Chapter 6

Inference for categorical data

6.3 Testing for goodness of fit using chi-square

In this section, we develop a method for assessing a null model when the data are binned. This technique is commonly used in two circumstances:

- Given a sample of cases that can be classified into several groups, determine if the sample is representative of the general population.
- Evaluate whether data resemble a particular distribution, such as a normal distribution or a geometric distribution.

Each of these scenarios can be addressed using the same statistical test: a chi-square test.

In the first case, we consider data from a random sample of 275 jurors in a small county. Jurors identified their racial group, as shown in Table 6.5, and we would like to determine if these jurors are racially representative of the population. If the jury is representative of the population, then the proportions in the sample should roughly reflect the population of eligible jurors, i.e. registered voters.

Race	White	Black	Hispanic	Other	Total
Representation in juries	205	26	25	19	275
Registered voters	0.72	0.07	0.12	0.09	1.00

Table 6.5: Representation by race in a city's juries and population.

While the proportions in the juries do not precisely represent the population proportions, it is unclear whether these data provide convincing evidence that the sample is not representative. If the jurors really were randomly sampled from the registered voters, we might expect small differences due to chance. However, unusually large differences may provide convincing evidence that the juries were not representative.

A second application, assessing the fit of a distribution, is presented at the end of this section. Daily stock returns from the S&P500 for the years 1990-2011 are used to assess whether stock activity each day is independent of the stock's behavior on previous days.

In these problems, we would like to examine all bins simultaneously, not simply compare one or two bins at a time, which will require us to develop a new test statistic.

6.3.1 Creating a test statistic for one-way tables

● **Example 6.20** Of the people in the city, 275 served on a jury. If the individuals are randomly selected to serve on a jury, about how many of the 275 people would we expect to be white? How many would we expect to be black?

About 72% of the population is white, so we would expect about 72% of the jurors to be white: $0.72 \times 275 = 198$.

Similarly, we would expect about 7% of the jurors to be black, which would correspond to about $0.07 \times 275 = 19.25$ black jurors.

⊙ **Guided Practice 6.21** Twelve percent of the population is Hispanic and 9% represent other races. How many of the 275 jurors would we expect to be Hispanic or from another race? Answers can be found in Table 6.6.

Race	White	Black	Hispanic	Other	Total
Observed data	205	26	25	19	275
Expected counts	198	19.25	33	24.75	275

Table 6.6: Actual and expected make-up of the jurors.

The sample proportion represented from each race among the 275 jurors was not a precise match for any ethnic group. While some sampling variation is expected, we would expect the sample proportions to be fairly similar to the population proportions if there is no bias on juries. We need to test whether the differences are strong enough to provide convincing evidence that the jurors are not a random sample. These ideas can be organized into hypotheses:

H_0: The jurors are a random sample, i.e. there is no racial bias in who serves on a jury, and the observed counts reflect natural sampling fluctuation.

H_A: The jurors are not randomly sampled, i.e. there is racial bias in juror selection.

To evaluate these hypotheses, we quantify how different the observed counts are from the expected counts. Strong evidence for the alternative hypothesis would come in the form of unusually large deviations in the groups from what would be expected based on sampling variation alone.

6.3.2 The chi-square test statistic

In previous hypothesis tests, we constructed a test statistic of the following form:

$$Z = \frac{\text{point estimate} - \text{null value}}{\text{SE of point estimate}}$$

This construction was based on (1) identifying the difference between a point estimate and an expected value if the null hypothesis was true, and (2) standardizing that difference using the standard error of the point estimate. These two ideas will help in the construction of an appropriate test statistic for count data.

In this example we have four categories: white, black, hispanic, and other. Because we have four values rather than just one or two, we need a new tool to analyze the data. Our strategy will be to find a test statistic that measures the overall deviation between the observed and the expected counts. We first find the difference between the observed and expected counts for the four groups:

	White	*Black*	*Hispanic*	*Other*
observed - expected	$205 - 198$	$26 - 19.25$	$25 - 33$	$19 - 24.75$

Next, we square the differences:

	White	*Black*	*Hispanic*	*Other*
$(\text{observed - expected})^2$	$(205 - 198)^2$	$(26 - 19.25)^2$	$(25 - 33)^2$	$(19 - 24.75)^2$

We must standardize each term. To know whether the squared difference is large, we compare it to what was expected. If the expected count was 5, a squared difference of 25 is very large. However, if the expected count was 1,000, a squared difference of 25 is very small. We will divide each of the squared differences by the corresponding expected count.

