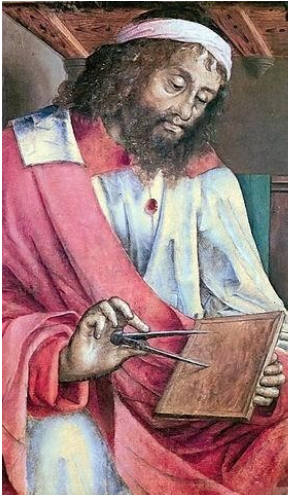
Penn Engineering
ONLINE LEARNING

# Video 1.1
Sampath Kannan

# What is an algorithm?

Muhammad ibn Musa al-Khwarizmi: gave rise to the word "algorithm"



Euclid: Inventor of an algorithm for computing greatest common divisors

# Why study algorithms?

As programs get complicated, thinking algorithmically allows us to:

- reason about their correctness and efficiency before implementing them
- focus on techniques for solving problems
- understand relationship between different computational problems

# Induction + Algorithm Design

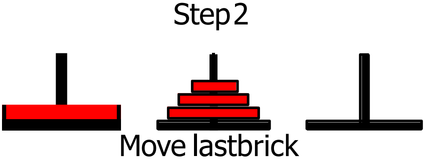
- A fundamental idea in algorithm design–solve a problem on bigger data sets using your knowledge of how to solve it on smaller ones.
- This idea embodies the proof technique of Mathematical Induction.
- Example: Towers of Hanoi

Step 1

Move $n - 1$ recursively

# Induction + Algorithm Design

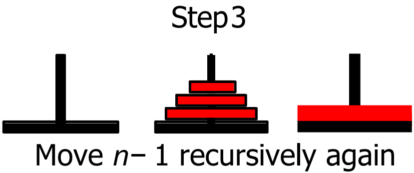
- A fundamental idea in algorithm design–solve a problem on bigger data sets using your knowledge of how to solve it on smaller ones.
- This idea embodies the proof technique of Mathematical Induction.
- Example: Towers of Hanoi




- Move top n-1 disks from rod A to rod B

# Induction + Algorithm Design

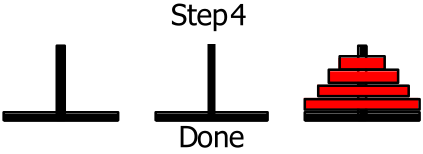
- A fundamental idea in algorithm design–solve a problem on bigger data sets using your knowledge of how to solve it on smaller ones.
- This idea embodies the proof technique of Mathematical Induction.
- Example: Towers of Hanoi




- Move top n-1 disks from rod A to rod B
- Move disk 1 from rod A to rod C

# Induction + Algorithm Design

- A fundamental idea in algorithm design–solve a problem on bigger data sets using your knowledge of how to solve it on smaller ones.
- This idea embodies the proof technique of Mathematical Induction.
- Example: Towers of Hanoi




- Move top n-1 disks from rod A to rod B
- Move disk 1 from rod A to rod C
- Move the n-1 disks from rod B to rod C

# Induction + Algorithm Design

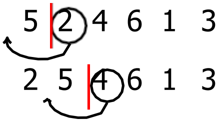
- A fundamental idea in algorithm design–solve a problem on bigger data sets using your knowledge of how to solve it on smaller ones.
- This idea embodies the proof technique of Mathematical Induction.
- Example: Towers of Hanoi

  - Move top n-1 disks from rod A to rod B
  - Move disk 1 from rod A to rod C
  - Move the n-1 disks from rod B to rod C
  - How long does this take? How can this be analyzed with **induction**?

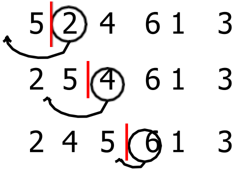
# Another Example: Insertion Sort

5 | 2 4 6 1 3

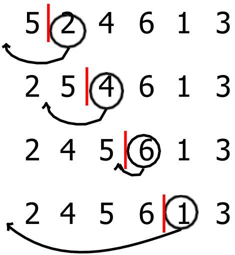
# Another Example: Insertion Sort



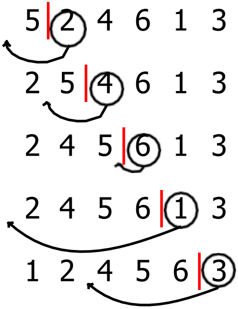
# Another Example: Insertion Sort



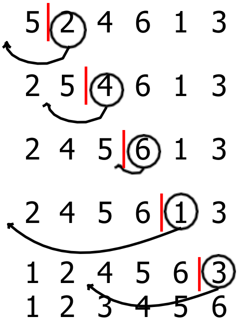
# Another Example: Insertion Sort



# Another Example: Insertion Sort

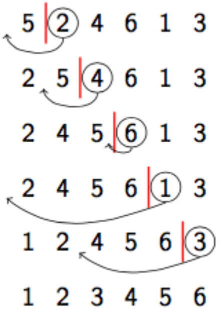


# Another Example: Insertion Sort



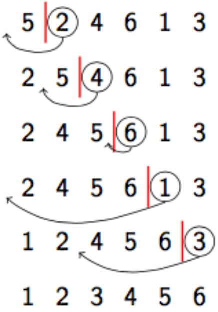
# Another Example: Insertion Sort

```
insertion-sort A:
    for i <- 1 to length(A)
        j <- i
        while j > 0 and A[j-1] > A[j]:
            swap A[j] and A[j-1]
            j <- j-1
```



# Another Example: Insertion Sort

```
insertion-sort A:
    for i <- 1 to length(A)
        j <- i
        while j > 0 and A[j-1] > A[j]:
            swap A[j] and A[j-1]
            j <- j-1
```




If we've already sorted the first **k elements** of the array, how long does it take to place the next element?

# Recurrence Relations

- How can we analyze the runtime of an algorithm that is recursive?

# Recurrence Relations

- How can we analyze the runtime of an algorithm that is recursive?
- Recurrence relation: a function defined in terms of itself

# Recurrence Relations

- How can we analyze the runtime of an algorithm that is recursive?
- Recurrence relation: a function defined in terms of itself
- How can we write the runtime of Towers of Hanoi using a recurrence?

# Recurrence Relations

- $T(n)$ = # operations required to solve a tower with n disks

# Recurrence Relations

- $T(n)$ = # operations required to solve a tower with n disks
- disks $T(n-1)$ = # operations required to solve a tower with $n-1$ disks

# Recurrence Relations

- $T(n)$ = # operations required to solve a tower with n disks
- $T(n-1)$ = # operations required to solve a tower with $n-1$ disks
- Can we write $T(n)$ using $T(n-1)$?

# Recurrence Relations

Towers of Hanoi recurrence: $T(n) = 2T(n-1) + 1$

# Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

- $T(n) = 2T(n-1) + 1$

# Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

- $T(n) = 2T(n-1) + 1$
- $T(n-1) = 2T(n-2) + 1$

# Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

- substituting in for $T(n-1)$:

# Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

- substituting in for $T(n-1)$:
- $T(n) = 2(2T(n-2) + 1) + 1$

# Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

- substituting in for $T(n-1)$:
- $T(n) = 2(2T(n-2) + 1) + 1$
- $T(n) = 4T(n-2) + 2 + 1$

# Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

› substituting in again for $T(n-2)$:

# Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

- substituting in again for $T(n-2)$:
- $T(n) = 8T(n-3) + 4 + 2 + 1$

# Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

- substituting in againfor $T(n-2)$:
- $T(n) = 8T(n-3) + 4 + 2 + 1$
- Can we generalize this to k?

# Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

› $T(n) = 2^k T(n-k) + (\sum_{i=0}^{k-1} 2^i)$

# Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

- $T(n) = 2^k T(n-k) + (\sum_{i=0}^{k-1} 2^i)$
- $T(n) = 2^k T(n-k) + (2^k - 1)$

# Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

- $T(n) = 2^k T(n-k) + (\sum_{i=0}^{k-1} 2^i)$
- $T(n) = 2^k T(n-k) + (2^k - 1)$
- What is $T(1)$?

# Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

- $T(n) = 2^k T(n-k) + \left(\sum_{i=0}^{k-1} 2^i\right)$
- $T(n) = 2^k T(n-k) + (2^k - 1)$
- What is $T(1)$?
  - How long does it take to solve a tower with 1 ring?

# Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

- $T(n) = 2^k T(n-k) + (\sum_{i=0}^{k-1} 2^i)$
- $T(n) = 2^k T(n-k) + (2^k - 1)$
- What is $T(1)$?
  - How long does it take to solve a tower with 1 ring?
  - $T(1) = 1$. Substitute $k = n - 1$
- $T(n) = 2^{n-1} + 2^{n-1} - 1$

# Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

- $T(n) = 2^k T(n-k) + (\sum_{i=0}^{k-1} 2^i)$
- $T(n) = 2^k T(n-k) + (2^k - 1)$
- What is $T(1)$?
  - How long does it take to solve a tower with 1 ring?
  - $T(1) = 1$. Substitute $k = n - 1$
- $T(n) = 2^{n-1} + 2^{n-1} - 1$
- $T(n) = 2^n - 1$

# Towers of Hanoi: Runtime

Result: Solving Towers of Hanoi requires $2^n - 1$ operations!

# Towers of Hanoi: Runtime

We can expand this recurrence out through **telescoping**

- $T(n) = 2T(n-1) + 1$
- $T(n-1) = 2T(n-2) + 1$
- substituting in for $T(n-1)$:
- $T(n) = 2(2T(n-2) + 1) + 1$
- $T(n) = 4T(n-2) + 2 + 1$
- substituting in again for $T(n-2)$:
- $T(n) = 8T(n-3) + 4 + 2 + 1$
- Can we generalize this to k?
- $T(n) = 2^k T(n-k) + (\sum_{i=0}^{k-1} 2^i)$
- $T(n) = 2^k T(n-k) + (2^k - 1)$
- What is $T(1)$?
  - How long does it take to solve a tower with 1 ring?
  - $T(1) = 1$. Substitute $k = n - 1$
- $T(n) = 2^{n-1} + 2^{n-1} - 1$
- $T(n) = 2^n - 1$

Result: Solving Towers of Hanoi requires $2^n - 1$ operations!

# Recurrence Relations: Back to Insertion Sort

- Can we write Insertion Sort using a recurrence?

# Recurrence Relations: Back to Insertion Sort

- Can we write Insertion Sort using a recurrence?
- Not really, it isn't recursive! Instead, we can analyze how long each iteration of the loop takes.

# Recurrence Relations: Back to Insertion Sort

- Can we write Insertion Sort using a recurrence?
- Not really, it isn't recursive! Instead, we can analyze how long each iteration of the loop takes.
- Key observation: At the kth iteration of the loop, the first $k - 1$ elements of the array are in sorted order

# Recurrence Relations: Back to Insertion Sort

- Can we write Insertion Sort using a recurrence?
- Not really, it isn't recursive! Instead, we can analyze how long each iteration of the loop takes.
- Key observation: At the kth iteration of the loop, the first $k - 1$ elements of the array are in sorted order
- First iteration of the loop: 0 swaps required (first elementis trivially sorted)
- Last iteration of the loop: at most $n - 1$ swaps required

# Recurrence Relations: Back to Insertion Sort

- Can we write Insertion Sort using a recurrence?
- Not really, it isn’t recursive !Instead,we can analyze how long each iteration of the loop takes.
- Key observation: At the kth iteration of theloop, the firs $k - 1$ elements of the array are in sorted order
- First iteration of the loop: 0 swaps required (first elementis trivially sorted)
- Last iteration of the loop: at most $n - 1$ swaps required
- In general, kth iteration of theloop: at most $k - 1$ swaps required

# Recurrence Relations: Back to Insertion Sort

Finding the total number of swaps:

- total number of swaps = $\sum_{i=0}^{n} i - 1$

# Recurrence Relations: Back to Insertion Sort

Finding the total number of swaps:

- total number of swaps = $\sum_{i=0}^{n} i - 1$
- $= 1 + 2 + \ldots + n - 1$
- $= \frac{n(n-1)}{2}$

# Recurrence Relations: Back to Insertion Sort

Finding the total number of swaps:

- total number of swaps = $\sum_{i=0}^{n} i - 1$
- $= 1 + 2 + \ldots + n - 1$
- $= \frac{n(n-1)}{2}$

