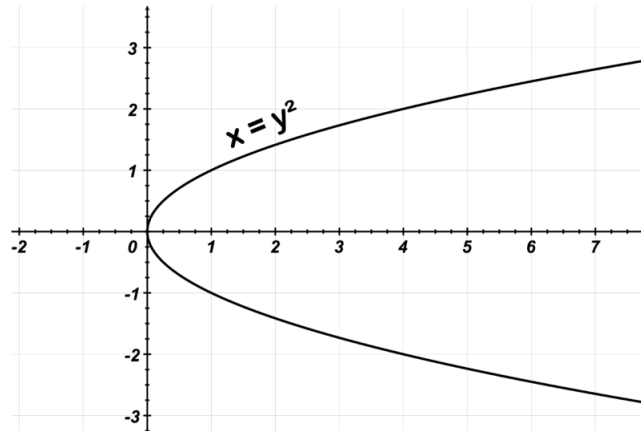
The third representation above is a graph. A good way to determine whether a relation is a function when looking at a graph is by doing a "vertical line test". If a vertical line can be drawn anywhere on the graph such that the line crosses the relation in two places, then the relation is not a function. If all possible vertical lines will only cross the relation in one place, then the relation is a function.



The vertical line test works because if a vertical line crosses a relation in more than one place it means that there must be two  $y$  values corresponding to one  $x$  value in that relation.

The graph above of  $y = 3x$  shows it is a function because any vertical line that is drawn only crosses the relation in one place.

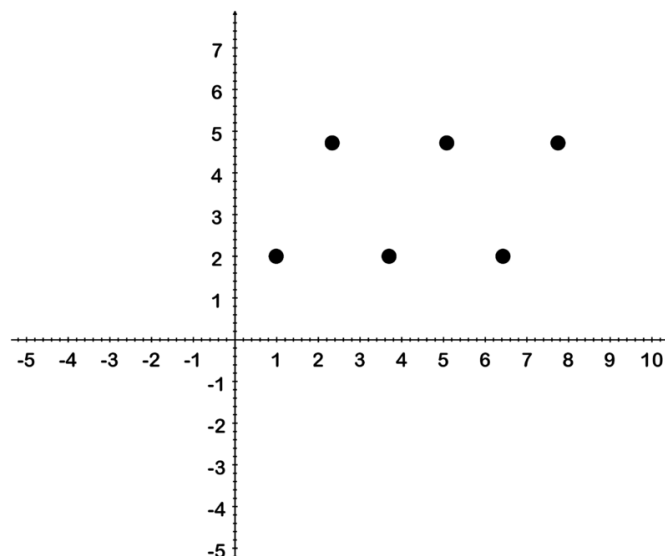
Conversely, the graph below of  $x = y^2$  shows it is not a function because a vertical line can be drawn that crosses the relation in two places.



### Example B

Determine if each relation is a function:

- (2, 4), (3, 9), (5, 11), (5, 12)
- see graph below



### Solutions

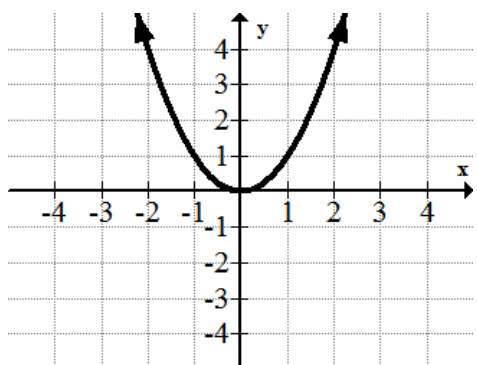
- This relation is not a function because 5 is paired with 11 and with 12.
- This relation is a function because every  $x$  is paired with only one  $y$ . A vertical line through the graph will always only encounter a single point.

## Function Families

Functions come in all different shapes, and there is great value in being able to recognize a function pattern. If mathematicians are cooks, then families of functions are their ingredients. Each family of functions has its own flavor and personality. Before you learn to combine functions to create an infinite number of potential models, you need to get a clear idea of the name of each function family and how it acts.