	White	*Black*	*Hispanic*	*Other*
$\frac{(\text{observed - expected})^2}{\text{expected}}$	$\frac{(205 - 198)^2}{198}$	$\frac{(26 - 19.25)^2}{19.25}$	$\frac{(25 - 33)^2}{33}$	$\frac{(19 - 24.75)^2}{24.75}$

Finally, to arrive at the overall measure of deviation between the observed counts and the expected counts, we add up the terms.

$$\begin{aligned} X^2 &= \sum \frac{(\text{observed - expected})^2}{\text{expected}} \\ &= \frac{(205 - 198)^2}{198} + \frac{(26 - 19.25)^2}{19.25} + \frac{(25 - 33)^2}{33} + \frac{(19 - 24.75)^2}{24.75} \end{aligned}$$

The test statistic X^2 is generally used for these reasons. We can write an equation for X^2 using the observed counts and expected counts:

X^2
chi-square
test statistic

$$X^2 = \frac{(\text{observed count}_1 - \text{expected count}_1)^2}{\text{expected count}_1} + \cdots + \frac{(\text{observed count}_4 - \text{expected count}_4)^2}{\text{expected count}_4}$$

The final number X^2 summarizes how strongly the observed counts tend to deviate from the null counts.

In Section 6.3.4, we will see that if the null hypothesis is true, then X^2 follows a new distribution called a *chi-square distribution*. Using this distribution, we will be able to obtain a p-value to evaluate whether there appears to be racial bias in the juries for the city we are considering.

6.3.3 The chi-square distribution and finding areas

The **chi-square distribution** is sometimes used to characterize data sets and statistics that are always positive and typically right skewed. Recall the normal distribution had two parameters – mean and standard deviation – that could be used to describe its exact characteristics. The chi-square distribution has just one parameter called **degrees of freedom (df)**, which influences the shape, center, and spread of the distribution.

⊙ **Guided Practice 6.22** Figure 6.7 shows three chi-square distributions. (a) How does the center of the distribution change when the degrees of freedom is larger? (b) What about the variability (spread)? (c) How does the shape change?[12]

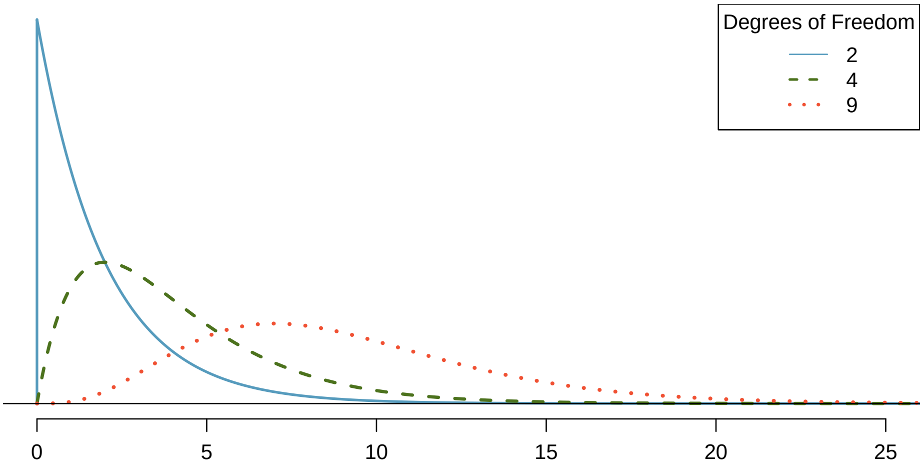


Figure 6.7: Three chi-square distributions with varying degrees of freedom.

Figure 6.7 and Guided Practice 6.22 demonstrate three general properties of chi-square distributions as the degrees of freedom increases: the distribution becomes more symmetric, the center moves to the right, and the variability inflates.

