
CHAPTER **10**

Logistic Models

Chapter Outline

- 10.1 LOGISTIC GROWTH MODEL**
 - 10.2 CONDITIONS FAVORING LOGISTIC GROWTH**
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10.1 Logistic Growth Model

Here you will explore the graph and equation of the logistic function.

Learning Objectives

- Recognize logistic functions.
- Identify the carrying capacity and inflection point of a logistic function.

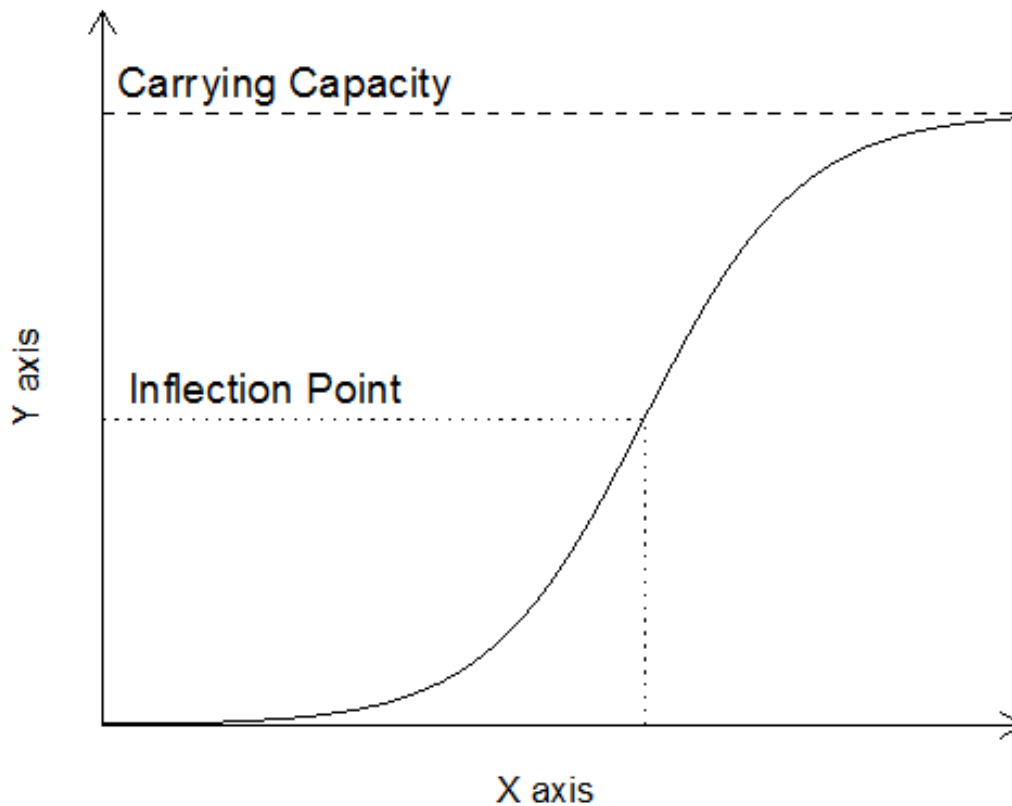
Logistic Models

Exponential growth increases without bound. This is reasonable for some situations; however, for populations there is usually some type of upper bound. This can be caused by limitations on food, space or other scarce resources. The effect of this limiting upper bound is a curve that grows exponentially at first and then slows down and hardly grows at all. This is characteristic of a logistic growth model.

The logistic equation is of the form:

$$f(t) = \frac{C}{1+ab^{-t}} = \frac{C}{1+ae^{-kt}}$$

The above equations represent the logistic function, and it contains three important pieces: C , a , and b . C determines that maximum value of the function, also known as the **carrying capacity**. C is represented by the dashed line in the graph below.



The constant of a in the logistic function is used much like a in the exponential function: it helps determine the value of the function at $t=0$. Specifically:

$$f(0) = \frac{C}{1+a}$$

The constant of b also follows a similar concept to the exponential function: it helps dictate the rate of change at the beginning and the end of the function. Just as in the exponential case, b must be positive. Just as in the exponential function, when b is greater than 1, we have growth. When b is $0 < b < 1$, then the function is decaying. Remember that $b = e^k$.