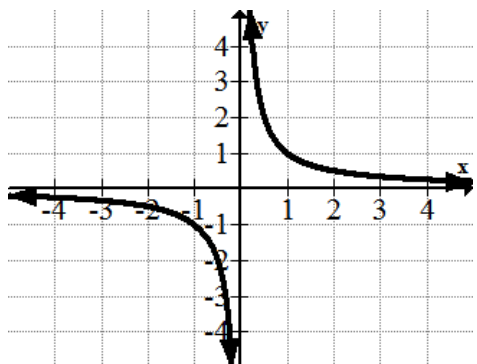
The following are functions that you may have seen in previous math courses:

**The Squaring Function:**  $f(x) = x^2$



The squaring function is commonly called a parabola and is useful for modeling the motion of falling objects. All parabolas are transformations of this squaring function.

**The Reciprocal Function:**  $f(x) = x^{-1} = \frac{1}{x}$

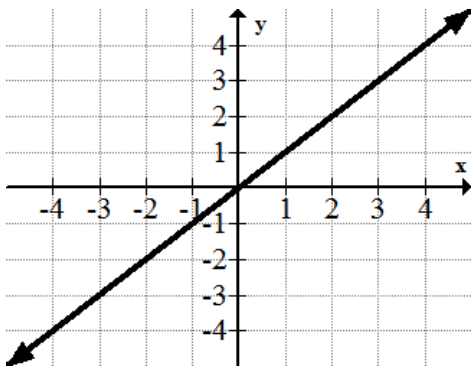


The reciprocal function is also known as a hyperbola and a rational function. It has two parts that are disconnected and is not defined at zero. Simple electric circuits are modeled with the reciprocal function.

## Modeling Data With Functions

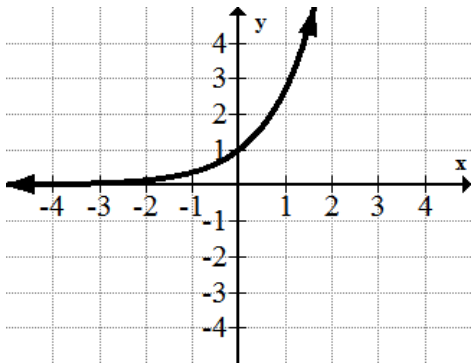
In this course, we will be working with three function families that are quite common in the kind of data that we would like to model. These include linear, exponential and logistic functions.

**The Identity (Linear) Function:**  $f(x) = x$



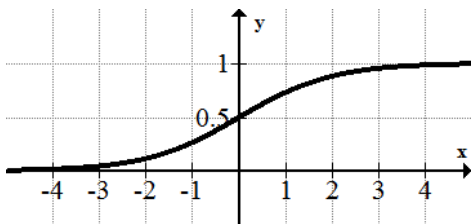
The identity function is the simplest function and all straight lines are transformations of the identity function family.

**The Exponential Function Family:**  $f(x) = e^x$



The exponential function family is one of the first functions you see where  $x$  is not the base of the exponent. This function eventually grows much faster than any power function.  $f(x) = 2^x$  is a very common exponential function as well. Many applications like biology and finance require the use of exponential growth.

**The Logistic Function:**  $f(t) = \frac{C}{1+ab^{-t}} = \frac{C}{1+ae^{-kt}}$



The logistic function is a combination of the exponential function and the reciprocal function. This curve is very powerful because it models population growths where the maximum population is limited by environmental resources.

## Vocabulary

A **relation** is a comparison of two or more sets of values.

A **function** is a relation of two or more sets of values in which each input number corresponds to one and only one output number.

A **function family** is a group of functions that all have the same basic shape.

### Guided Practice

Determine if each relation is a function:

- $(-1, 4)(0, 3)(1, 5)(1, 7)(2, 15)$
- $y = x$
- $(2, 0)(4, -1)(2.1, 4)(1, 4)(4, -1)$
- $y = 4x$
- $x = |y|$

### Solutions

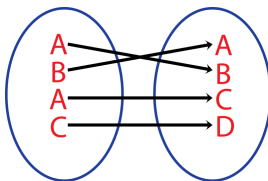
- There are two different 'outputs' (or  $y$ -values) for the 'input' (or  $x$ -value) of 1. Because we cannot know whether 1 should go with 5 or 7 at any given time, this relation is *not* a function.
- Since  $y = x$ , any time a number is chosen to represent  $x$ , that, and only that, number becomes  $y$ . From this it is apparent that each input has one and only one output: This relation *is* a function.
- Don't be fooled! This is a function, there is only one unique output for each input. The fact that both  $x$  values 2.1 and 1 are associated with  $y$  value 4 does not mean that 2.1 and 1 do not have a specific associated value. Also, no matter how close two  $x$ 's (2 and 2.1, for instance) may be, if they are not *exactly* the same, they don't affect the definition of a function.
- This is a function, very similar to #2. Any value chosen for  $x$  has one and only one associated value for  $y$  (4 times as big).
- This is *not* a function. This graph looks like a "<", with the point on the origin. Any value chosen for  $x$  will have 2 associated  $y$  values. For instance:  $4 = |-4|$  and  $4 = |4|$ .