Our principal interest in the chi-square distribution is the calculation of p-values, which (as we have seen before) is related to finding the relevant area in the tail of a distribution. To do so, a new table is needed: the **chi-square table**, partially shown in Table 6.8. A more complete table is presented in Appendix B.3 on page 392. This table is very similar to the t table from Sections 7.1 and 7.3: we identify a range for the area, and we examine a

[12] (a) The center becomes larger. If we look carefully, we can see that the center of each distribution is equal to the distribution's degrees of freedom. (b) The variability increases as the degrees of freedom increases. (c) The distribution is very strongly skewed for $df = 2$, and then the distributions become more symmetric for the larger degrees of freedom $df = 4$ and $df = 9$. In fact, as the degrees of freedom increase, the X^2 distribution approaches a normal distribution.

particular row for distributions with different degrees of freedom. One important difference from the t table is that the chi-square table only provides upper tail values.

Upper tail		0.3	0.2	0.1	0.05	0.02	0.01	0.005	0.001
df	2	2.41	**3.22**	**4.61**	5.99	7.82	9.21	10.60	13.82
	3	*3.66*	*4.64*	6.25	*7.81*	*9.84*	*11.34*	*12.84*	*16.27*
	4	4.88	5.99	7.78	9.49	11.67	13.28	14.86	18.47
	5	6.06	7.29	9.24	11.07	13.39	15.09	16.75	20.52
	6	7.23	8.56	10.64	12.59	15.03	16.81	18.55	22.46
	7	8.38	9.80	12.02	14.07	16.62	18.48	20.28	24.32

Table 6.8: A section of the chi-square table. A complete table is in Appendix B.3 on page 392.

● **Example 6.23** Figure 6.9(a) shows a chi-square distribution with 3 degrees of freedom and an upper shaded tail starting at 6.25. Use Table 6.8 to estimate the shaded area.

This distribution has three degrees of freedom, so only the row with 3 degrees of freedom (df) is relevant. This row has been italicized in the table. Next, we see that the value – 6.25 – falls in the column with upper tail area 0.1. That is, the shaded upper tail of Figure 6.9(a) has area 0.1.

● **Example 6.24** We rarely observe the *exact* value in the table. For instance, Figure 6.9(b) shows the upper tail of a chi-square distribution with 2 degrees of freedom. The lower bound for this upper tail is at 4.3, which does not fall in Table 6.8. Find the approximate tail area.

The cutoff 4.3 falls between the second and third columns in the 2 degrees of freedom row. Because these columns correspond to tail areas of 0.2 and 0.1, we can be certain that the area shaded in Figure 6.9(b) is between 0.1 and 0.2.

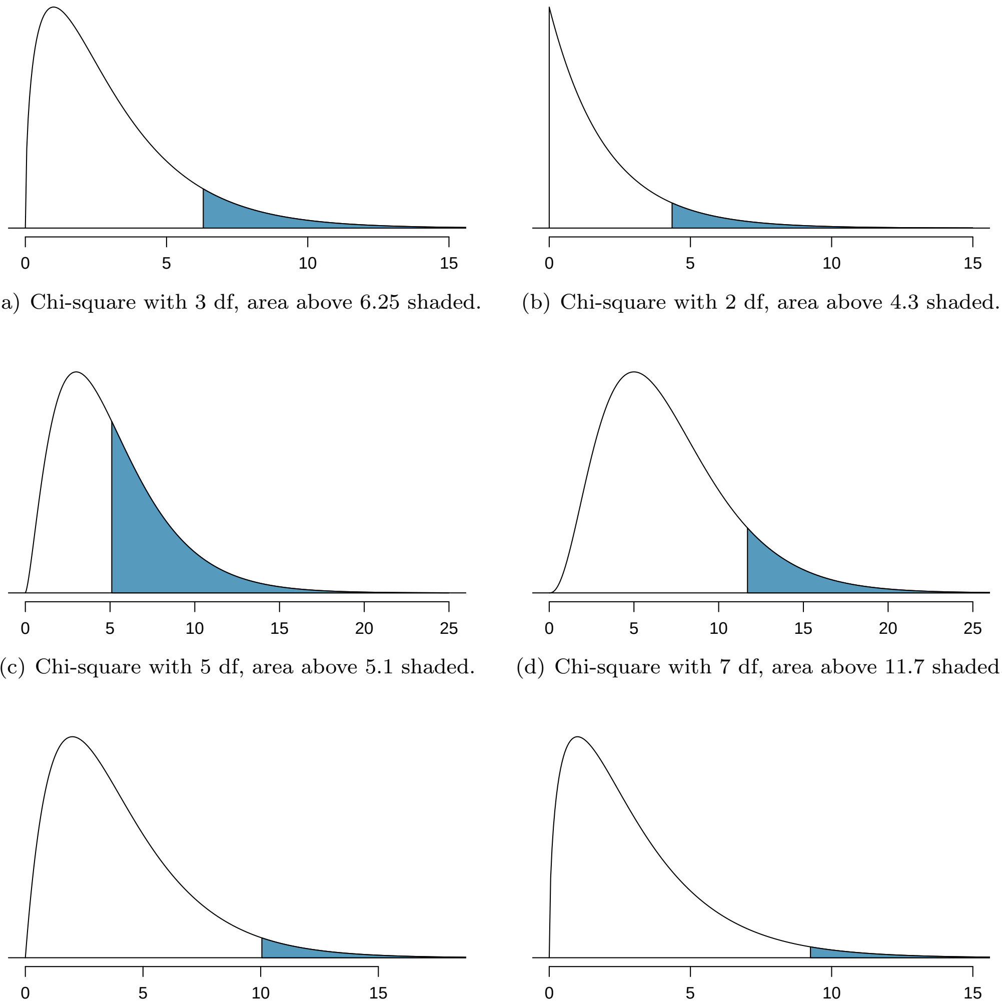
Using a calculator or statistical software allows us to get more precise areas under the chi-square curve than we can get from the table alone.