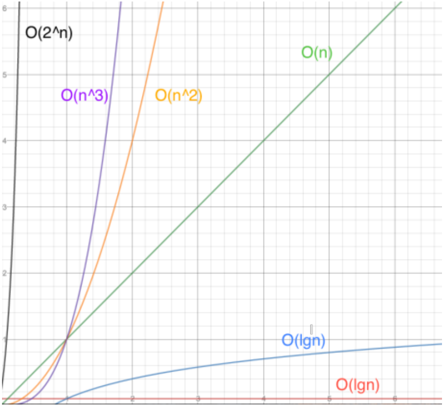
Number ofswaps required for Insertion sort: $\frac{n(n-1)}{2}$

Penn Engineering
ONLINE LEARNING

Video 1.2
Sampath Kannan

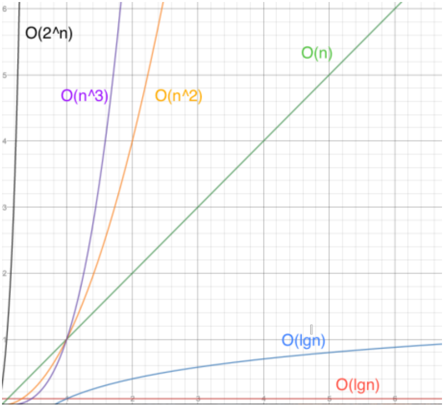
# Asymptotic Bounds

**Motivation:**



# Asymptotic Bounds

**Motivation:**



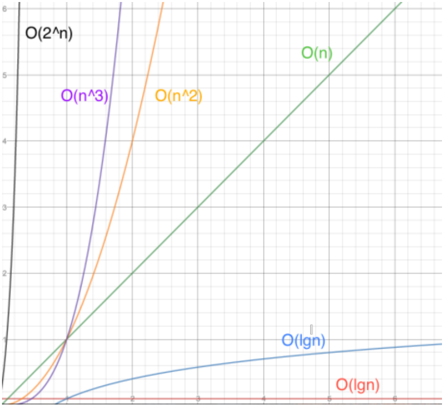

- Essentially a way to compare functions without worrying about their behavior on small *n*.

  Big-Oh is like ≤ (ignoring constant factors), and Big-Omega is like ≥

- Gives us an idea of how fast a function grows

# Asymptotic Bounds

**Motivation:**




- Essentially a way to compare functions without worrying about their behavior on small *n*. In this sense Big-Oh is like ≤ (ignoring constant factors), and Big-Omega is like ≥
- Gives us an idea of how fast a function grows

- Note: $O(f(n))$ is a **set**.
  - $O(n^2)$: the set of all function that do not grow faster than $n^2$

# Asymptotic Bounds: Examples

Some elements of $O(n^2)$:

- $2n^2 \in O(n^2)$
- $100n^2 + n + 1 \in O(n^2)$
- $n \in O(n^2)$

Some elements of $\Omega(n^2)$:

- $2n^2 \in \Omega(n^2)$
- $\frac{n^2}{1000} \quad n \in \Omega(n^2)$
- $n^3 \in \Omega(n)^2$

What is the complexity of insertion sort?

- $T(n) = T(n\ 1) + n$
- $T(n) = \frac{n^2+n}{2}$
- $T(n) \in O(n^2)$

Insertion sort has a runtime of $O(n^2)$

# Complexity

More on Insertion Sort...

```
insertion-sort A:
    for i <- 1 to length(A)
        j <- i
        while j > 0 and A[j-1] > A[j]:
            swap A[j] and A[j-1]
            j <- j-1
```

# Complexity

More on Insertion Sort...

```
insertion-sort A:
    for i <- 1 to length(A)
        j <- i
        while j > 0 and A[j-1] > A[j]:
            swap A[j] and A[j-1]
            j <- j-1
```

How does insertion sort perform on an already-sorted array?

# Complexity

More on Insertion Sort...

```
insertion-sort A:
    for i <- 1 to length(A)
        j <- i
        while j > 0 and A[j-1] > A[j]:
            swap A[j] and A[j-1]
            j <- j-1
```

How does insertion sort perform on an already-sorted array?

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |

# Complexity

More on Insertion Sort...

```
insertion-sort A:
    for i <- 1 to length(A)
        j <- i
        while j > 0 and A[j-1] > A[j]:
            swap A[j] and A[j-1]
            j <- j-1
```

How does insertion sort perform on an already-sorted array?

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |

time for inner loop?

# Complexity

More on Insertion Sort...

```
insertion-sort A:
    for i <- 1 to length(A)
        j <- i
        while j > 0 and A[j-1] > A[j]:
            swap A[j] and A[j-1]
            j <- j-1
```

How does insertion sort perform on an already-sorted array?

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |

- time for inner loop?
- Each iteration requires no swaps! (constant time to check the first element)
- $\sum_{i=1}^{n} 1 = n \in O(n)$

# Complexity

More on Insertion Sort...

```
insertion-sort A:
    for i <- 1 to length(A)
        j <- i
        while j > 0 and A[j-1] > A[j]:
            swap A[j] and A[j-1]
            j <- j-1
```

How does insertion sort perform on an already-sorted array?

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |

- time for inner loop?
- Each iteration requires no swaps! (constant time to check the first element)
- $\sum_{i=1}^{n} 1 = n \in O(n)$
- is insertion sort $O(n)$, or $O(n^2)$?

# Complexity

More on Insertion Sort...

```
insertion-sort A:
    for i <- 1 to length(A)
        j <- i
        while j > 0 and A[j-1] > A[j]:
            swap A[j] and A[j-1]
            j <- j-1
```

How does insertion sort perform on an already-sorted array?

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |

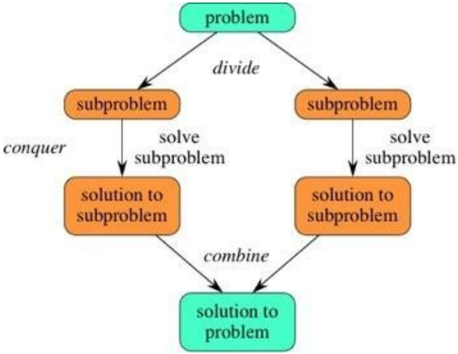
- time for inner loop?
- Each iteration requires no swaps! (constant time to check the first element)

Takeaway: Can't assume anything about the input. Always assume the worst case!

- $\sum_{i=1}^{n} 1 = n \in O(n)$
- is insertion sort $O(n)$, or $O(n^2)$?

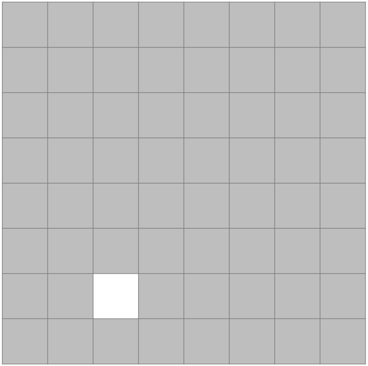
# Algorithm Design: Divide and Conquer Paradigm

Idea: Solve a problem by splitting it into pieces, solving those pieces recursively, and merging them to solve the larger problem



# Divide and Conquer Example: Triominos

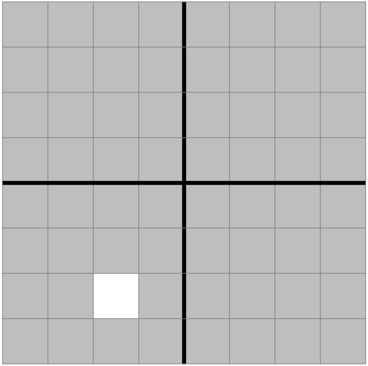
- Input: *NxN* grid (assume n is a power of 2) with a single square removed, and a supply of corner shaped triomino tiles
- Goal: Fill the grid without any overlapping tiles



Algorithm:

# Divide and Conquer Example: Triominos

- Input: *NxN* grid (assume n is a powerof 2) with a single square removed, and a supply of corner shaped triomino tiles
- Goal: Fill the grid without any overlapping tiles



Algorithm:

- **Divide** the grid into 4 squares(size $2^{n-1} x\ 2^{n-1}$).
- note: 1 of these 4 squares contains the missing piece

Video 1.3
Sampath Kannan

# Binary Search

- How long does it take to search for an element in an array? O(n)
- Idea: Can we do better if we know that the array is sorted?

```
Binary-search(A,val):
    return Binary-search(A, val, 0, (length(A)-1))
```

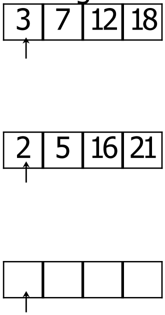
```
Binary-search(A, val, low, high):if
   high < low
      return -1 (not found)
   mid <- (low + high) / 2  if
   A[mid] > val
      return Binary-search(A, val, low, mid-1)
   else if A[mid] <val
      return Binary-search(A, val, mid+1, hi)
   else returnmid
```

- Each step of the algorithm, the size of the input halves.
- $T(n) = T(\frac{n}{2}) + 1$
- How to solve this recurrence: How many times can we halve N before reaching1? $\frac{N}{2}, \frac{N}{4}, \ldots$
- $\frac{N}{2^k} = 1, k = lg_2 N$
- binary search runs in $O(lgN)$

# Merging two sorted lists

Input: two sorted arrays of size n and m

Output: a single sorted array of size n+m



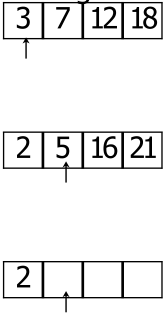

```
merge(A, B):
    C = new array[len(A) + len(B)]
    i, j, k <- 0
    while i < len(A) and j < len(B):
        if A[i] < B[j]:
            C[k] <-A[i]
            i++, k++
        else:
            C[k] <-B[j]
            j++, k++
    while i < len(A):
        C[k++] <-A[i++]
    while j < len(B):
        C[k++]<- B[j++]
    return C
```

- How long does this take?
- Every time a comparison is made, either i or j is incremented
- Total number of comparisons is n + m
- merging runs in $O(n + m)$ time

# Merging two sorted lists

Input: two sorted arrays of size n and m

Output: a single sorted array of size n+m



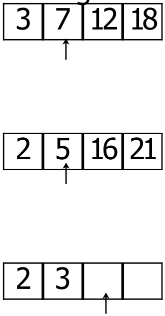

```
merge(A, B):
    C = new array[len(A) + len(B)]
    i, j, k <- 0
    while i < len(A) and j < len(B):
        if A[i] < B[j]:
            C[k] <-A[i]
            i++, k++
        else:
            C[k] <-B[j]
            j++, k++
    while i < len(A):
        C[k++] <-A[i++]
    while j < len(B):
        C[k++]<- B[j++]
    return C
```

- How long does this take?
- Every time a comparison is made, either i or j is incremented
- Total number of comparisons is n + m
- merging runs in $O(n + m)$ time

# Merging two sorted lists

Input: two sorted arrays of size n and m

Output: a single sorted array of size n+m



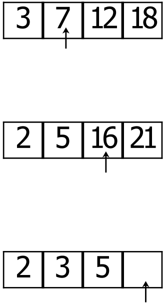

```
merge(A, B):
    C = new array[len(A) + len(B)]
    i, j, k <- 0
    while i < len(A) and j < len(B):
        if A[i] < B[j]:
            C[k] <-A[i]
            i++, k++
        else:
            C[k] <-B[j]
            j++, k++
    while i < len(A):
        C[k++] <-A[i++]
    while j < len(B):
        C[k++]<- B[j++]
    return C
```

- How long does this take?
- Every time a comparison is made, either i or j is incremented
- Total number of comparisons is n + m
- merging runs in $O(n + m)$ time

# Merging two sorted lists

Input: two sorted arrays of size n and m

Output: a single sorted array of size n+m




```
merge(A, B):
    C = new array[len(A) + len(B)]
    i, j, k <- 0
    while i < len(A) and j < len(B):
        if A[i] < B[j]:
            C[k] <-A[i]
            i++, k++
        else:
            C[k] <-B[j]
            j++, k++
    while i < len(A):
        C[k++] <-A[i++]
    while j < len(B):
        C[k++]<- B[j++]
    return C
```

- How long does this take?
- Every time a comparison is made, either i or j is incremented
- Total number of comparisons is n + m
- merging runs in $O(n + m)$ time

# More on Divide and Conquer: Mergesort

Input: An array of size n, Output: A sorted array of size n

Can we apply the Divide and Conquer paradigm to sorting?
Idea: Split the array, sort halves recursively, **merge** the result