### More Practice

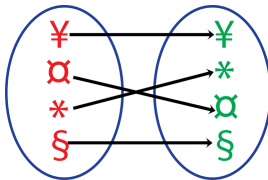
- What is the definition of a function?
- Can a function definition be written in the form  $x = 3y$  instead of  $y = 3x$ ?
- Is it mandatory for a function to have both an input and an output?
- Can a statement be a function if there is only one input and output?
- Give an example of a relation that is not a function, and explain why it is not a function.

For Questions 6 - 14, identify each relation as either a function, or not a function:

- $(2, 4)(4, 6)(6, 8)(3, 4)(5, 7)(8, 2)$
- $(-1, 6)(0, 4)(-4, 0)(-1, -6)(-3, -8)$
- 



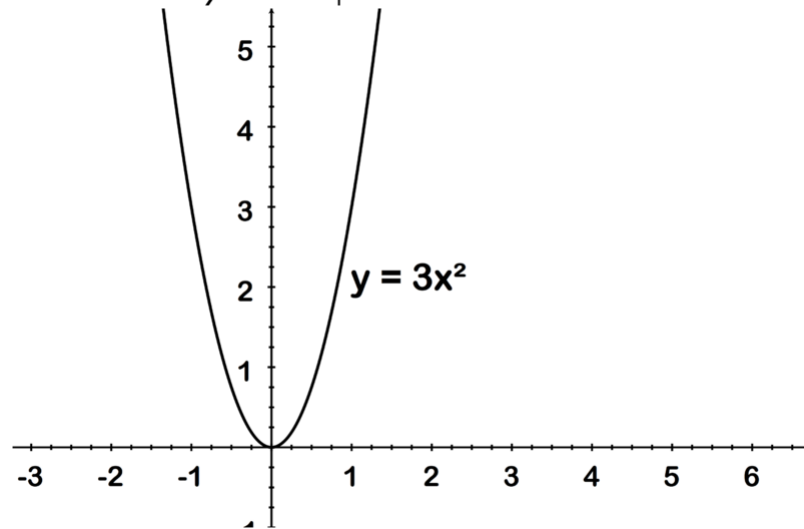
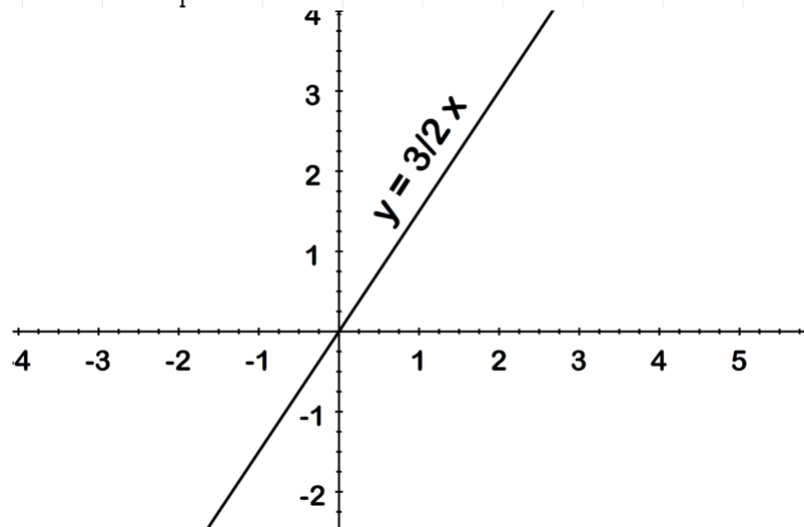
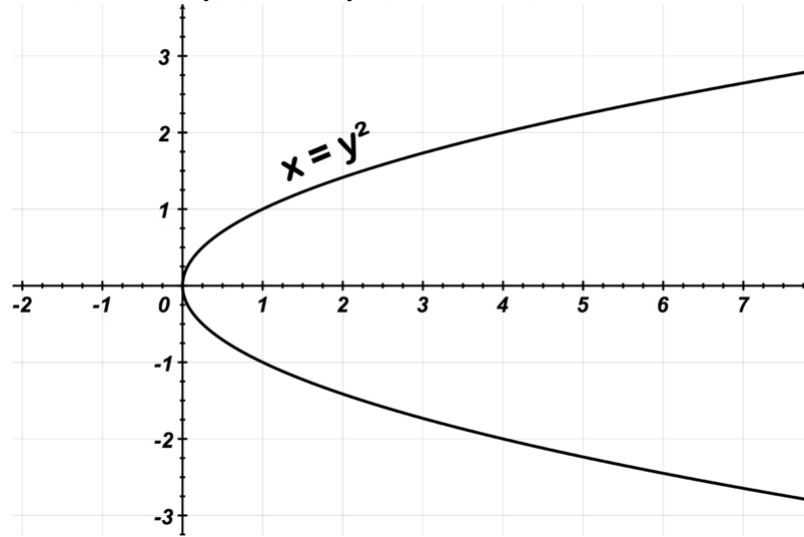
9.