TI Calculator: finding areas under the chi-square curve

Use the X^2**cdf** command to find areas under the chi-square curve.

1. Hit 2ND VARS (i.e. DISTR).
2. Choose 8: X^2cdf.
3. Enter the lower bound (generally the chi-square value).
4. Enter the upper bound (use a large number, such as 1000).
5. Enter the degrees of freedom.
6. Choose Paste and hit ENTER.

TI-83: Do steps 1 - 2, then type the lower bound, upper bound, and degrees of freedom separated by commas. e.g. X^2cdf(5, 1000, 3), and hit ENTER.



(a) Chi-square with 3 df, area above 6.25 shaded.

(b) Chi-square with 2 df, area above 4.3 shaded.

(c) Chi-square with 5 df, area above 5.1 shaded.

(d) Chi-square with 7 df, area above 11.7 shaded.

(e) Chi-square with 4 df, area above 10 shaded.

(f) Chi-square with 3 df, area above 9.21 shaded.

Figure 6.9: (a) Six chi-square distributions with different right tail areas shaded.

⊙ **Guided Practice 6.25** Figure 6.9(c) shows an upper tail for a chi-square distribution with 5 degrees of freedom and a cutoff of 5.1. Find the tail area using a calculator.[13]

⊙ **Guided Practice 6.26** Figure 6.9(d) shows a cutoff of 11.7 on a chi-square distribution with 7 degrees of freedom. Find the area of the upper tail.[14]

⊙ **Guided Practice 6.27** Figure 6.9(e) shows a cutoff of 10 on a chi-square distribution with 4 degrees of freedom. Find the area of the upper tail.[15]

⊙ **Guided Practice 6.28** Figure 6.9(f) shows a cutoff of 9.21 with a chi-square distribution with 3 df. Find the area of the upper tail.[16]

6.3.4 Finding a p-value for a chi-square distribution

In Section 6.3.2, we identified a new test statistic (X^2) within the context of assessing whether there was evidence of racial bias in how jurors were sampled. The null hypothesis represented the claim that jurors were randomly sampled and there was no racial bias. The alternative hypothesis was that there was racial bias in how the jurors were sampled.

We determined that a large X^2 value would suggest strong evidence favoring the alternative hypothesis: that there was racial bias. However, we could not quantify what the chance was of observing such a large test statistic ($X^2 = 5.89$) if the null hypothesis actually was true. This is where the chi-square distribution becomes useful. If the null hypothesis was true and there was no racial bias, then X^2 would follow a chi-square distribution, with three degrees of freedom in this case. Under certain conditions, the statistic X^2 follows a chi-square distribution with $k - 1$ degrees of freedom, where k is the number of bins or categories of the variable.