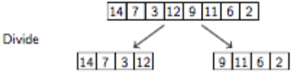
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 14 | 7 | 3 | 12 | 9 | 11 | 6 | 2 |

```
mergesort(A):
    mergesort(A, 0,len(A)-1)


mergesort(A, aux, lo, hi):
    if (hi - lo <= 1) return
    mid = (lo + hi) / 2
    mergesort(A, lo, mid)
    mergesort(A,mid+1,hi)
    C = merge(A[lo:mid], A[mid+1:hi])
    copy elements from C back into A
```

# More on Divide and Conquer: Mergesort

Input: An array of size n, Output: A sorted array of size n

Can we apply the Divide and Conquer paradigm to sorting?
Idea: Split the array, sort halves recursively, **merge** the result

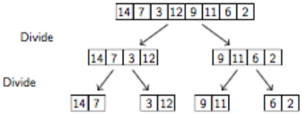



```
mergesort(A):
    mergesort(A, 0,len(A)-1)


mergesort(A, aux, lo, hi):
    if (hi - lo <= 1) return
    mid = (lo + hi) / 2
    mergesort(A, lo, mid)
    mergesort(A,mid+1,hi)
    C = merge(A[lo:mid],A[mid+1:hi])
    copy elements from C back into A
```

# More on Divide and Conquer: Mergesort

Input: An array of size n, Output: A sorted array of size n

Can we apply the Divide and Conquer paradigm to sorting?
Idea: Split the array, sort halves recursively, **merge** the result



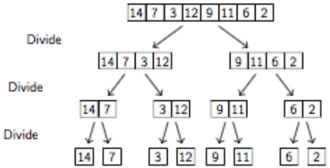

```
mergesort(A):
    mergesort(A, 0,len(A)-1)


mergesort(A, aux, lo, hi):
    if (hi - lo <= 1) return
    mid = (lo + hi) / 2
    mergesort(A, lo, mid)
    mergesort(A, mid+1, hi)
    C = merge(A[lo:mid], A[mid+1:hi])
    copy elements from C back into A
```

# More on Divide and Conquer: Mergesort

Input: An array of size n, Output: A sorted array of size n

Can we apply the Divide and Conquer paradigm to sorting?
Idea: Split the array, sort halves recursively, **merge** the result

```
mergesort(A):
    mergesort(A, 0,len(A)-1)


mergesort(A, aux, lo, hi):
    if (hi - lo <= 1) return
    mid = (lo + hi) / 2
    mergesort(A, lo, mid)
    mergesort(A, mid+1,hi)
    C = merge(A[lo:mid], A[mid+1:hi])
    copy elements from C back into A
```

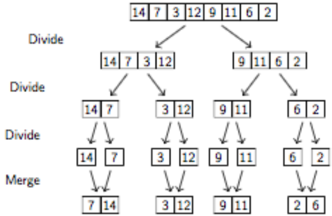


# More on Divide and Conquer: Mergesort

Input: An array of size n, Output: A sorted array of size n

Can we apply the Divide and Conquer paradigm to sorting?
Idea: Split the array, sort halves recursively, **merge** the result

```
mergesort(A):
    mergesort(A, 0,len(A)-1)


mergesort(A, aux, lo, hi):
    if (hi - lo <= 1) return
    mid = (lo + hi) / 2
    mergesort(A, lo, mid)
    mergesort(A,mid+1,hi)
    C = merge(A[lo:mid], A[mid+1:hi])
    copy elements from C back into A
```

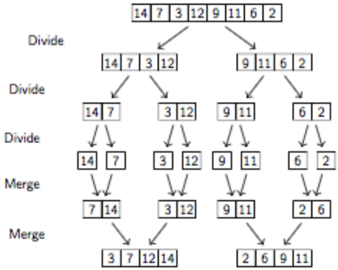


# More on Divide and Conquer: Mergesort

Input: An array of size n, Output: A sorted array of size n

Can we apply the Divide and Conquer paradigm to sorting?
Idea:  Split the array, sort halves recursively, **merge** the result

```
mergesort(A):
    mergesort(A, 0,len(A)-1)


mergesort(A, aux, lo, hi):
    if (hi - lo <= 1) return
    mid = (lo + hi) / 2
    mergesort(A, lo, mid)
    mergesort(A,mid+1,hi)
    C = merge(A[lo:mid], A[mid+1:hi])
    copy elements from C back into A
```

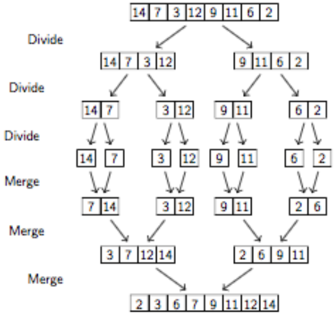


# More on Divide and Conquer: Mergesort

Input: An array of size n, Output: A sorted array of size n

Can we apply the Divide and Conquer paradigm to sorting?
Idea: Split the array, sort halves recursively, **merge** the result

```
mergesort(A):
    mergesort(A, 0,len(A)-1)


mergesort(A, aux, lo, hi):
    if (hi - lo <= 1) return
    mid = (lo + hi) / 2
    mergesort(A, lo, mid)
    mergesort(A,mid+1,hi)
    C = merge(A[lo:mid], A[mid+1:hi])
    copy elements from C back into A
```



Video 1.4
Sampath Kannan

# Algorithm Design: Using Randomness

- Remember from Insertion Sort: Algorithm performance can depend on the input:
  - on a sorted list: $O(n)$ comparisons
  - on a reversed list: $O(n^2)$ comparisons
  - In general: somewhere between $n$ and $\frac{n(n+1)}{2}$ comparisons
  - However, the **worst-case** is still $O(n^2)$
- An "adversary" can repeatedly construct an input to our algorithm that causes it to perform as poorly as possible
- Can we prevent our algorithm performance from depending on the input?
  - Shift the dependency: from **input** to **randomization**
- Idea: Write algorithms that toss a coin!

# First: An Introduction to Probability

- For a stronger introduction,see:
  https://www.coursera.org/learn/probability- intro

- *Random Variable*: A function X from the results of an experiment to numbers
- $E[X]$: the expected value of the random variable X (a "weighted average" )
  - Formula: $E[X] = \Sigma\, i * P(X = i)$ (for all values i that X can take on)
  - Example:
    - Roll a 6-sided die. Let $X =$ the value that the die lands on. What is $E[X]$?
    - X can take on each of the values 1 through 6, each with probability $\frac{1}{6}$
    - $E[X] = 1\frac{1}{6} + 2\frac{1}{6} + .. + 6\frac{1}{6} = \frac{21}{6} = 3.5$

# Intro to Probability: Continued

- What is the expected sum of two dice?
- *X* = the sum of two dice. Want to find *E* [*X* ].
- X can take on values from 2...12
- E.x. *P* (*X* = 5). Can result from two die rolls of:
  - (1, 4)
  - (4, 1)
  - (2, 3)
  - (3, 2)

- $P(X = 5) = 4\frac{1}{36} = \frac{1}{9}$
- $E[X] = \sum_{i=2}^{12} i * P(X = i) = 2\frac{1}{36} + 3\frac{2}{36} + ... + 12\frac{1}{36}$

| sum | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| probability | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

Calculation is not trivial. Solution: Linearity of Expectation!

# Intro to Probability: Continued

- *Linearity of Expectation*: For n random variables, $X_1, .., X_n$, $E[X_1 + .. + X_n] = E[X_1] + .. + E[X_n]$
- Example:
  - What is the expected sum of rolling 2 dice?
  - let $X_i$ be the random variable denoting the value of the i'th die rolled
  - let $X$ be the r.v. denoting the sum of all 2 dice
  - then $X = X_1 + X_2$
  - $E[X] = E[X_1 + X_2]$
  - $E[X] = E[X_1] + E[X_2]$ (by lin. of exp.)
  - as shown above, for each i, $E[X_i] = 3.5$
  - $E[X] = 3.5 + 3.5 = 7$

# Expectation Example: Hat Checking

N people go to a restaurant, take off their hats and throw them in a pile. Afterwards, they each take a hat from the pile at random. What is the expected number of people who get their hat back?

- We can analyze using random variables!
- Let: $X$ = the number of people who get their hats back

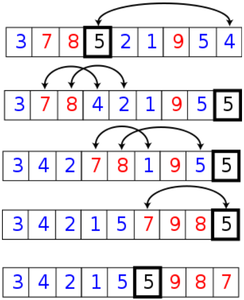
$$X_i = \begin{cases} 1 & \text{person } i \text{ chooses their own hat} \\ 0 & \text{person } i \text{ doesn't choose their own hat} \end{cases}$$

- What is $E[X_i]$?
  - From the definition:
  - $E[X_i] = 1 * P(\text{choose their hat}) + 0P(\text{don't choose their hat})$
  - $E[X_i] = P(\text{choose hat}) = \frac{1}{n}$
- Again, $X = X_1 + X_2 + .. + X_n$
- $E[X] = E[X_1 + .. + X_n] = E[X_1] + .. + E[X_n]$ by lin. of exp.
- $E[X] = n\frac{1}{n} = 1$

In expectation, one person will correctly take their own hat!

# Quicksort: An Introduction

Goal: Another sorting algorithm that uses divide-and-conquer

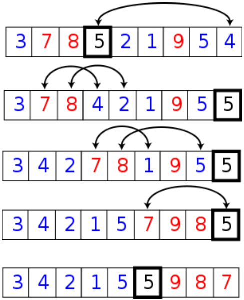



- Idea:
  - Select an element in the array
  - Partition the other elements of the array around it
- Is the array more sorted than it was before?
- Answer: yes!
- Next step: recursively sort the left and right sides of the array as well.

Problem: What about ”adversarial inputs”? This algorithm will perform better on some inputs than others.

# Quicksort: Randomized

Can we write an algorithm for sorting that uses coin tossing (randomness)?




- Idea:
  - *Randomly* select an element in the array
  - Partition the other elements of the array around it
  - Recursively sort the left and right sides of the array

Result: Another divide and conquer algorithm for sorting, that uses **randomness.**

Video 1.5
Sampath Kannan

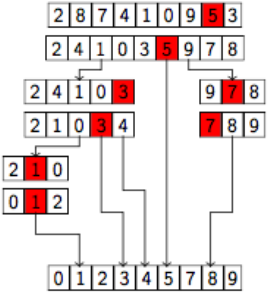
# Quicksort

Idea: Choose an element at random. Partition the array around this element. Recursively sort the left and right side.

# Quicksort

Idea: Choose an element at random. Partition the array around this element. Recursively sort the left and right side.

```
quicksort(A):
    quicksort(A, 0, len(A)-1)

quicksort(A, lo, hi):
    if(lo >= hi) return
    pivot_location <- partition(A, lo, hi)
    quicksort(A, lo, pivot_location - 1)
    quicksort(A, pivot_location +1, hi)

partition(A, lo, hi):
    pivot_index <- random(lo, hi)
    swap(A, pivot_index, hi)
    pivot <- A[hi]
    I <- lo, j <- hi, C <- new array
    for k = lo to hi -1
         if A[k] <= pivot:
             C[i++] <- A[k]
         else:
             C[j--] <- A[k]
    C[i] <- A[hi] (copy the pivot in)
    copy C[lo : hi] back into A
    return i
```



# Quicksort

Quicksort (compare to Mergesort)

# Quicksort

Quicksort (compare to Mergesort)

- divide-and-conquer algorithm
- First partition, then sort recursively

# Quicksort

Quicksort (compare to Mergesort)

- divide-and-conquer algorithm
- First partition, then sort recursively
- Can be done with no extra space
- runtime: See next slide

# Quicksort: Analysis

- First: the recurrence for quicksort
- Step 1: Partition requires $O(n)$
- Step 2: Recursively sort left and right sides of the array

# Quicksort: Analysis

- First: the recurrence for quicksort
- Step 1: Partition requires $O(n)$
- Step 2: Recursively sort left and right sides of the array
  - What are the sizes of these two arrays?
    - $k$ and $n - k - 1$, for some k
  - $T(n) = T(k) + T(n - k - 1) + O(n)$

# Quicksort: Analysis

- First: the recurrence for quicksort
- Step 1: Partition requires $O(n)$
- Step 2: Recursively sort left and right sides of the array
  - What are the sizes of these two arrays?
  - $k$ and $n - k - 1$, for some k
- $T(n) = T(k) + T(n - k - 1) + O(n)$

<u>Worstcase</u> (bad partition):

- partition does not split array at all

at every step($k = 1$ or $n - 1$)

- $T(n) = T(1) + T(n - 1) + n$
- $T(n) = O(n^2)$ (similar to insertion sort)