10. (Jim, Kitty) (Joe, Betty) (Brian, Alice) (Jesus, Anissa) (Ken, Kelli)

11. (Jim, Alice) (Joe, Alice) (Brian, Betty) (Jim, Kitty) (Ken, Anissa)



15. At a Prom dance, each boy pins a corsage on his date. Is this an example of a function?

16. Later, at the same dance, Cory shows up with two dates, does this change the answer?

## 7.2 Choosing a Function Model

### Learning Objectives

- Read and understand a given problem situation.
- Develop a strategy for selecting a function model.

### Mathematical Modeling

As you learn more and more mathematical methods and skills, it is important to think about the purpose of mathematics and how it works as part of a bigger picture. Mathematics is used to solve problems that often arise from real-life situations. **Mathematical modeling** is a process by which we start with a real-life situation and arrive at a quantitative solution using the tools of mathematics.

Modeling involves creating a set of mathematical equations that describes a situation, solving those equations, and using them to understand the real-life problem. Often the model needs to be adjusted because it does not describe the situation as well as we wish.

A mathematical model can be used to gain understanding of a real-life situation by learning how the system works, which variables are important in the system, and how they are related to each other. Models can also be used to predict what a system will do for different values of the independent variable. Lastly, a model can be used to estimate quantities that are difficult to evaluate exactly.

Mathematical models are like other types of models. The goal is not to produce an exact copy of the “real” object but rather to give a representation of some aspect of the real thing. One of the most difficult parts of the modeling process is determining which function best describes a situation. We often find that the function we choose is not appropriate. Then we must choose a different one. Keep this in mind: if we try to fit a function to real data, it will rarely fit perfectly. However, if the function fits the data fairly well, we can choose it as our *model*. We can then use the *properties* of that function to describe what is happening in our data. That is what it means to model data.

### Examples of How Models are Fit

Below are several examples of how mathematical models can be fit to data to help explain the behavior of the data and allow us to predict what may happen if the values were to change. For now, read along and follow the thought process that is used to select a model and evaluate its fit. In the next few sections, you will be learning how to fit linear, exponential and logistic models to data and interpret what those models tell us.

#### Example A

You have a cylinder that is filled with water to a height of 50 centimeters. The cylinder has a hole at the bottom which is covered with a stopper. The stopper is released at time  $t = 0$  seconds and allowed to empty. The following data shows the height of the water in the cylinder at different times.

|            |    |      |      |      |      |      |      |      |     |     |     |     |     |
|------------|----|------|------|------|------|------|------|------|-----|-----|-----|-----|-----|
| Time(sec)  | 0  | 2    | 4    | 6    | 8    | 10   | 12   | 14   | 16  | 18  | 20  | 22  | 24  |
| Height(cm) | 50 | 42.5 | 35.7 | 29.5 | 23.8 | 18.8 | 14.3 | 10.5 | 7.2 | 4.6 | 2.5 | 1.1 | 0.2 |

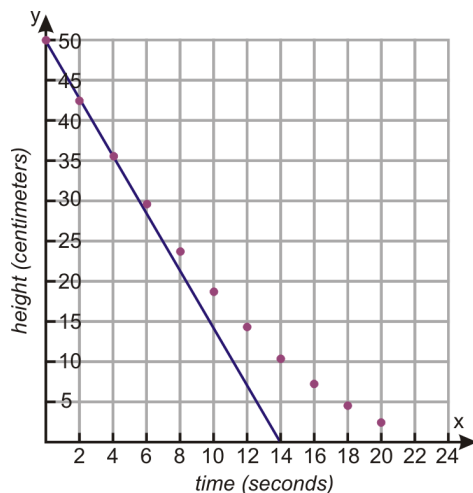
- Find the height (in centimeters) of water in the cylinder as a function of time in seconds.
- Find the height of the water when  $t = 5$  seconds.
- Find the height of the water when  $t = 13$  seconds.