Inflection Point

All logistic functions have a point at which things “turn over,” called the **inflection point**, indicated by the horizontal dotted line on the graph. It is at the inflection point that the graph transitions from curving up (concave up) to curving down (concave down). This point occurs halfway to the carrying capacity:

$$f(t) = \frac{C}{2}$$

The vertical dotted line on the graph shows the corresponding value of t for the inflection point. We can find this value of t by using the equation:

$$t = \frac{\ln(a)}{\ln(b)} = \frac{\ln(a)}{k}$$

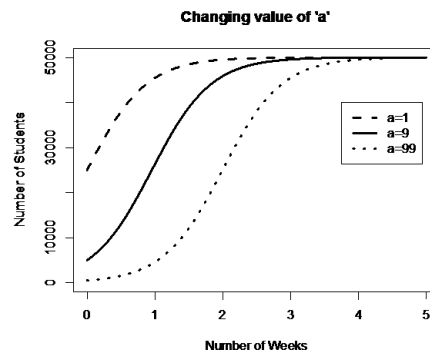
Model Parameters

Let’s take a closer look at the parameters a and b . Imagine the following: A sociology major is researching the spread of rumors on campus. This campus has 50,000 students, and when the rumor starts, only 500 students know about it. In this example, C is 50,000 and 500 is the number of students that know the rumor at time 0.

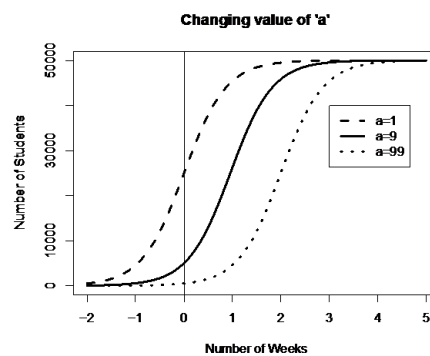
The a parameter effectively controls the starting point of the logistic model, just as it did in the exponential model. For a given value of C , a tells us where the function crosses the y -axis. Knowing the logistic equation, our C value, and the initial value of y , we can solve for a when time is 0:

$$\begin{aligned} f(t) &= \frac{C}{1+ab^{-t}} \\ 500 &= \frac{50,000}{1+ab^{-0}} \\ 500 &= \frac{50,000}{1+a} \\ a &= \frac{50,000}{500} - 1 = 99 \end{aligned}$$

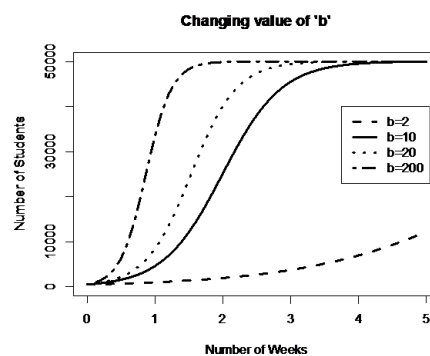
When C and b stay constant, a will shift the graph horizontally. The figure below shows what happens if we change the value of a for the same carrying capacity and $b=10$. For values of a equal to 1, 9, and 99, the starting values are 25,000, 5,000, and 500, respectively. We see that all three curves have the same shape and scale; they’re just shifted horizontally. The larger the value of a is, the more to the right the curve is shifted.



If we take a closer look at the graph and extend the x-axis to the left, we can see that the three lines are actually the same shape (because they all have the same C and b parameter values), but simply moved over horizontally:



Now, let's do the same examination on the parameter of b . If we keep the C and a values constant, what happens to the logistic model as we alter the value of b ? The graph below shows just that:



From this figure, we can graphically see some of the properties for b mentioned earlier. The dashed line ($b = 2$) doesn't even hit the Carrying Capacity at the end of 5 weeks. The other lines get progressively steeper as b increases. This means that higher values of b show a quicker growth to Carrying Capacity –assuming C and a are constant.

Example A

A rumor is spreading at a school that has a total student population of 1200. Four people know the rumor when it starts and three days later 300 people know the rumor. About how many people at the school know the rumor by the fourth day?

Solution

In a limited population, the count of people who know a rumor is an example of a situation that can be modeled using the logistic function. The population is 1200 so this will be the carrying capacity.