● **Example 6.29** How many categories were there in the juror example? How many degrees of freedom should be associated with the chi-square distribution used for X^2?

In the jurors example, there were $k = 4$ categories: white, black, Hispanic, and other. According to the rule above, the test statistic X^2 should then follow a chi-square distribution with $k - 1 = 3$ degrees of freedom if H_0 is true.

Just like we checked sample size conditions to use the normal model in earlier sections, we must also check a sample size condition to safely apply the chi-square distribution for X^2. Each expected count must be at least 5. In the juror example, the expected counts were 198, 19.25, 33, and 24.75, all easily above 5, so we can apply the chi-square model to the test statistic, $X^2 = 5.89$.

● **Example 6.30** If the null hypothesis is true, the test statistic $X^2 = 5.89$ would be closely associated with a chi-square distribution with three degrees of freedom. Using this distribution and test statistic, identify the p-value and state whether or not there is evidence of racial bias in the juror selection.

[13] Using X^2cdf(5.1, 1000, 5) gives 0.4038.

[14] The area is 0.1109.

[15] The area is 0.4043.

[16] The area is 0.0266.

The chi-square distribution and p-value are shown in Figure 6.10. Because larger chi-square values correspond to stronger evidence against the null hypothesis, we shade the upper tail to represent the p-value. Using the chi-square table in Appendix B.3 or the short table on page 238, we can determine that the area is between 0.1 and 0.2. That is, the p-value is larger than 0.1 but smaller than 0.2. Generally we do not reject the null hypothesis with such a large p-value. In other words, the data do not provide convincing evidence of racial bias in the juror selection.

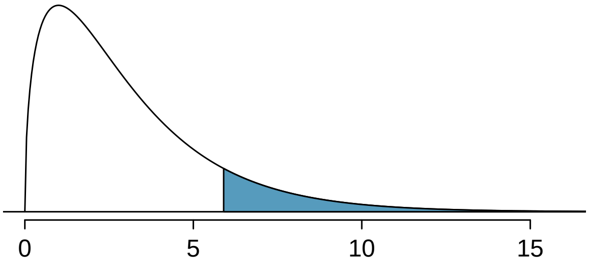


Figure 6.10: The p-value for the juror hypothesis test is shaded in the chi-square distribution with *df* = 3.

The test that we just carried out regarding jury selection is known as the X^2 **goodness of fit test**. It is called "goodness of fit" because we test whether or not the proposed or expected distribution is a good fit for the observed data.

Chi-square goodness of fit test for one-way table

Suppose we are to evaluate whether there is convincing evidence that a set of observed counts O_1, O_2, ..., O_k in k categories are unusually different from what might be expected under a null hypothesis. Call the *expected counts* that are based on the null hypothesis E_1, E_2, ..., E_k. If each expected count is at least 5 and the null hypothesis is true, then the test statistic below follows a chi-square distribution with $k-1$ degrees of freedom:

$$X^2 = \frac{(O_1 - E_1)^2}{E_1} + \frac{(O_2 - E_2)^2}{E_2} + \cdots + \frac{(O_k - E_k)^2}{E_k}$$

The p-value for this test statistic is found by looking at the upper tail of this chi-square distribution. We consider the upper tail because larger values of X^2 would provide greater evidence against the null hypothesis.

TIP: Conditions for the chi-square goodness of fit test

There are two conditions that must be checked before performing a chi-square goodness of fit test. If these conditions are not met, this test should not be used.

Simple random sample. The data must be arrived at by taking a simple random sample from the population of interest. The observed counts can then be organized into a list or one-way table.

All Expected Counts at least 5 Each particular scenario (i.e. cell count) must have at least 5 expected cases.

6.3.5 Evaluating goodness of fit for a distribution

Goodness of fit test for a one-way table

1. State the name of the test being used: X^2 goodness of fit test.
2. Verify conditions.
   - a random sample
   - all expected counts ≥ 5 (calculate and record expected counts)
3. Write the hypotheses in plain language. No mathematical notation is needed for this test.
   - H_0: The distribution of [...] matches [the expected distribution].
   - H_A: The distribution of [....] does not match [the expected distribution]
4. Identify the significance level α.
5. Calculate the test statistic and degrees of freedom.

$$X^2 = \sum \frac{(\text{observed counts - expected counts})^2}{\text{expected counts}}$$

$$df = (\#\text{ of categories} - 1)$$

6. Find the p-value and compare it to α to determine whether to reject or not reject H_0.
7. Write the conclusion in the context of the question.