### Solution

Begin by graphing to get a visual image of what the relationship looks like. Let's begin by defining the variables.

Define  $x =$  the time in seconds

$y =$  height of the water in centimeters



Notice that most of the points seem to fit on a straight line when the water level is high. Assume that a function relating the height of the water to the time is linear.

### Now Solve for the Equation of the Line

We will find the equation of the line from two data points from the time/height data. We have selected  $(0, 50)$  and  $(4, 35.7)$ , as these two points fall very close to the line.

Using the formula  $y = mx + b$ , we begin by solving for the slope ( $m$ ) and then for the  $y$ -intercept ( $b$ ).

$$m = \frac{-14.3}{4} = -3.58$$

$$35.7 = -3.58(4) + b$$

$$b = 50$$

- We obtain the function:  $y = -3.58x + 50$
- The height of the water when  $t = 5$  seconds is

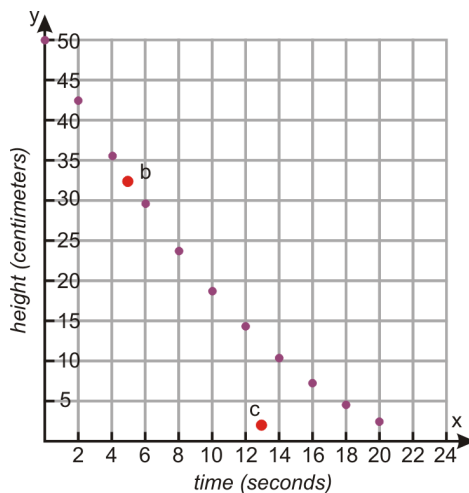
$$y = -3.58(5) + 50 = 32.1 \text{ centimeters}$$

- c. The height of the water when  $t = 13$  seconds is

$$y = -3.58(13) + 50 = 3.46 \text{ centimeters}$$

### Check

To check the validity of the solutions, let's plot the answers to b) and c) on the scatter plot. We see that the answer to b) is close to the rest of the data, but the answer to c) does not seem to follow the linear trend.



We can conclude that when the water level is high, the relationship between the height of the water and the time is a linear function. When the water level is low, we must change our assumption. There must be a non-linear relationship between the height and the time.

### Solve with new assumptions

If you are familiar with quadratic equations, you might recognize the form here and use a graphing calculator or other tool to help you find the equation of the function.

- We obtain the function  $y = .075x^2 - 3.87x + 50$
- The height of the water when  $t = 5$  seconds is

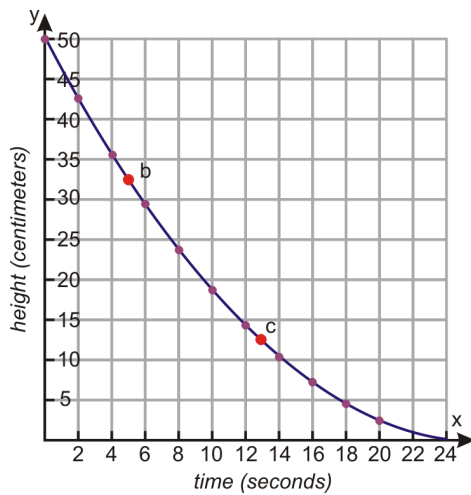
$$y = .075(5)^2 - 3.87(5) + 50 = 32.53 \text{ centimeters}$$

- The height of the water when  $t = 13$  seconds is

$$y = .075(13)^2 - 3.87(13) + 50 = 12.37 \text{ centimeters}$$

### Check

To check the validity of the solutions let's plot the answers to b) and c) on the scatterplot. We see that the answer to both b) and c) are close to the rest of the data.



We conclude that a quadratic function represents the situation more accurately than a linear function. However, for high water levels the linear function is an equally good representation.