# Quicksort: Analysis

- First: the recurrence for quicksort
- Step 1: Partition requires $O(n)$
- Step 2: Recursively sort left and right sides of the array
  - What are the sizes of these two arrays?
  - $k$ and $n - k - 1$, for some k
- $T(n) = T(k) + T(n - k - 1) + O(n)$

<u>Worstcase</u> (bad partition):

- partition does not split array at all at every step ($k = 1$ or $n - 1$)
- $T(n) = T(1) + T(n-1) + n$
- $T(n) = O(n^2)$ (similar to insertion sort)

<u>Best case</u> (good partition):

- partition splits array evenly at every step ($k = \frac{n}{2}$)
- $T(n) = T(\frac{n}{2}) + T(\frac{n}{2}) + O(n)$
- $T(n) = O(n \lg n)$ (recall from merge sort)

# Quicksort: Analysis

- First: the recurrence for quicksort
- Step 1: Partition requires $O(n)$
- Step 2: Recursively sort left and right sides of the array
  - What are the sizes of these two arrays?
  - $k$ and $n - k - 1$, for some k
- $T(n) = T(k) + T(n - k - 1) + O(n)$

<u>Worstcase</u> (bad partition):

- partition does not split array at all at every step ($k = 1$ or $n - 1$)
- $T(n) = T(1) + T(n - 1) + n$
- $T(n) = O(n^2)$ (similar to insertion sort)

<u>Best case</u> (good partition):

- partition splits array evenly at every step ($k = \frac{n}{2}$)
- $T(n) = T(\frac{n}{2}) + T(\frac{n}{2}) + O(n)$
- $T(n) = O(n\lg n)$ (recall from merge sort)

How does the algorithm perform on average?
We can analyze with **expectation**

# Quicksort: Analysis

Recurrence for quicksort:

# Quicksort: Analysis

Recurrence for quicksort:

- taking the expected value overall possible i:

# Quicksort: Analysis

Recurrence for quicksort:

- taking the expected value overall possible i:
- $T(n) = \frac{1}{n} \Sigma_{i=1}^{n} T(i) + T(n-i) + O(n)$

# Quicksort: Analysis

Recurrence for quicksort:

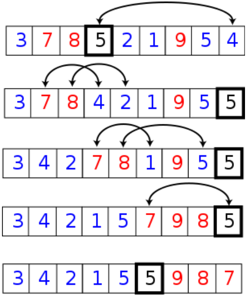
- taking the expected value overall possible i:
- $T(n) = \frac{1}{n} \sum_{i=1}^{n} T(i) + T(n-i) + O(n)$
- This is difficult to analyze! Can we find a better way to analyze quicksort?

# Quicksort: Analysis

Recurrence for quicksort:

- taking the expected value overall possible i:
- $T(n) = \frac{1}{n} \sum_{i=1}^{n} T(i) + T(n-i) + O(n)$
- This is difficult to analyze! Can we find a better way to analyze quicksort?

Partition step




Idea: Any two elements are never compared more than once

- What happens after an element is compared to the partitioning element?
  - these two elements won't be compared again

# Quicksort: Analysis

Analyze with random variables:

- denote the kth smallest element in the array as $e_k$

# Quicksort: Analysis

Analyze with random variables:

- denote the kth smallest element in the array as $e_k$
- X = total number of comparisons
-

$$X_{ij} = \begin{cases} 1 & e_i \text{ and } e_j \text{ are compared} \\ 0 & e_i \text{ and } e_j \text{ are not compared} \end{cases}$$

# Quicksort: Analysis

Analyze with random variables:

- denote the kth smallest element in the array as $e_k$
- X = total number of comparisons
- $$X_{ij} = \begin{cases} 1 & e_i \text{ and } e_j \text{ are compared} \\ 0 & e_i \text{ and } e_j \text{ are not compared} \end{cases}$$
- Then $X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$

# Quicksort: Analysis

Analyze with random variables:

- denote the kth smallest element in the array as $e_k$
- X = total number of comparisons
-

$$X_{ij} = \begin{cases} 1 & e_i \text{ and } e_j \text{ are compared} \\ 0 & e_i \text{ and } e_j \text{ are not compared} \end{cases}$$

- Then $X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$
- $E[X] = E[\Sigma\,\Sigma\, X_{ij}] = \Sigma\,\Sigma\, E[X_{ij}]$ by lin. of exp.
- Recall: $E[X_{ij}] = 1 * P(X_{ij} = 1) + 0 * P(X_{ij} = 0)$
- $E[X_{ij}] = P(X_{ij} = 1)$

# Quicksort: Analysis

Analyze with random variables:

What is the probability that $e_i$ and $e_j$ are compared?

# Quicksort: Analysis

Analyze with random variables:

What is the probability that $e_i$ and $e_j$ are compared?

- $e_i$ and $e_j$ will be compared if either is selected as a pivot

# Quicksort: Analysis

Analyze with random variables:

What is the probability that $e_i$ and $e_j$ are compared?

- $e_i$ and $e_j$ will be compared if either is selected as a pivot
- $e_i$ and $e_j$ will not be compared if some $e_k$, $i < k < j$ is selected as a pivot **first**
  - $e_i$ will be to the left of $e_k$, and $e_j$ will be to the right.

# Quicksort: Analysis

- Which pivots must be chosen for $e_i$ and $e_j$ to be compared?
  - either $e_i$ or $e_j$ (2 total)

# Quicksort: Analysis

- Which pivots must be chosen for $e_i$ and $e_j$ to be compared?
  - either $e_i$ or $e_j$ (2 total )
- Which pivots for $e_i$ and $e_j$ not to be compared?
  - $e_{i+1}$, $e_{i+2}$, ..., or $e_{j-1}$ ($j - i - 1$ total)

# Quicksort: Analysis

- Which pivots must be chosen for $e_i$ and $e_j$ to be compared?
  - either $e_i$ or $e_j$ (2 total )
- Which pivots for $e_i$ and $e_j$ not to be compared?
  - $e_{i+1}$, $e_{i+2}$, ..., or $e_{j-1}$ ($j - i - 1$ total)
- Elements are chosen as pivots randomly
- $E[X_{ij}] = \frac{2}{(j-i-1)+2} = \frac{2}{j-i+1}$
- $E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$
- $E[X] \leq 2n\lg n \in O(n\lg n)$

# Quicksort: Analysis

- Which pivots must be chosen for $e_i$ and $e_j$ to be compared?
  - either $e_i$ or $e_j$ (2 total )
- Which pivots for $e_i$ and $e_j$ not to be compared?
  - $e_{i+1}$, $e_{i+2}$, ..., or $e_{j-1}$ ($j - i - 1$ total)

- $E[X_{ij}] = \frac{2}{(j-i-1)+2} = \frac{2}{j-i+1}$
- $E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$
- $E[X] \leq 2n \lg n \in O(n \lg n)$

Result: Randomized Quicksort makes an expected $O(n \lg n)$ comparisons!

# Quick Select

Goal: select the kth smallest element of an array

# Quick Select

Goal: select the kth smallest element of an array

Option 1:

- Use quicksort to sort the array A
- Select the kth smallest element ($A[k-1]$)
- Time required: $O(nlgn)$ to sort the array
- Are we doing unnecessary work? Can we do better?

# Quick Select

Goal: select the kth smallest element of an array

Option 1:

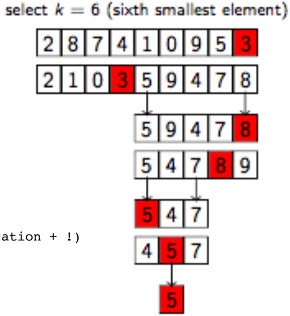
- Use quicksort to sort the array A
- Select the kth smallest element ($A[k-1]$)
- Time required: $O(nlgn)$ to sort the array
- Are we doing unnecessary work? Can we do better?

Key Idea:

- When we partition the array, the kth smallest element will only be on one side of this partition
- No need to recursively sort both sides of the array: Only the side containing the element we want

# Quick select

```
quicksort(A):
    quicksort(A, 0, len(A)-1)

quicksort(A, lo, hi):
    if(lo == hi) return A[lo]
    pivot_location <- partition(A, lo, hi)
    if pivot_location == k:
        return A[k]
    else if pivot_location < k:
        return quickselect(A, lo, pivot_location -1, k)
    else:
        return quickselect(A, pivot_location +1, hi, k-pivot_location + !)
```

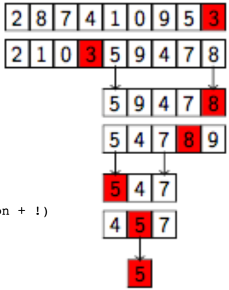


# Quickselect

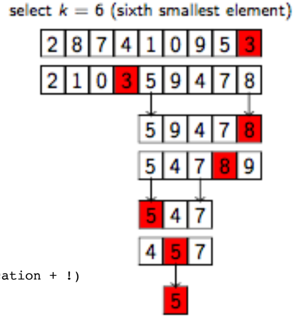
```
quicksort(A):
    quicksort(A, 0, len(A)-1)

quicksort(A, lo, hi):
    if(lo == hi) return A[lo]
    pivot_location <- partition(A, lo, hi)
    if pivot_location == k:
        return A[k]
    else if pivot_location < k:
        return quickselect(A, lo, pivot_location -1, k)
    else:
        return quickselect(A, pivot_location +1, hi, k-pivot_location + !)
```

- Analysis?

select $k = 6$ (sixth smallest element)



# Quickselect




```
quicksort(A):
    quicksort(A, 0, len(A)-1)

quicksort(A, lo, hi):
    if(lo == hi) return A[lo]
    pivot_location <- partition(A, lo, hi)
    if pivot_location == k:
        return A[k]
    else if pivot_location < k:
        return quickselect(A, lo, pivot_location -1, k)
    else:
        return quickselect(A, pivot_location +1, hi, k-pivot_location + !)
```

- Analysis?
  - We will use a similar analysis to Quicksort
  - What will change? Are certain elements less likely to be compared?

# Quickselect: Analysis

Analyze with random variables:

# Quickselect: Analysis

Analyze with random variables:

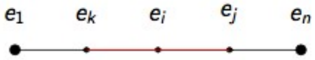
- denote the kth smallest element in the array as $e_k$
- What is the probability that $e_i$ and $e_j$ are compared *when selecting* $e_k$?
- 3 cases:

# Quickselect: Analysis

Analyze with random variables:

- denote the kth smallest element in the array as $e_k$
- What is the probability that $e_i$ and $e_j$ are compared *when selecting* $e_k$?
- 3 cases:

- case 1: $k < i < j$.
  - *Compared when*: $e_i$ or $e_j$ are selected as pivots
  - *Not compared when*: any other element between $e_k$ and $e_j$ are selected
  - $P(e_i\ e_j \text{ compared}) = \frac{2}{j-k+1}$

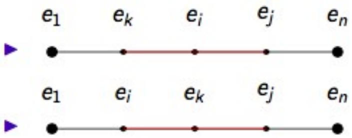


# Quickselect: Analysis

Analyze with random variables:

- denote the kth smallest element in the array as $e_k$
- What is the probability that $e_i$ and $e_j$ are compared *when selecting* $e_k$?
- 3 cases:

- case 1: $k < i < j$.
  - *Compared when:* $e_i$ or $e_j$ are selected as pivots
  - *Not compared when:* any other element between $e_k$ and $e_j$ are selected
  - $P(e_i\ e_j\ compared) = \frac{2}{j-k+1}$
- case 2: $i < k < j$.
  - Similarly: $P(e_i\ e_j\ compared) = \frac{2}{j-i+1}$

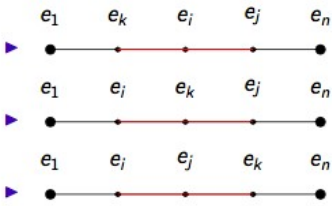


# Quickselect: Analysis

Analyze with random variables:

- denote the kth smallest element in the array as $e_k$
- What is the probability that $e_i$ and $e_j$ are compared *when selecting* $e_k$?
- 3 cases:

- case 1: $k < i < j$.
  - *Compared when*: $e_i$ or $e_j$ are selected as pivots
  - *Not compared when*: any other element between $e_k$ and $e_j$ are selected
  - $P(e_i\ e_j\ compared) = \frac{2}{j-k+1}$
- case 2: $i < k < j$.
  - Similarly: $P(e_i\ e_j\ compared) = \frac{2}{j-i+1}$
- case 3: $i < j < k$.
  - Similarly: $P(e_i\ e_j\ compared) = \frac{2}{k-i+1}$



# Quickselect: Analysis

Runtime:

- Similar to quick sort analysis, how many total comparisons are we making?