Identifying information:

$$c = 1200; (0, 4); (3, 300)$$

First, use the point (0, 4) so solve for a .

$$4 = \frac{1200}{1+ab^{-0}}$$

$$1 + ab^{-0} = \frac{1200}{4}$$

$$1 + ab^{-0} = 300$$

(Remember, any number raised to an exponent of zero –even a negative zero –is one.)

$$1 + a = 300$$

$$a = 299$$

Next, use the point (3, 300) to solve for b.

$$300 = \frac{1200}{1+299b^{-3}}$$

$$1 + 299b^{-3} = 4$$

$$299b^{-3} = 3$$

$$b^{-3} = \frac{3}{299}$$

(Knowing $b^{-x} = \frac{1}{b^x}$ helps here)

$$\frac{299}{3} = b^3$$

$$99.667 = b^3$$

$$4.636 = b$$

The modeling equation when $x = 4$ is:

$$f(t) = \frac{1200}{1+299 \cdot 4.636^{-t}}$$

Substituting in 4 for t in the logistic equation we get 728.470 or approximately 729 people.

Alternate solution, using equivalent formula and solving for k:

$$4 = \frac{1200}{1+ae^{-k0}}$$

$$1 + ae^{-k0} = \frac{1200}{4}$$

$$1 + ae^{-k0} = 300$$

(Remember, any number (even the number e) raised to an exponent of zero –even a negative zero –is one.)

$$1 + a = 300$$

$$a = 299$$

Next...

$$300 = \frac{1200}{1+299e^{-k3}}$$

$$1 + 299e^{-k3} = 4$$

$$299e^{-k3} = 3$$

$$e^{-k3} = \frac{3}{299}$$

$$\ln(e^{-k3}) = \ln\left(\frac{3}{299}\right)$$

$$-k3 = -4.602$$

$$k = 1.534$$

Now, if you substitute the values for a and k into the original equation and solve for $x=4$, you will get the same solution: approximately 729 people!

Example B

Long Island has roughly 8 million people. A hundred years ago, it had 2 million people. Suppose that the resources and infrastructure of the island could only support 20 million people. When will the population reach ten million inhabitants?

Solution

Identify known points and the carrying capacity. (0, 8,000,000) and (-100, 2,000,000).

$$c = 20,000,000$$

Use the first point to solve for a .

$$80,000,000 = \frac{20,000,000}{1+ab^{-0}}$$

$$1 + ab^{-0} = \frac{20,000,000}{80,000,000}$$

$$1 + ab^{-0} = 2.5$$

$$ab^{-0} = 1.5$$

$$a = 1.5$$

Now use the other point to solve for b :

$$2,000,000 = \frac{20,000,000}{1+1.5b^{-(-100)}}$$

$$1 + 1.5b^{-(-100)} = \frac{20,000,000}{2,000,000}$$

$$1 + 1.5b^{-(-100)} = 10$$

$$1.5b^{-(-100)} = 9$$

$$b^{-(-100)} = \frac{9}{1.5}$$

$$b^{100} = 6$$

$$b = 1.018$$

The question asks for the x value when $f(x) = 10,000,000$.

$$10,000,000 = \frac{20,000,000}{1+1.5 \cdot 1.018^{-t}}$$

$$1 + 1.5 \cdot 1.018^{-t} = \frac{20,000,000}{10,000,000}$$

$$1 + 1.5 \cdot 1.018^{-t} = 2$$

$$1.5 \cdot 1.018^{-t} = 1$$

$$1.018^{-t} = 0.667$$

$$\ln(1.018^{-t}) = \ln(0.667)$$

$$-t \cdot \ln(1.018) = \ln(0.667)$$

$$-t = \frac{\ln(0.667)}{\ln(1.018)}$$

$$-t = -22.70$$

$$t = 22.70$$

This means that according to your assumption and the two population data points you used, the predicted time from now that the population of Long Island will reach 10 million inhabitants is about 22.6 years.

Example C

A special kind of algae is grown in giant clear plastic tanks and can be harvested to make biofuel. The algae are given plenty of food, water and sunlight to grow rapidly and the only limiting resource is space in the tank. The algae are harvested when 95% of the tank is full leaving the tank 5% full of algae to reproduce and refill the tank. Currently the time between harvests is twenty days and the payoff is 90% harvest. Would you recommend a more optimal harvest schedule?