Section 4.3 would be useful background reading for this example, but it is not a prerequisite.

We can apply our new chi-square testing framework to the second problem in this section: evaluating whether a certain statistical model fits a data set. Daily stock returns from the S&P500 for 1990-2011 can be used to assess whether stock activity each day is independent of the stock's behavior on previous days. This sounds like a very complex question, and it is, but a chi-square test can be used to study the problem. We will label each day as `Up` or `Down` (`D`) depending on whether the market was up or down that day. For example, consider the following changes in price, their new labels of up and down, and then the number of days that must be observed before each `Up` day:

Change in price	2.52	-1.46	0.51	-4.07	3.36	1.10	-5.46	-1.03	-2.99	1.71
Outcome	Up	D	Up	D	Up	Up	D	D	D	Up
Days to Up	1	-	2	-	2	1	-	-	-	4

If the days really are independent, then the number of days until a positive trading day should follow a geometric distribution. The geometric distribution describes the probability of waiting for the k^{th} trial to observe the first success. Here each up day (Up) represents a success, and down (D) days represent failures. In the data above, it took only one day until the market was up, so the first wait time was 1 day. It took two more days before we observed our next `Up` trading day, and two more for the third `Up` day. We would like to determine if these counts (1, 2, 2, 1, 4, and so on) follow the geometric distribution.

Table 6.11 shows the number of waiting days for a positive trading day during 1990-2011 for the S&P500.

Days	1	2	3	4	5	6	7+	Total
Observed	1532	760	338	194	74	33	17	2948

Table 6.11: Observed distribution of the waiting time until a positive trading day for the S&P500, 1990-2011.

We consider how many days one must wait until observing an `Up` day on the S&P500 stock exchange. If the stock activity was independent from one day to the next and the probability of a positive trading day was constant, then we would expect this waiting time to follow a *geometric distribution*. We can organize this into a hypothesis framework:

H_0: The stock market being up or down on a given day is independent from all other days. We will consider the number of days that pass until an `Up` day is observed. Under this hypothesis, the number of days until an `Up` day should follow a geometric distribution.

H_A: The stock market being up or down on a given day is not independent from all other days. Since we know the number of days until an `Up` day would follow a geometric distribution under the null, we look for deviations from the geometric distribution, which would support the alternative hypothesis.

There are important implications in our result for stock traders: if information from past trading days is useful in telling what will happen today, that information may provide an advantage over other traders.

We consider data for the S&P500 from 1990 to 2011 and summarize the waiting times in Table 6.12 and Figure 6.13. The S&P500 was positive on 53.2% of those days.

Because applying the chi-square framework requires expected counts to be at least 5, we have *binned* together all the cases where the waiting time was at least 7 days to ensure each expected count is well above this minimum. The actual data, shown in the *Observed* row in Table 6.12, can be compared to the expected counts from the *Geometric Model* row. The method for computing expected counts is discussed in Table 6.12. In general, the expected counts are determined by (1) identifying the null proportion associated with each bin, then (2) multiplying each null proportion by the total count to obtain the expected counts. That is, this strategy identifies what proportion of the total count we would expect to be in each bin.

Days	1	2	3	4	5	6	7+	Total
Observed	1532	760	338	194	74	33	17	2948
Geometric Model	1569	734	343	161	75	35	31	2948

Table 6.12: Distribution of the waiting time until a positive trading day. The expected counts based on the geometric model are shown in the last row. To find each expected count, we identify the probability of waiting D days based on the geometric model ($P(D) = (1 - 0.532)^{D-1}(0.532)$) and multiply by the total number of streaks, 2948. For example, waiting for three days occurs under the geometric model about $0.468^2 \times 0.532 = 11.65\%$ of the time, which corresponds to $0.1165 \times 2948 = 343$ streaks.