# Quickselect: Analysis

Runtime:

- Similar to quick sort analysis, how many total comparisons are we making?
- Sum over all pairs of elements $e_i$ , $e_j$ (split among the 3 cases)

$$E[X] = \sum_{i<j<k} \frac{2}{k-i+1} + \sum_{i<k<j} \frac{2}{j-i+1} + \sum_{k<i<j} \frac{2}{j-k+1}$$

- Non obvious sum, but yields $E[X] = O(n)$!

# Quickselect: Analysis

Runtime:

- Similar to quick sort analysis, how many total comparisons are we making?
- Sum over all pairs of elements $e_i$ , $e_j$ (split among the 3 cases)

$$E[X] = \sum_{i<j<k} \frac{2}{k-i+1} + \sum_{i<k<j} \frac{2}{j-i+1} + \sum_{k<i<j} \frac{2}{j-k+1}$$

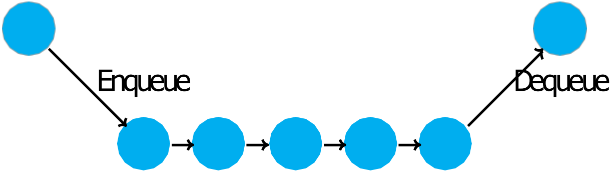
- Non obvious sum, but yields $E[X] = O(n)$!

Outcome:

- Quick select is faster than quick sort!
- Note: quick select is randomized
- Can we make it deterministic, and still keep the worstcase $O(n)$?
- Yes, with some extra work

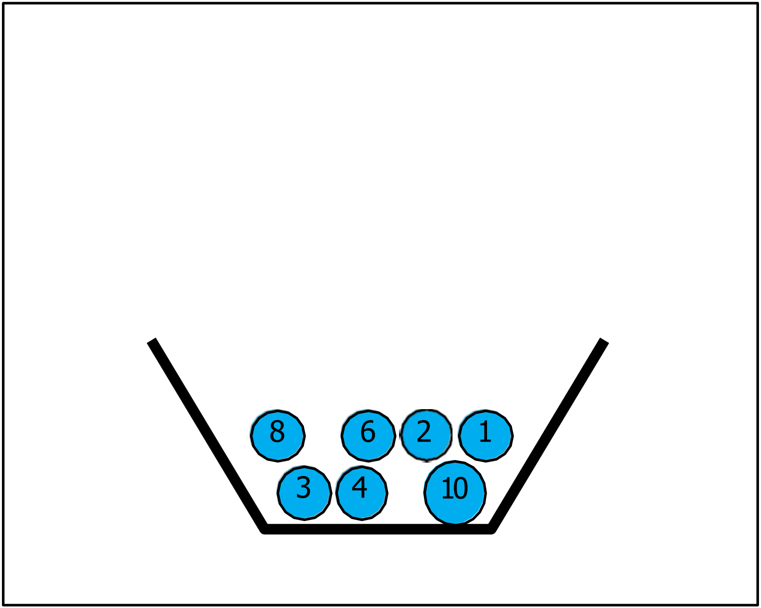
Video 1.6
Sampath Kannan

# Queues

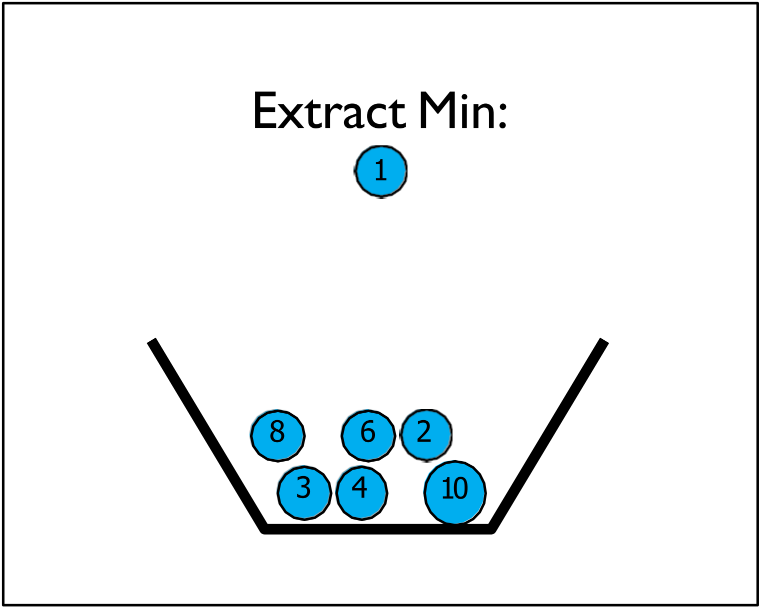



- Sometimes we want to extract elements not in the order we insert them but instead in the order of some given keys. We call this a *priorityqueue*
- For example your operating systemis constantly getting jobs to complete, it needs a fast way of getting the highest priority job to schedule next

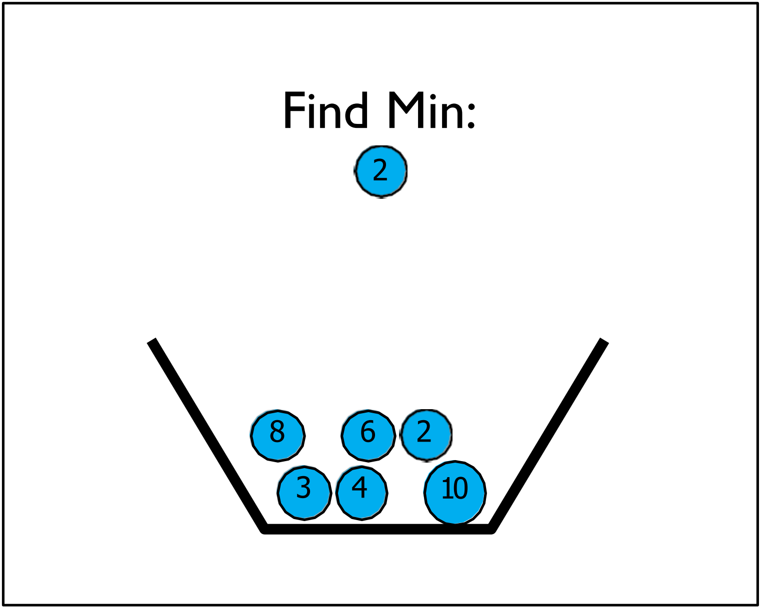
# Operations of Priority Queues



# Operations of Priority Queues

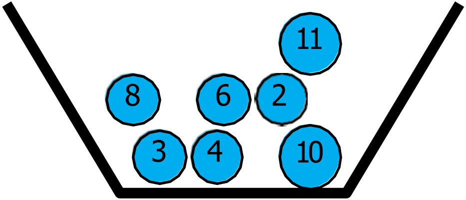


# Operations of Priority Queues



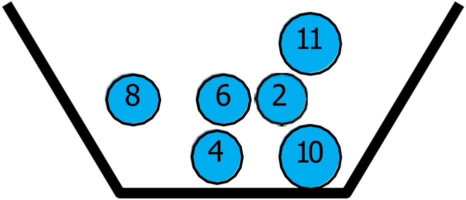
# Operations of Priority Queues

Insert(11):



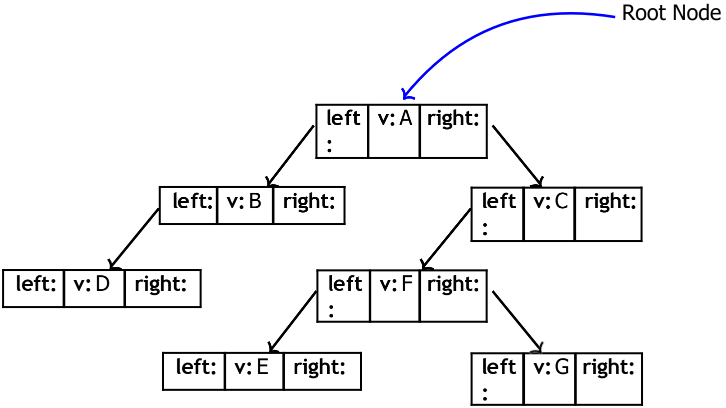
# Operations of Priority Queues

Delete(3):



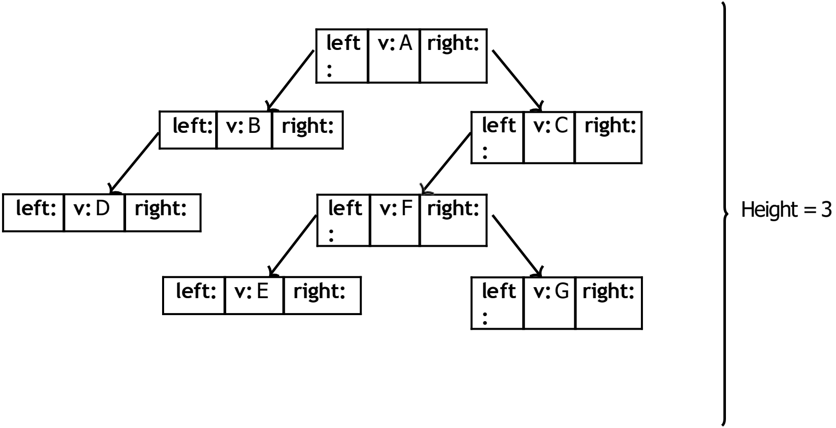
# Trees

In order to make an efficient priority heap we will usea more  general data structure called a tree.



# Trees

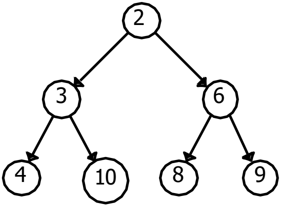
In order to make an efficient priority heap we will usea more  general data structure called a tree.



# Heaps as trees

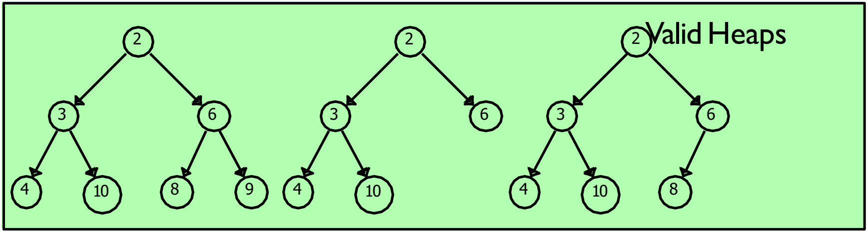
We can use a tree to make a heap by enforcing the properties that node will have a key value that is less than both of it's children, and that the tree will always be complete except for the last layer.

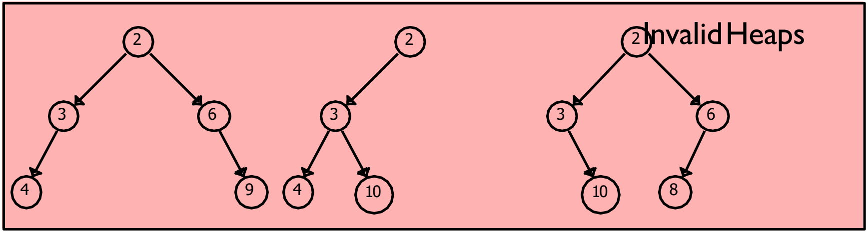
- This makes finding the minimum very easy. It's always on top!




- We will see that removing the root (minimum) element can be done in a number of operations proportional to the height.
- However if we want to find an arbitrary element we will have to search the whole tree.

# Heaps Shapes

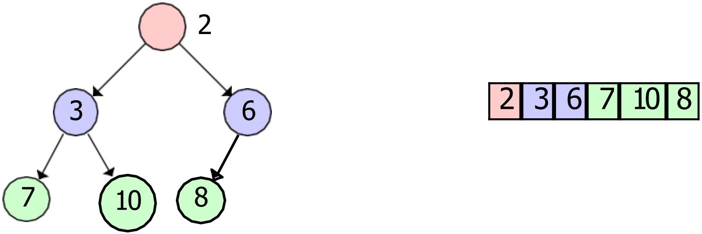


Video 1.7
Sampath Kannan

# Heap Representation

Since the tree for a heap will always been contiguous we can represent the m implicitly with anarray



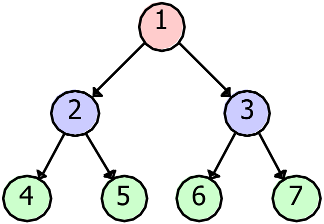

So the *i* th level of the tree will occupy spots $2^{i-1}$ to $2^{i} - 1$ (we are using 1 based indexing for convenience)

# Heap Representation

We need to be able to compute positions of the left and right children of a given element.