### Example B

A scientist counts 2,000 fish in a lake. The fish population increases at a rate of 1.5 fish per generation but the lake has space and food for only 2,000,000 fish. The following table gives the number of fish (in thousands) in each generation.

|                    |   |    |    |     |      |      |      |      |
|--------------------|---|----|----|-----|------|------|------|------|
| Generation         | 0 | 4  | 8  | 12  | 16   | 20   | 24   | 28   |
| Number (thousands) | 2 | 15 | 75 | 343 | 1139 | 1864 | 1990 | 1999 |

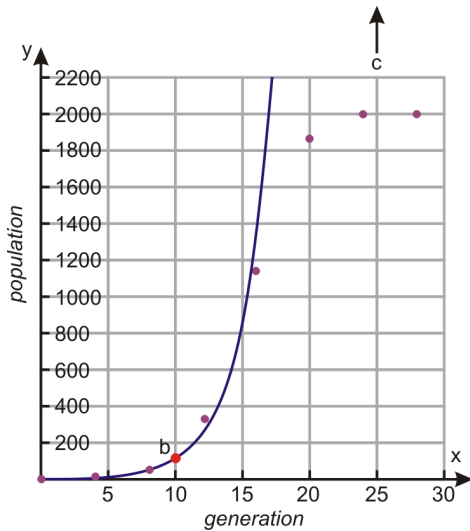
- Find the number of fish as a function of generation.
- Find the number of fish in generation 10.
- Find the number of fish in generation 25.

### Solution

We will define the variables and graph the relationship again.

Define  $x$  = the generation number

$y$  = the number of fish in the lake.



We know that a population can increase exponentially. So, maybe we assume that we can use an exponential function to describe the relationship between the generation number and the number of fish.

**Solve**

- a. Since the population increases at a rate of 1.5 per generation, assume the function  $y = 2(1.5)^x$
- b. The number of fish in generation 10 is:  $y = 2(1.5)^{10} = 115$  thousand fish
- c. The number of fish in generation 25 is:  $y = 2(1.5)^{25} = 50502$  thousand fish

**Check**

To check the validity of the solutions, let's plot the answers to b) and c) on the scatter plot (see the red dots above). We see that the answer to b) fits the data well but the answer to c) does not seem to follow the trend very closely. The result is not even on our graph!

When the population of fish is high, the fish compete for space and resources so they do not increase as fast. We must change our assumptions.

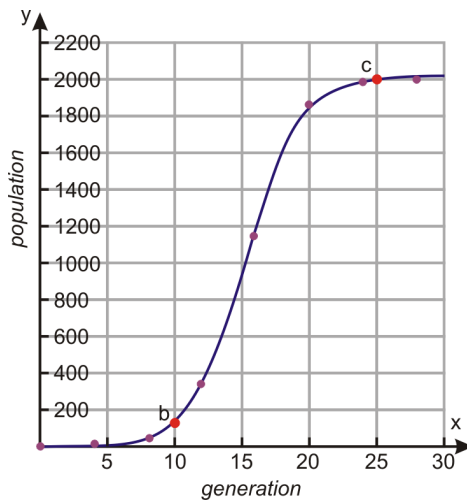
**Solve with new assumptions**

When we try different models, we find that a logistic function fits the data the best.

- a. We obtain the function  $y = \frac{2023.6}{1+1706.3(2.71)^{-.484x}}$
- b. The number of fish in generation 10 is  $y = \frac{2023.6}{1+1706.3(2.71)^{-.484(10)}} = 139.6$  thousand fish
- c. The number of fish in generation 25 is  $y = \frac{2023.6}{1+1706.3(2.71)^{-.484(25)}} = 2005$  thousand fish

**Check**

To check the validity of the solutions, let's plot the answers to b) and c) on the scatter plot. We see that the answer to both b) and c) are close to the rest of the data.



We conclude that a logistic function represents the situation more accurately than an exponential function. However, for small populations the exponential function is an equally good representation, and it is much easier to use in most cases.