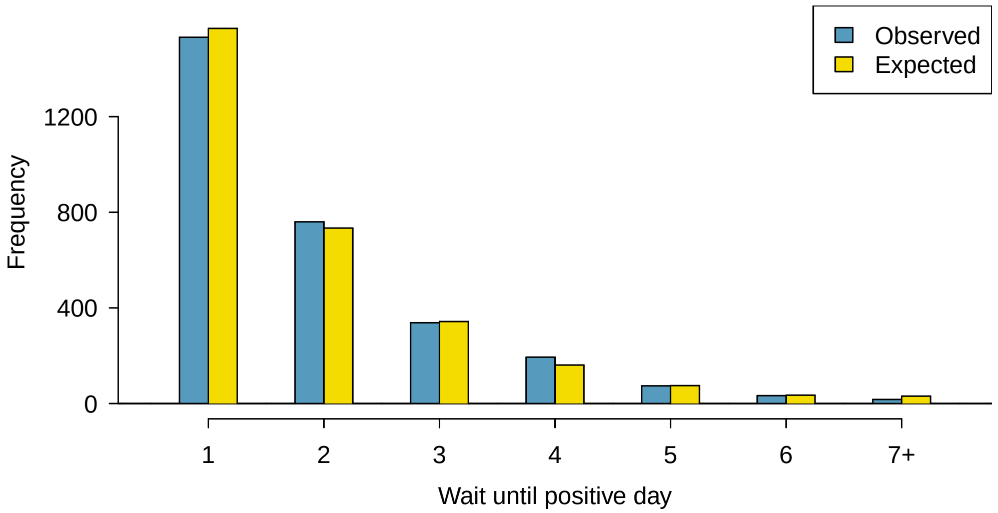


Figure 6.13: Side-by-side bar plot of the observed and expected counts for each waiting time.

● **Example 6.31** Do you notice any unusually large deviations in the graph? Can you tell if these deviations are due to chance just by looking?

It is not obvious whether differences in the observed counts and the expected counts from the geometric distribution are significantly different. That is, it is not clear whether these deviations might be due to chance or whether they are so strong that the data provide convincing evidence against the null hypothesis. However, we can perform a chi-square test using the counts in Table 6.12.

⊙ **Guided Practice 6.32** Table 6.12 provides a set of count data for waiting times ($O_1 = 1532$, $O_2 = 760$, ...) and expected counts under the geometric distribution ($E_1 = 1569$, $E_2 = 734$, ...). Compute the chi-square test statistic, X^2.[17]

⊙ **Guided Practice 6.33** Because the expected counts are all at least 5, we can safely apply the chi-square distribution to X^2. However, how many degrees of freedom should we use?[18]

● **Example 6.34** If the observed counts follow the geometric model, then the chi-square test statistic $X^2 = 15.08$ would closely follow a chi-square distribution with $df = 6$. Using this information, compute a p-value.

Figure 6.14 shows the chi-square distribution, cutoff, and the shaded p-value. If we look up the statistic $X^2 = 15.08$ in Appendix B.3, we find that the p-value is between 0.01 and 0.02. In other words, we have sufficient evidence to reject the notion that the wait times follow a geometric distribution, i.e. trading days are not independent and past days may help predict what the stock market will do today.

[17] $X^2 = \frac{(1532-1569)^2}{1569} + \frac{(760-734)^2}{734} + \cdots + \frac{(17-31)^2}{31} = 15.08$

[18] There are $k = 7$ groups, so we use $df = k - 1 = 6$.

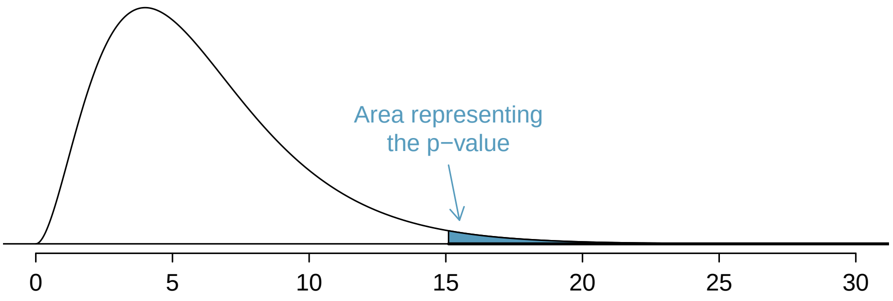


Figure 6.14: Chi-square distribution with 6 degrees of freedom. The p-value for the stock analysis is shaded.