# Heap Representation

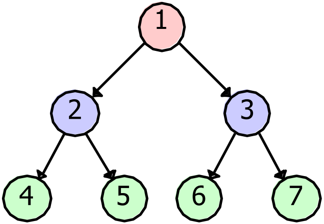
We need to be able to compute positions of the left and right children of a given element.




- Left child of 1 is 2, left child of 2 is 4, left child of 3 is 6, etc...

# Heap Representation

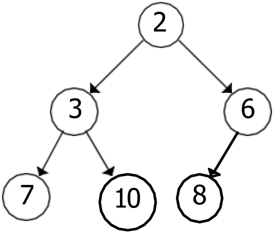
We need to be able to compute positions of the left and right children of a given element.




- Left childof 1 is 2, left childof 2 is 4, left child of 3 is 6, etc...
- In general the left child of node $k$ is at position $2k$ . So the right child is at $2k + 1$

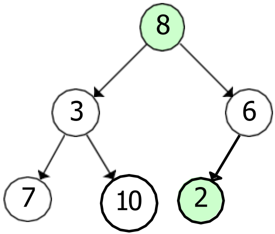
# Operations on Heaps: Extract Min

We want to remove the minimum element (root) while maintaining the two heap properties: order and shape



# Operations on Heaps: Extract Min

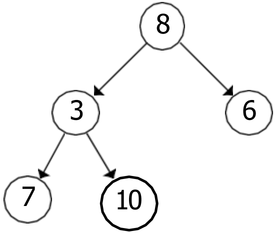
Step 1: Swap the root node with the node in the bottom right




```
sink(A, k):
  N = length(A)
  while 2*k <= N
    smallest = 2*k
    if A[2*k] < A[2*k+1]
       smallest = 2*k+1
    if A[k] < smallest: break
    swap(A[k], A[smallest])
    k = smallest

extract-min(A, k):
  N = length(A)
  min = A[1]
  A[1] = A[N]
  sink(A, 1)
  return min
```

# Operations on Heaps: Extract Min

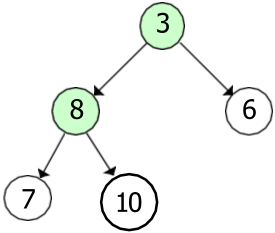
Step 2: Now we can remove(2) while maintaining the shape property




```
sink(A, k):
  N = length(A)
  while 2*k <= N
    smallest = 2*k
    if A[2*k] < A[2*k+1]
       smallest = 2*k+1
    if A[k] < smallest: break
    swap(A[k], A[smallest])
    k = smallest

extract-min(A, k):
  N = length(A)
  min = A[1]
  A[1] = A[N]
  sink(A, 1)
  return min
```

# Operations on Heaps: Extract Min

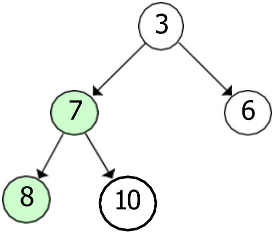
Step 3: We will fix the order property by swapping (8) with it's smallest child




```
sink(A, k):
  N = length(A)
  while 2*k <= N
    smallest = 2*k
    if A[2*k] < A[2*k+1]
       smallest = 2*k+1
    if A[k] < smallest: break
    swap(A[k], A[smallest])
    k = smallest

extract-min(A, k):
  N = length(A)
  min = A[1]
  A[1] = A[N]
  sink(A, 1)
  return min
```

# Operations on Heaps: Extract Min

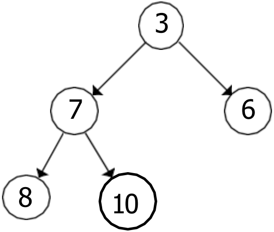
Step 4: Keep fixing the order property by swapping (8) with it's smallest child again




```
sink(A, k):
  N = length(A)
  while 2*k <= N
    smallest = 2*k
    if A[2*k] < A[2*k+1]
       smallest = 2*k+1
    if A[k] < smallest: break
    swap(A[k], A[smallest])
    k = smallest

extract-min(A, k):
  N = length(A)
  min = A[1]
  A[1] = A[N]
  sink(A, 1)
  return min
```

# Operations on Heaps: Extract Min

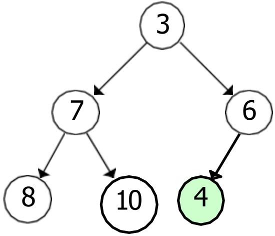
Step 5: The heap properties have been preserved so we're done!




```
sink(A, k):
  N = length(A)
  while 2*k <= N
    smallest = 2*k
    if A[2*k] < A[2*k+1]
       smallest = 2*k+1
    if A[k] < smallest: break
    swap(A[k], A[smallest])
    k = smallest

extract-min(A, k):
  N = length(A)
  min = A[1]
  A[1] = A[N]
  sink(A, 1)
  return min
```

# Operations on Heaps: Insert

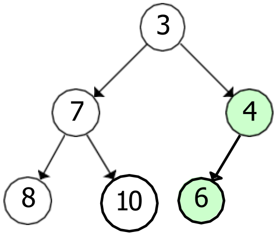
Step 1: Preserve the shape property by inserting the new element at the bottom right




```
swim(A, k) :
   while k > 1 and A[k/ 2] < A[k]:
       swa p(A[k], A[k/ 2])
       k = k/2

in s e r t(A , k,  va l ):
     N= le ngth(A)
     A[N+1] = va l
     swim(A, N+1)
```

# Operations on Heaps: Insert

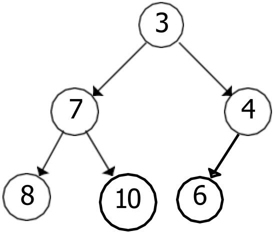
Step 2: Fix the order property by swapping (4) with its parent since it’s smaller




```
swim(A, k) :
    while k > 1 and A[k/ 2] < A[k]:
        swa p(A[k], A[k/ 2])
        k = k/2

in s e r t(A , k,  va l ):
    N= le ngth(A)
    A[N+1] = va l
    swim(A, N+1)
```

# Operations on Heaps: Insert

Step 3: (4) is bigger than its parent now so we're done!




```
swim(A, k) :
    while k > 1 and A[k/ 2] < A[k]:
        swa p(A[k], A[k/ 2])
        k = k/2

in s e r t(A , k,  va l ):
    N= le ngth(A)
    A[N+1] = va l
    swim(A, N+1)
```

# Heap efficiency

- All operations on the heap are a combination of a constant number of operations and sink or swim operation.
- Swim operation executes as long as $k > 1$ and divides it by 2 on every iteration
- Can execute at most $\log_2 k$ times. Since $k$ is initially at most $n$, the number of elements, swim has a run time that is $O(\log n)$
- By the same logic sink has run time that is $O(\log n)$ as well.
- So all the operations are $O(\log n)$. Except for delete which must first take potentially $O(n)$ steps to locate the given element in the array.

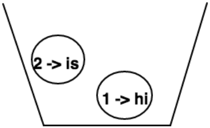
Video 1.8
Sampath Kannan

# Dynamic Dictionaries

Dynamic Dictionaries support three main operations:

- **insert** into adictionary
- **delete** from adictionary
- **search** for an element in a dictionary

Abstract representation:




Dynamic dictionaries are used in applications everywhere:

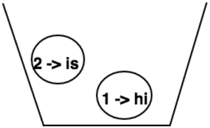
- Databases
- Router lookup tables, ids of IP packets
- Any application that involves storing information!

# Dynamic Dictionaries

Dynamic Dictionaries support three main operations:

- **insert** into adictionary
- **delete** from adictionary
- **search** for an element in a dictionary

Abstract representation:




next: insert the pair (3, ”the”)

Dynamic dictionaries are used in applications everywhere:

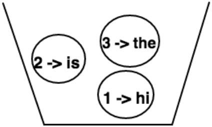
- Databases
- Router lookup tables, ids of IP packets
- Any application that involves storing information!

# Dynamic Dictionaries

Dynamic Dictionaries support three main operations:

- **insert** into adictionary
- **delete** from adictionary
- **search** for an element in a dictionary

Abstract representation:




Dynamic dictionaries are used in applications everywhere:

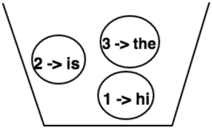
- Databases
- Router lookup tables, ids of IP packets
- Any application that involves storing information!

# Dynamic Dictionaries

Dynamic Dictionaries support three main operations:

- **insert** into adictionary
- **delete** from adictionary
- **search** for an element in a dictionary

Abstract representation:




next: lookup 1

Dynamic dictionaries are used in applications everywhere:

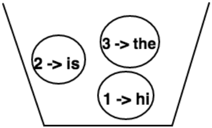
- Databases
- Router lookup tables, ids of IP packets
- Any application that involves storing information!

# Dynamic Dictionaries

Dynamic Dictionaries support three main operations:

- **insert** into adictionary
- **delete** from adictionary
- **search** for an element in a dictionary

Abstract representation:




next: lookup 1 returns "hi"

Dynamic dictionaries are used in applications everywhere:

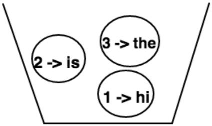
- Databases
- Router lookup tables, ids of IP packets
- Any application that involves storing information!

# Dynamic Dictionaries

Dynamic Dictionaries support three main operations:

- **insert** into adictionary
- **delete** from adictionary
- **search** for an element in a dictionary

Abstract representation:




next: lookup 1 returns "hi"
next: delete 3 from dictionary

Dynamic dictionaries are used in applications everywhere:

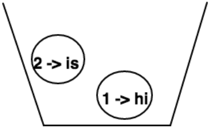
- Databases
- Router lookup tables, ids of IP packets
- Any application that involves storing information!

# Dynamic Dictionaries

Dynamic Dictionaries support three main operations:

- **insert** into adictionary
- **delete** from adictionary
- **search** for an element in a dictionary

Abstract representation:




Dynamic dictionaries are used in applications everywhere:

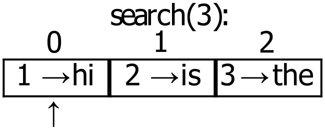
- Databases
- Router lookup tables, ids of IP packets
- Any application that involves storing information!

# Implementations of Dictionaries

Can we find an efficient implementation for dictionaries?

Attempt 1: Arrays

- search: $O(n)$
  - Entire array must be traversed
- insertion, deletion: $O(n)$
  - Array may need to be resized (requires copying all elements to a new array)

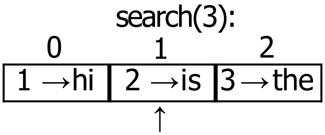


# Implementations of Dictionaries

Can we find an efficient implementation for dictionaries?

Attempt 1:Arrays

- search: $O(n)$
  - Entire array must be traversed
- insertion, deletion: $O(n)$
  - Array may need to be resized (requires copying all elements to a new array

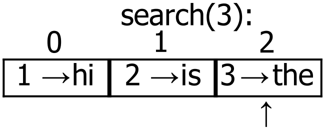


# Implementations of Dictionaries

Can we find an efficient implementation for dictionaries?

Attempt 1:Arrays

- search: $O(n)$
  - Entire array must be traversed
- insertion, deletion: $O(n)$
  - Array may need to be resized (requires copying all elements to a new array

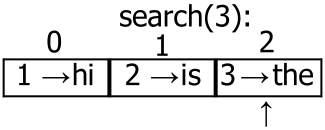


# Implementations of Dictionaries

Can we find an efficient implementation for dictionaries?

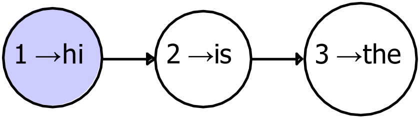
Attempt 1: Arrays

- search: $O(n)$
  - Entire array must be traversed
- insertion, deletion: $O(n)$
  - Array may need to be resized (requires copying all elements to a new array




Attempt 2: Linked Lists

- search, deletion: $O(n)$
  - Entire list must be traversed
- insertion: $O(1)$
  - Can easily insert at the front of the list

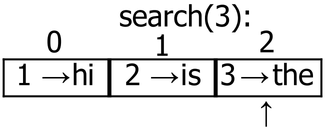


# Implementations of Dictionaries

Can we find an efficient implementation for dictionaries?