Solution

Identify known quantities and model the growth of the algae.

Known quantities:

$$(0, 0.05); (20, 0.95); c = 1 \text{ or } 100\%$$

The question asks about optimal harvest schedule. Currently the harvest is 90% per 20 day or a unit rate of 4.5% per day. If you shorten the time between harvests where the algae are growing the most efficiently, then potentially this unit rate might be higher. Suppose you leave 15% of the algae in the tank and harvest when it reaches 85%. How much time will that take to yield 70%?

Solve for a:

$$0.05 = \frac{1}{1+ab^{-0}}$$

$$1 + ab^{-0} = \frac{1}{0.05}$$

$$1 + ab^{-0} = 20$$

$$a = 19$$

Solve for b:

$$0.95 = \frac{1}{1+19b^{-20}}$$

$$1 + 19b^{-20} = \frac{1}{0.95}$$

$$19b^{-20} = 0.053$$

$$b^{-20} = \frac{0.053}{19}$$

$$\frac{19}{0.053} = b^{20}$$

$$358.491 = b^{20}$$

$$1.34 = b$$

Model for algae growth is:

$$f(t) = \frac{1}{1+19 \cdot 1.34^{-t}}$$

The day at which you have 15% algae:

$$0.15 = \frac{1}{1+19 \cdot 1.34^{-t_1}}$$

$$t_1 = 4.137$$

The day at which you have 85% algae (so that you have a 70% yield):

$$0.85 = \frac{1}{1+19 \cdot 1.34^{-t_2}}$$

$$t_2 = 16.10$$

$$t_2 - t_1 = 16.10 - 4.137 = 11.963$$

It takes about 12 days for the batches to yield 70% harvest which is a unit rate of about 6% per day. This is a significant increase in efficiency. A harvest schedule that maximizes the time where the logistic curve is steepest creates the fastest overall algae growth.

Vocabulary

Carrying capacity is the maximum sustainable population that the environmental factors will support. In other words, it is the population limit.

The **logistic model** is appropriate whenever the total count has an upper limit and the initial growth is exponential. Examples are the spread of rumors and disease in a limited population and the growth of bacteria or human population when resources are limited.

Guided Practice

- Determine the logistic model given $c = 12$ and the points (0, 9) and (1, 11).
- Determine the logistic model given $c = 7$ and the points (0, 2) and (3, 5).
- Solve for t when A=14, given C=20, a=4 and b=1.1.

Solutions

- The two points give two equations, and the logistic model has two variables.

$$9 = \frac{12}{1+ab^{-t}}$$

$$1+a = 1.333$$

$$a = 0.333$$

$$11 = \frac{12}{1+0.333b^{-1}}$$

$$1+0.333b^{-1} = \frac{12}{11}$$

$$0.333b^{-1} = 0.091$$

$$b^{-1} = \frac{0.091}{0.333}$$

$$3.659 = b$$

The approximate model is:

$$f(t) = \frac{12}{1+0.333 \cdot 3.659^{-t}}$$

- The two points give two equations, and the logistic model has two variables.

$$2 = \frac{7}{1+ab^{-t}}$$

$$a = 2.5$$

Then,

$$5 = \frac{7}{1+2.5 \cdot b^{-3}}$$

$$1.842 = b$$

The approximate model is:

$$f(t) = \frac{7}{1+2.5 \cdot 1.842^{-t}}$$

3. You will need to solve for an exponent using logs.

$$f(t) = \frac{20}{1+4 \cdot 1.11^{-t}}$$

$$14 = \frac{20}{1+4 \cdot 1.11^{-t}}$$

$$1 + 4 \cdot 1.11^{-t} = \frac{20}{14}$$

$$4 \cdot 1.11^{-t} = 0.429$$

$$1.11^{-t} = 0.107$$

$$-t \cdot \ln(1.11) = \ln(0.107)$$

$$-t = \frac{\ln(0.107)}{\ln(1.11)}$$

$$-t = -21.416$$

$$t = 21.416$$

More Practice

For 1-5, determine the logistic model given the carrying capacity and two points.