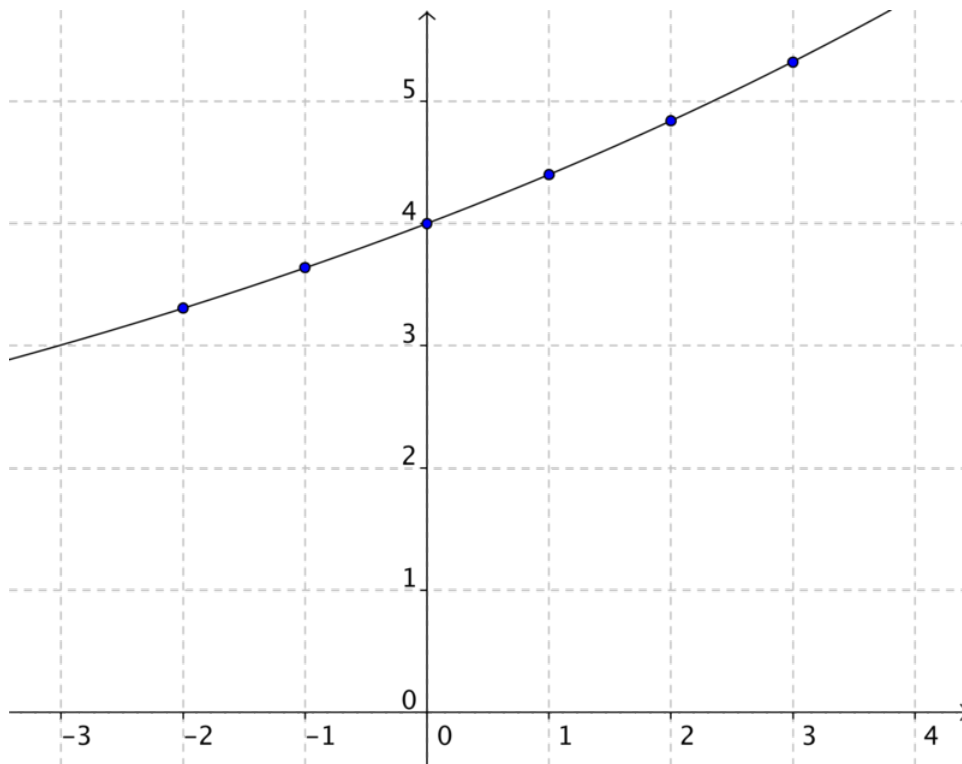
### Example C

Find a model to fit the following table of values:

|                      |      |      |   |     |      |       |
|----------------------|------|------|---|-----|------|-------|
| Dependent Variable   | -2   | -1   | 0 | 1   | 2    | 3     |
| Independent Variable | 3.31 | 3.64 | 4 | 4.4 | 4.84 | 5.324 |

### Solution

Start by graphing the points, to get a sense of the shape. This can help you rule out certain models, and lead you in the direction of which models might be a good fit.



The graph almost looks straight, but has a slight curve, so it can't be linear. It could be quadratic or cubic, but let's check if it's exponential:

|                               |      |                           |                        |                       |                          |                            |
|-------------------------------|------|---------------------------|------------------------|-----------------------|--------------------------|----------------------------|
| Independent Variable (x)      | -2   | -1                        | 0                      | 1                     | 2                        | 3                          |
| Dependent Variable (y)        | 3.31 | 3.64                      | 4                      | 4.4                   | 4.84                     | 5.324                      |
| Ratio of Consecutive y values |      | $\frac{3.64}{3.31} = 1.1$ | $\frac{4}{3.64} = 1.1$ | $\frac{4.4}{4} = 1.1$ | $\frac{4.84}{4.4} = 1.1$ | $\frac{5.324}{4.84} = 1.1$ |

Since the ratios are all the same and equal to 1.1, it is an exponential function with a growth factor of 1.1. Given the point (0,4), the initial value is 4. The function is:

$$f(x) = 4 \cdot (1.1)^x.$$

### Review Questions

A scientist counts 2,000 fish in a lake. The fish population increases at a rate of 1.5 fish per generation but the lake has space and food for only 2,000,000 fish. The following table gives the number of fish (in thousands) in each generation.

|                    |   |    |    |     |      |      |      |      |
|--------------------|---|----|----|-----|------|------|------|------|
| Generation         | 0 | 4  | 8  | 12  | 16   | 20   | 24   | 28   |
| Number (thousands) | 2 | 15 | 75 | 343 | 1139 | 1864 | 1990 | 1999 |

1. Which function seems to best fit the data: linear, quadratic, or exponential?
2. Find the model for the function of best fit.
3. Find the number of fish as a function of generation.



4. Find the number of fish in generation 10.
5. Find the number of fish in generation 25.