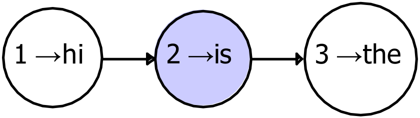
Attempt 1: Arrays

- search: $O(n)$
  - Entire array must be traversed
- insertion, deletion: $O(n)$
  - Array may need to be resized (requires copying all elements to a new array




Attempt 2: Linked Lists

- search, deletion: $O(n)$
  - Entire list must be traversed
- insertion: $O(1)$
  - Can easily insert at the front of the list

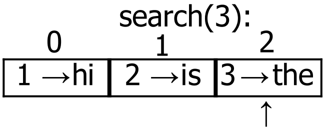


# Implementations of Dictionaries

Can we find an efficient implementation for dictionaries?

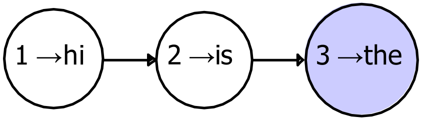
Attempt 1: Arrays

- search: $O(n)$
  - Entire array must be traversed
- insertion, deletion: $O(n)$
  - Array may need to be resized (requires copying all elements to a new array




Attempt 2: Linked Lists

- search, deletion: $O(n)$
  - Entire list must be traversed
- insertion: $O(1)$
  - Can easily insert at the front of the list



# Dictionaries: Binary Search Trees

Attempt 3: Binary Search Tree

- Store items in nodes of a binary tree

# Dictionaries: Binary Search Trees

Attempt 3:Binary Search Tree

- Store items in nodes of a binary tree
- Keys in the tree are ordered.

**search tree property:**

- All keys to the left of a node are < that node’s key
- All keys to the rightof a node are > that node’skey
- The left and right subtrees of the node also satisfy the search tree property

# Dictionaries: Binary Search Trees

Attempt 3:Binary Search Tree

- Store items in nodes of a binary tree
- Keys in the tree are ordered.

**search tree property:**

- All keys to the left of a node are < that node's key
- All keys to the rightof a node are > that node'skey
- The left and right subtrees of the node also satisfy the search tree property

insert (4, ”a”)

# Dictionaries: Binary Search Trees

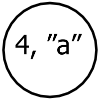
Attempt 3:Binary Search Tree

- Store items in nodes of a binary tree
- Keys in the tree are ordered.

**search tree property:**

- All keys to the left of a node are < that node's key
- All keys to the rightof a node are > that node'skey
- The left and right subtrees of the node also satisfy the search tree property

insert (2, "b")



# Dictionaries: Binary Search Trees

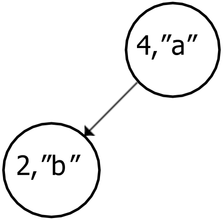
Attempt 3:Binary Search Tree

- Store items in nodes of a binary tree
- Keys in the tree are ordered.

**search tree property:**

- All keys to the left of a node are < that node’s key
- All keys to the rightof a node are > that node’skey
- The left and right subtrees of the node also satisfy the search tree property

insert (7, ”c”)

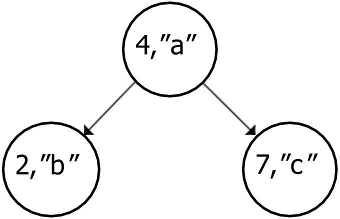


# Dictionaries: Binary Search Trees

Attempt 3:Binary Search Tree

- Store items in nodes of a binary tree
- Keys in the tree are ordered.

insert (1, ”d”)




**search tree property:**

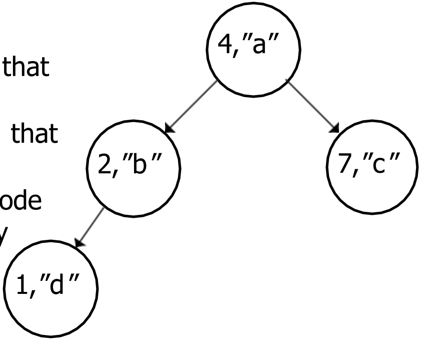
- All keys to the left of a node are < that node’s key
- All keys to the rightof a node are > that node’skey
- The left and right subtrees of the node also satisfy the search tree property

# Dictionaries: Binary Search Trees

Attempt 3:Binary Search Tree

- Store items in nodes of a binary tree
- Keys in the tree are ordered.

insert (3, ”e”)




**search tree property:**

- All keys to the left of a node are < that node’s key
- All keys to the rightof a node are > that node’skey
- The left and right subtrees of the node also satisfy the search tree property

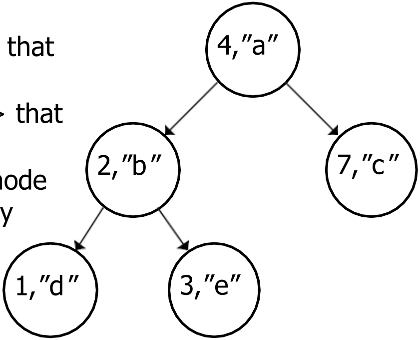
# Dictionaries: Binary Search Trees

Attempt 3:Binary Search Tree

- Store items in nodes of a binary tree
- Keys in the tree are ordered.

**search tree property:**

- All keys to the left of a node are < that node's key
- All keys to the rightof a node are > that node'skey
- The left and right subtrees of the node also satisfy the search tree property

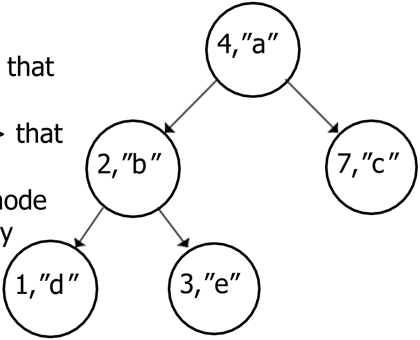


# Dictionaries: Binary Search Trees

Attempt 3:Binary Search Tree

- Store items in nodes of a binary tree
- Keys in the tree are ordered.

search(3)




**search tree property:**

- All keys to the left of a node are < that node's key
- All keys to the rightof a node are > that node'skey
- The left and right subtrees of the node also satisfy the search tree property

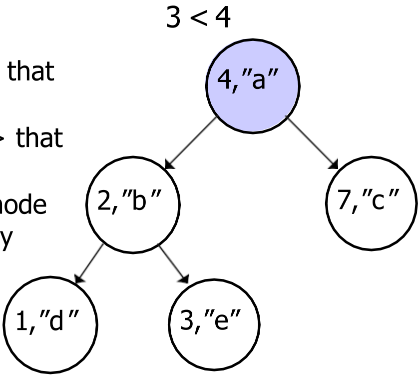
# Dictionaries: Binary Search Trees

Attempt 3:Binary Search Tree

- Store items in nodes of a binary tree
- Keys in the tree are ordered.

**search tree property:**

- All keys to the left of a node are < that node's key
- All keys to the rightof a node are > that node'skey
- The left and right subtrees of the node also satisfy the search tree property



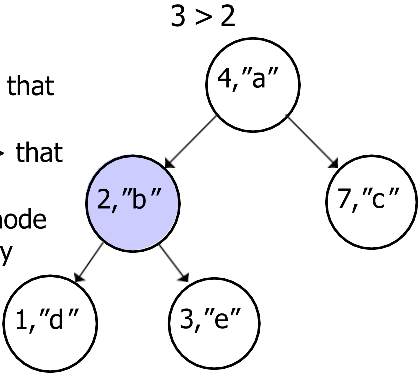
# Dictionaries: Binary Search Trees

Attempt 3:Binary Search Tree

- Store items in nodes of a binary tree
- Keys in the tree are ordered.

**search tree property:**

- All keys to the left of a node are < that node's key
- All keys to the rightof a node are > that node'skey
- The left and right subtrees of the node also satisfy the search tree property



# Dictionaries: Binary Search Trees

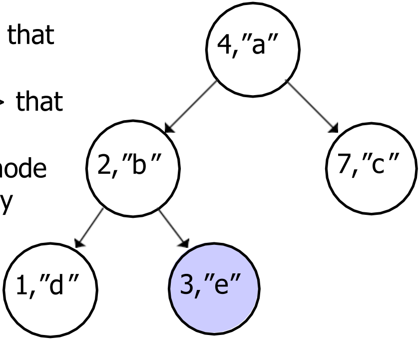
Attempt 3:Binary Search Tree

- Store items in nodes of a binary tree
- Keys in the tree are ordered.

**search tree property:**

- All keys to the left of a node are < that node's key
- All keys to the rightof a node are > that node'skey
- The left and right subtrees of the node also satisfy the search tree property

Search (3)
return "e"

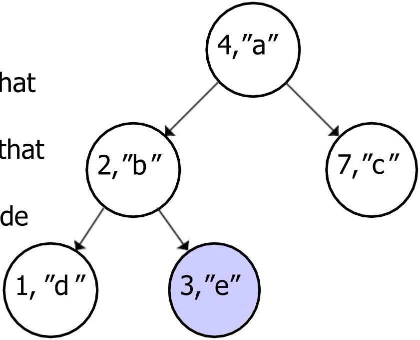


# Dictionaries: Binary Search Trees

Attempt 3:Binary Search Tree

- Store items in nodes of a binary tree
- Keys in the tree are ordered.

Search (3)
Return "e"




**search tree property:**

- All keys to the left of a node are < that node's key
- All keys to the rightof a node are > that node'skey
- The left and right subtrees of the node also satisfy the search tree property

Time to insert, search and delete is proportional to the height of the tree!

# Binary Search Trees: Runtime

- Insert, Deletion and Search take time proportional to height of the tree

# Binary Search Trees: Runtime

- Insert, Deletion and Search take time proportional to height of the tree
- But how bad can the height be?

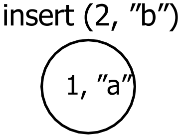
# Binary Search Trees: Runtime

- Insert, Deletion and Search take time proportional to height of the tree
- But how bad can the height be?

insert (1, ”a”)

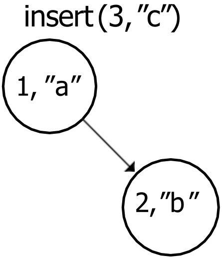
# Binary Search Trees: Runtime

- Insert, Deletion and Search take time proportional to height of the tree
- But how bad can the height be?

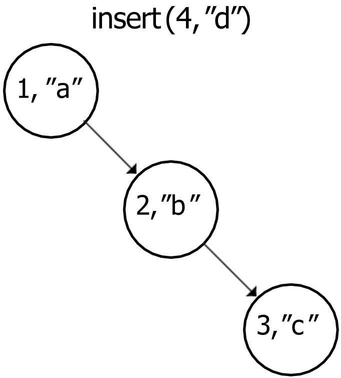


# Binary Search Trees: Runtime

- Insert, Deletion and Search take time proportional to height of the tree
- But how bad can the height be?

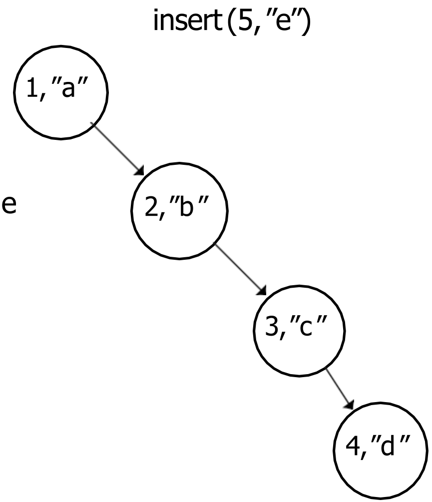


# Binary Search Trees: Runtime



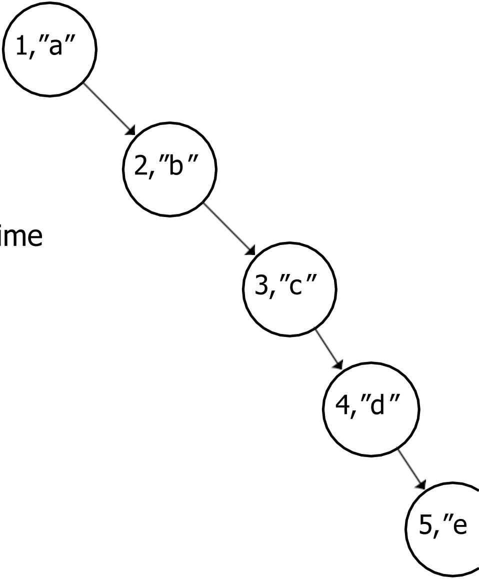

- Insert, Deletion and Search take time proportional to height of the tree
- But how bad can the height be?