- $c = 12; (0, 5); (1, 7)$
- $c = 200; (0, 150); (5, 180)$
- $c = 1500; (0, 150); (10, 1000)$
- $c = 1000000; (0, 100000); (-40, 20000)$
- $c = 30000000; (-60, 10000); (0, 8000000)$

For 6-8, use the logistic function.

$$f(t) = \frac{32}{1+20 \cdot e^{-0.45t}}$$

- What is the carrying capacity of the function?
- What is the y -intercept of the function?
- Use your answers to 6 and 7, with at least two values of x (you choose), to make a sketch of the function.

For 9-11, use the logistic function.

$$g(t) = \frac{25}{1+4.5^{-t}}$$

9. What is the carrying capacity of the function?
10. What is the y -intercept of the function?
11. Use your answers to 9 and 10, with at least two values of x (you choose), to make a sketch of the function.
12. Find and sketch the inflection point of the logistic function. At what value of t does it occur?

10.2 Conditions Favoring Logistic Growth

Lesson Objectives

- Examine real-life examples of logistic (S-curve) growth.
- Understand the concept of carrying capacity in terms of population growth and resource availability.

Introduction

Imagine a huge bowl of your favorite potato salad, ready for a picnic on a beautiful, hot, midsummer day. The cook was careful to prepare it under strictly sanitary conditions, using fresh eggs, clean organic vegetables, and new jars of mayonnaise and mustard. Familiar with food poisoning warnings, s/he was so thorough that only a single bacterium made it into that vast amount of food. While such a scenario is highly unrealistic without authentic canning, it will serve as an example as we begin our investigation of how populations change, or **population dynamics**. Because potato salad provides an ideal environment for bacterial growth, just as your mother may have warned, we can use this single bacterial cell in the potato salad to ask:

How Do Populations Grow Under Ideal Conditions?

Given food, warm temperatures, moisture, and oxygen, a single aerobic bacterial cell can grow and divide by binary fission to become two cells in about 20 minutes. The two new cells, still under those ideal conditions, can each repeat this performance, so that after 20 more minutes, four cells constitute the population. Given this modest doubling, how many bacteria do you predict will be happily feeding on potato salad after five hours at the picnic? After you've thought about this, compare your prediction with the "data" below.

Like many populations under ideal conditions, bacteria show exponential. Each bacterium can undergo binary fission every 20 minutes. After 5 hours, a single bacterium can produce a population of 32,768 descendants.

TABLE 10.1:

Time (Hours and Minutes)	Population Size (Number of Bacteria)
0	1
20 minutes	2
40 minutes	4
1 hour	8
1 hour 20 minutes	16
1 hour 40 minutes	32
2 hours	64
2 hours 20 minutes	128
2 hours 40 minutes	256
3 hours	512
3 hours 20 minutes	1024
3 hours 40 minutes	2048
4 hours	4096
4 hours 20 minutes	8192
4 hours 40 minutes	16,384

TABLE 10.1: (continued)

Time (Hours and Minutes)	Population Size (Number of Bacteria)
5 hours	32,768

Are you surprised? This phenomenal capacity for growth of living populations was first described by Thomas Robert Malthus in his 1798 *Essay on the Principle of Population*. Although Malthus focused on human populations, biologists have found that many populations are capable of this explosive reproduction, if provided with ideal conditions. This pattern of growth is **exponential**, as the population grows larger, the rate of growth increases. If you have worked compound interest problems in math or played with numbers for estimating the interest in your savings account, you can compare the growth of a population under ideal conditions to the growth of a savings account under a constant rate of compound interest. The graph below, using potato salad bacterial “data,” shows the pattern of exponential growth: the population grows very slowly at first, but more and more rapidly as time passes.