# Binary Search Trees:  Runtime



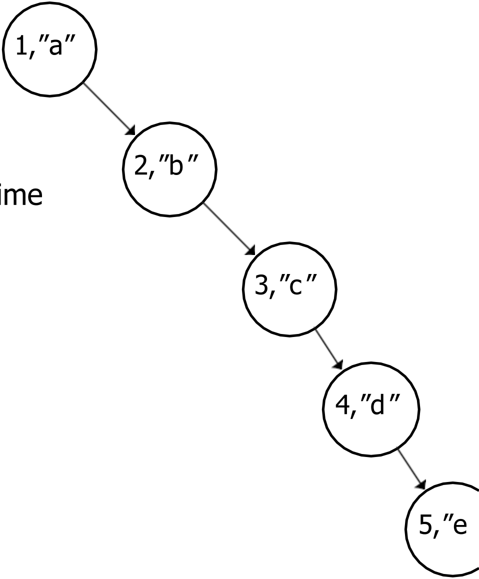

- Insert, Deletion and Search take time proportional to heightof the tree
- But how bad can the height be?

# Binary Search Trees: Runtime



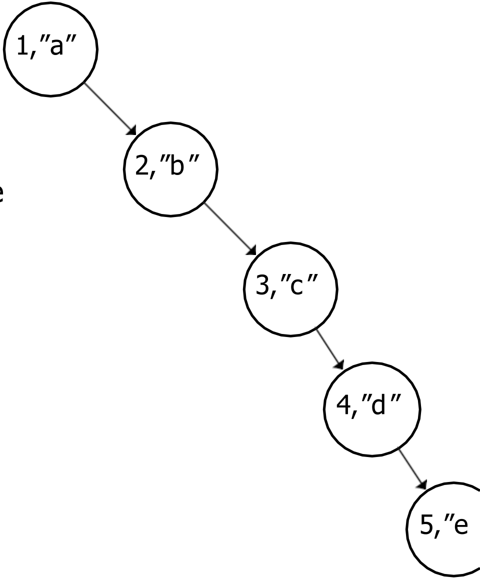

- Insert, Deletion and Search take time proportional to height of the tree
- But how bad can the height be?

# Binary Search Trees: Runtime




- Insert, Deletion and Search take time proportional to height of the tree
- But how bad can the height be?
- Worst case: height of tree is $O(n)$ (number of elements inserted)

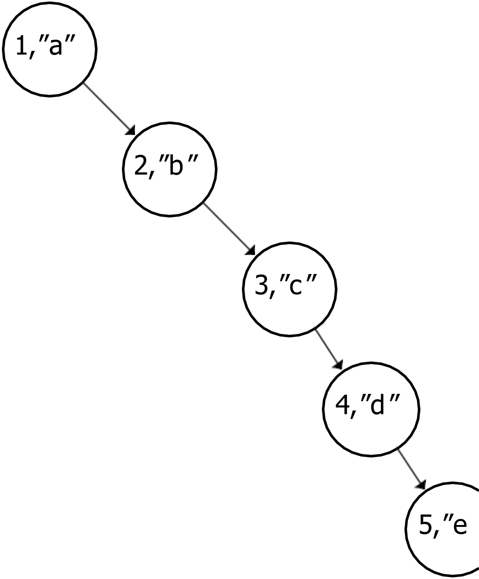
# Binary Search Trees: Runtime




- Insert, Deletion and Search take time proportional to height of the tree
- But how bad can the height be?
- Worst case: height of tree is $O(n)$ (number of elements inserted)
- However, common case: tree is balanced.

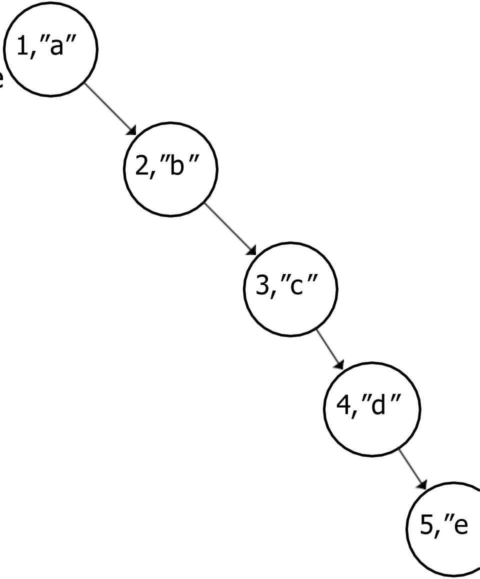
# Binary Search Trees: Runtime

- Insert, Deletion and Search take time proportional to height of the tree
- But how bad can the height be?
- Worst case: height of tree is $O(n)$ (number of elements inserted)
- However, common case: tree is balanced.
  - 1st level: 1 node
  - 2nd level: 2nodes
  - kth level: $2^k$ nodes
  - $n = 1 + 2 + 2^2 + \ldots + 2^k$
  - $2^{k+1} - 1 = n$, $k = O(lgn)$

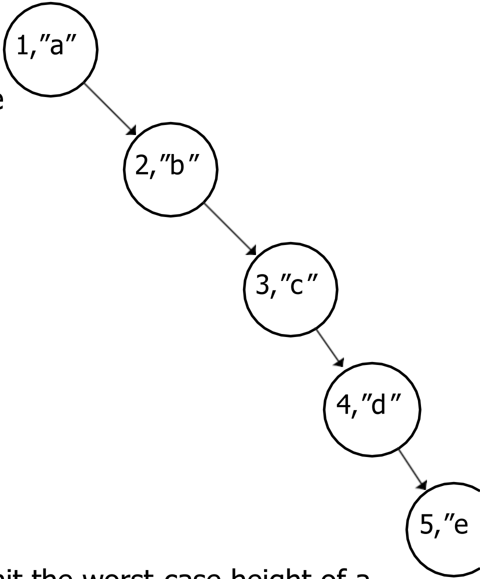


# Binary Search Trees: Runtime

- Insert, Deletion and Search take time proportional to height of the tree
- But how bad can the height be?
- Worst case: height of tree is $O(n)$ (number of elements inserted)
- However, common case: tree is balanced.
  - 1st level: 1 node
  - 2nd level: 2nodes
  - kth level: $2^k$ nodes
  - $n = 1 + 2 + 2^2 + \ldots + 2^k$
  - $2^{k+1} - 1 = n$, $k = O(lgn)$
- common case: height is $O(lgn)$



# Binary Search Trees: Runtime




- Insert, Deletion and Search take time proportional to height of the tree
- But how bad can the height be?
- Worst case: height of tree is $O(n)$ (number of elements inserted)
- However, common case: tree is balanced.
  - 1st level: 1 node
  - 2nd level: 2nodes
  - kth level: $2^k$ nodes
  - $n = 1 + 2 + 2^2 + \ldots + 2^k$
  - $2^{k+1} - 1 = n$, $k = O(lgn)$
- common case: height is $O\,(lgn)$

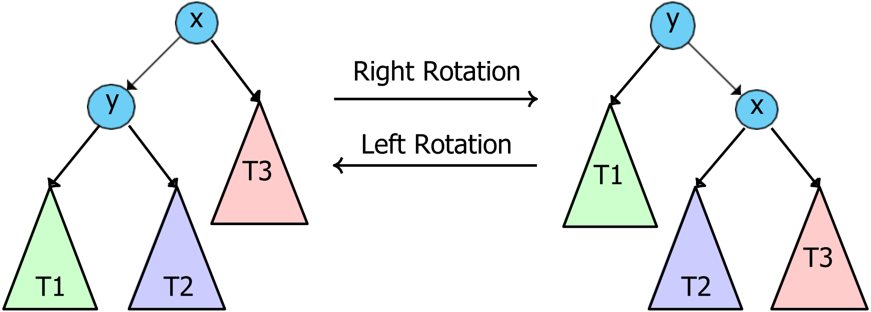
Is there anything we can do to limit the worst-case height of a binary search tree?

Video 1.9
Sampath Kannan

# Balanced Binary Search Trees

- BSTs can become unbalanced leading to $O(n)$ run times for operations.
- We need a way to modify them so that their height is $O(\log n)$ instead of $O(n)$.
- Intuitively we can get this property if the left and right sub-trees always have similar heights
- Modifications must preserve search tree property

# Rotations




We use rotations to keep left and right sub-trees balanced. In an AVL tree we maintain the invariant that all left and right sub-trees have a height difference of at most 1.

# Hashing

- To use an array to implement a dictionary we need a way to map elements from our universe to indices. This mapping is called a **hash function** and the array is called a **hash table**
- Example: If our universe is all the integers and we have a hash table of size 37 we could use $h(x) = x \bmod 37$ as our hash function.
- If only one item gets mapped to each index then all operations are $O(1)$!

# Load factor

- Suppose we have *m* different keys and a hash table of size *n*, and suppose that for each key we randomly choose an index to map it to.

# Load factor

- Suppose we have $m$ different keys and a hash table of size $n$, and suppose that for each key we randomly choose an index to map it to.
- $P(h(k) = i) = 1/n$.

# Load factor

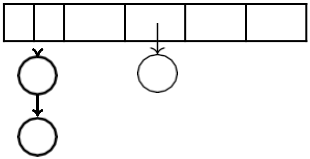
- Suppose we have *m* different keys and a hash table of size *n*, and suppose that for each key we randomly choose an index to map it to.
- $P(h(k) = i) = 1/n$.
- Let $X_i$ be the number of keys mapped to index *i* and

$$E[X] = \sum_{k} P(h(k) = i) * (1) = \sum_{k} (1/n) * (1) = m/n$$

- **load factor** = $a$ .

# Handling Collisions

- Can't get rid of collisions so we need to store multiple items in a single bin
- One approach to this is **chaining**:



- Instead of storing each item directly in the array, we store a linked list of all the items that map to that index
- Run-time of all operations is now proportional to the length of the linked lists at the index we are operating on. We just saw that this gives *expected* $O(\alpha)$ performance.
- Note that the worst case is still $O(m)$!

# Handling Collisions 2

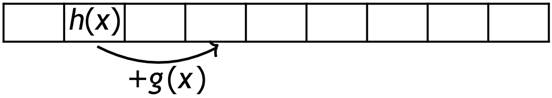
- Instead of chaining we can use **open addressing** where keys that map to the same index are stored in separate locations in the table.
- One approach to this is **double hashing**, where we use 2 hash functions $h(x)$ and $g(x)$.
- When there is a collision at $h(x)$ we try to insert at $h(x) + g(x)$, then $h(x) + 2g(x)$, ... etc

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $h(x)$ | | | | | | | |

- Pros: No extra storage required, we don't have to deal with pointers
- Cons: Deletion is very tricky and easy to mess up

# Handling Collisions 2

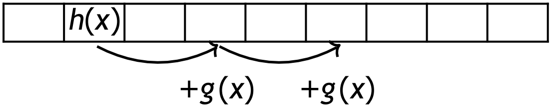
- Instead of chaining we can use **open addressing** where keys that map to the same index are stored in separate locations in the table.
- One approach to this is **double hashing**, where we use 2 hash functions $h(x)$ and $g(x)$.
- When there is a collision at $h(x)$ we try to insert at $h(x) + g(x)$, then $h(x) + 2g(x)$, ... etc




- Pros: No extra storage required, we don't have to deal with pointers
- Cons: Deletion is very tricky and easy to mess up

# Handling Collisions 2

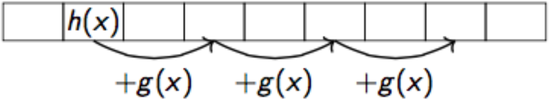
- Instead of chaining we can use **open addressing** where keys that map to the same index are stored in separate locations in the table.
- One approach to this is **double hashing**, where we use 2 hash functions $h(x)$ and $g(x)$.
- When there is a collision at $h(x)$ we try to insert at $h(x) + g(x)$, then $h(x) + 2g(x)$, ... etc




- Pros: No extra storage required, we don't have to deal with pointers
- Cons: Deletion is very tricky and easy to mess up

# Handling Collisions 2

- Instead of chaining we can use **open addressing** where keys that map to the same index are stored in separate locations in the table.
- One approach to this is **double hashing**, where we use 2 hash functions $h(x)$ and $g(x)$.
- When there is a collision at $h(x)$ we try to insert at $h(x) + g(x)$, then $h(x) + 2g(x)$, ... etc




- Pros: No extra storage required, we don't have to deal with pointers
- Cons: Deletion is very tricky and easy to mess up