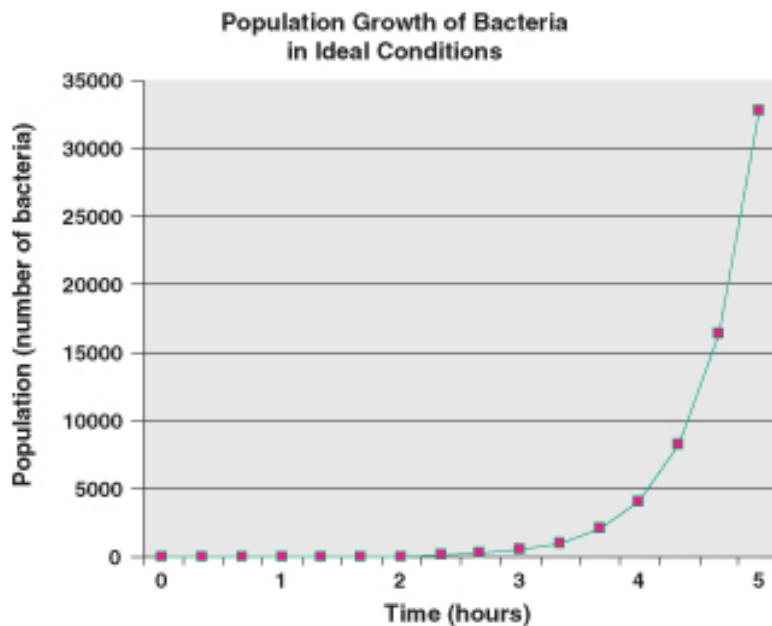


FIGURE 10.1

Exponential or geometric growth is very slow at first, but accelerates as the population grows. Because rate of growth depends on population size, growth rate increases as population increases. Most populations have the ability to grow exponentially, but such growth usually occurs only under ideal conditions that are not found in nature. Note the “J” shape of the curve.

Of course, if bacterial populations always grew exponentially, they would long ago have covered the Earth many times over. While Thomas Malthus emphasized the importance of exponential growth on population, he also stated that ideal conditions do not often exist in nature. A basic limit for all life is energy. Growth, survival, and reproduction require energy. Because energy supplies are limited, organisms must “spend” them wisely. We will end this lesson with a much more realistic model of population growth and the implications of its limits, but first, let’s look more carefully at the characteristics of populations which allow them to grow.

How Do Populations Grow in Nature?

You learned above that populations can grow exponentially if conditions are ideal. While exponential growth occurs when populations move into new or unfilled environments or rebound after catastrophes, most organisms do not live in ideal conditions very long, if at all. Let’s look at some data for populations growing under more realistic conditions.

Biologist Georgyi Gause studied the population growth of two species of *Paramecium* in laboratory cultures. Both species grew exponentially at first, as Malthus predicted. However, as each population increased, rates of growth slowed and eventually leveled off. Each species reached a different maximum, due to differences in size of individuals and space and nutrient needs, but both showed the same, S-shaped growth pattern.

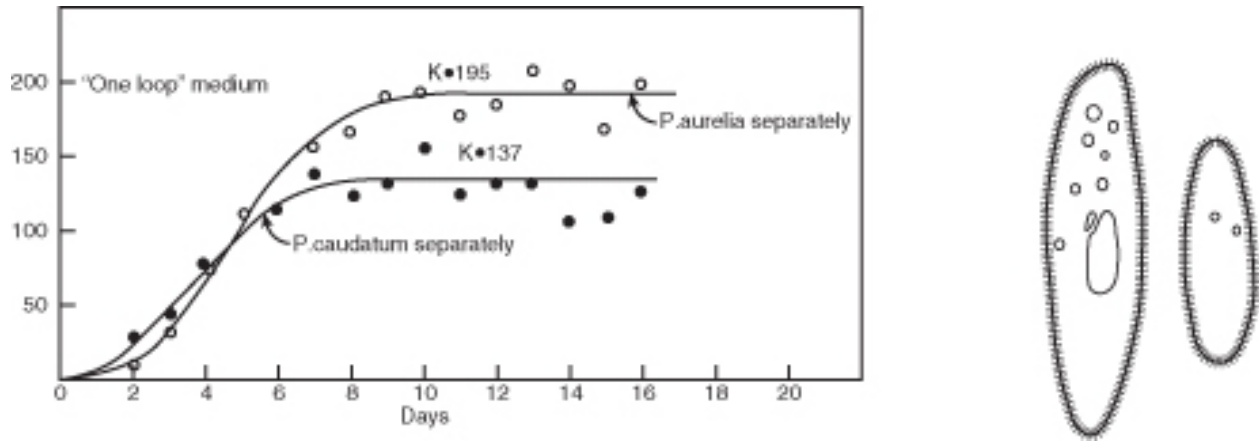


FIGURE 10.2

Two species of *Paramecium* illustrate logistic growth, with different plateaus due to differences in size and space and nutrient requirements. The growth pattern resembles and is often called an S-curve. Slow but exponential growth at low densities is followed by faster growth and then leveling.

Perhaps even more realistic is the growth of a sheep population, observed after the introduction of fourteen sheep to the island of Tasmania in 1800. Like the lab *Paramecia*, the sheep population at first grew exponentially. However, over the next 20 years, the population sharply declined by 1/3. Finally, the number of sheep increased slowly to a plateau. The general shape of the growth curve matched the S-shape of *Paramecium* growth, except that the sheep “overshot” their plateau at first.



FIGURE 10.3

Sheep introduced to Tasmania show logistic growth, except that they overshoot their carrying capacity before stabilizing.

As Malthus realized, no population can maintain exponential growth indefinitely. Inevitably, **limiting factors** such as reduced food supply or space lower birth rates, increase death rates, or lead to emigration, and lower the population growth rate. After reading Malthus' work in 1938, Pierre Verhulst derived a mathematical model of population growth which closely matches the S-curves observed under realistic conditions. In this **logistic (S-curve) model**, growth rate is proportional to the size of the population but also to the amount of available resources. At higher population densities, limited resources lead to competition and lower growth rates. Eventually, the growth rate declines to zero and the population becomes stable.

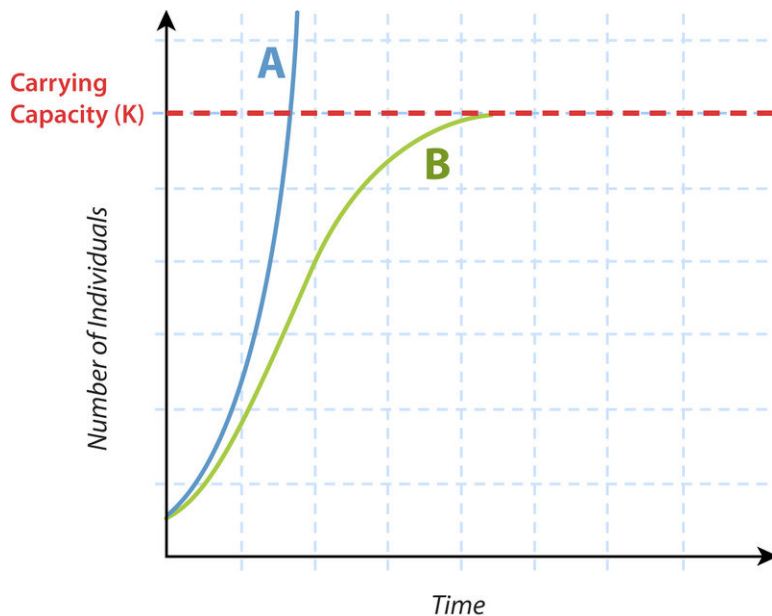


FIGURE 10.4

Growth of populations according to Malthus' exponential model (A) and Verhulst's logistic model (B). Both models assume that population growth is proportional to population size, but the logistic model also assumes that growth depends on available resources. A models growth under ideal conditions and shows that all populations have a capacity to grow infinitely large. B limits exponential growth to low densities; at higher densities, competition for resources or other limiting factors inevitably cause growth rate to slow to zero. At that point, the population reaches a stable plateau, the carrying capacity (K).

The logistic model describes population growth for many populations in nature. Some, like the sheep in Tasmania, “overshoot” the plateau before stabilizing, and some fluctuate wildly above and below a plateau average. A few may crash and disappear. However, the plateau itself has become a foundational concept in population biology known as **carrying capacity (K)**. Carrying capacity is the maximum population size that a particular environment can support without habitat degradation. Limiting factors determine carrying capacity, and often these interact. In the next section, we will explore in more detail the kinds of factors which restrict populations to specific carrying capacities and some adaptations that limit growth.

Lesson Summary

- Few populations in nature grow exponentially. No population can continue such growth indefinitely.
- The logistic (S-curve) model best describes the growth of many populations in nature.
- In the logistic model, growth rate depends on both population size and availability of resources. Growth is slow at first, but as size increases, growth accelerates. At higher densities, limited resources cause growth rate to decline, and populations stabilize at carrying capacity.