● **Example 6.35** In Example 6.34, we rejected the null hypothesis that the trading days are independent. Why is this so important?

Because the data provided strong evidence that the geometric distribution is not appropriate, we reject the claim that trading days are independent. While it is not obvious how to exploit this information, it suggests there are some hidden patterns in the data that could be interesting and possibly useful to a stock trader.

6.3.6 Calculator: chi-square goodness of fit test

TI calculator: Carrying out the chi-square goodness of fit test

Use **STAT, TESTS, X^2GOF-Test**.

1. Enter the observed counts into list L1 and the expected counts into list L2.
2. Choose STAT.
3. Right arrow to TESTS.
4. Down arrow and choose D: X^2GOF-Test.
5. Leave Observed: L1 and Expected: L2.
6. Enter the degrees of freedom after df:
7. Choose Calculate and hit ENTER, which returns:

X^2	chi-square value
p	p-value
df	degrees of freedom

TI-83: Unfortunately the TI-83 does not have this test built in. To carry out the test manually, make list L3 = (L1 - L2)^2 / L2 and do 1-Var-Stats on L3. The sum of L3 will correspond to the value of X^2 for this test.

⊙ **Guided Practice 6.36** Use the data above and a calculator to find the X^2 statistic, df, and p-value for chi-square goodness of fit test.[19]

[19] First enter the observed values into L1 and the expected values into L2. Use STAT, TESTS, X^2GOF-Test. $X^2 = 15.08$, $df = 6$, p-value= 0.0196.

Days	1	2	3	4	5	6	7+	Total
Observed	1532	760	338	194	74	33	17	2948
Geometric Model	1569	734	343	161	75	35	31	2948

Table 6.15: Distribution of the waiting time until a positive trading day. The expected counts based on the geometric model are shown in the last row.

6.4 Homogeneity and independence in two-way tables

Google is constantly running experiments to test new search algorithms. For example, Google might test three algorithms using a sample of 10,000 google.com search queries. Table 6.16 shows an example of 10,000 queries split into three algorithm groups.[20] The group sizes were specified before the start of the experiment to be 5000 for the current algorithm and 2500 for each test algorithm.

Search algorithm	current	test 1	test 2	Total
Counts	5000	2500	2500	10000

Table 6.16: Google experiment breakdown of test subjects into three search groups.

● **Example 6.37** What is the ultimate goal of the Google experiment? What are the null and alternative hypotheses, in regular words?

The ultimate goal is to see whether there is a difference in the performance of the algorithms. The hypotheses can be described as the following:

H_0: The algorithms each perform equally well.

H_A: The algorithms do not perform equally well.

In this experiment, the explanatory variable is the search algorithm. However, an outcome variable is also needed. This outcome variable should somehow reflect whether the search results align with the user's interests. One possible way to quantify this is to determine whether (1) there was no new, related search, and the user clicked one of the links provided, or (2) there was a new, related search performed by the user. Under scenario (1), we might think that the user was satisfied with the search results. Under scenario (2), the search results probably were not relevant, so the user tried a second search.

Table 6.17 provides the results from the experiment. These data are very similar to the count data in Section 6.3. However, now the different combinations of two variables are binned in a *two-way* table. In examining these data, we want to evaluate whether there is strong evidence that at least one algorithm is performing better than the others. To do so, we apply a chi-square test to this two-way table. The ideas of this test are similar to those ideas in the one-way table case. However, degrees of freedom and expected counts are computed a little differently than before.

[20] Google regularly runs experiments in this manner to help improve their search engine. It is entirely possible that if you perform a search and so does your friend, that you will have different search results. While the data presented in this section resemble what might be encountered in a real experiment, these data are simulated.

	Search algorithm			
	current	test 1	test 2	Total
No new search	3511	1749	1818	7078
New search	1489	751	682	2922
Total	5000	2500	2500	10000

Table 6.17: Results of the Google search algorithm experiment.

TIP: What is so different about one-way tables and two-way tables?
A one-way table describes counts for each outcome in a single variable. A two-way table describes counts for *combinations* of outcomes for two variables. When we consider a two-way table, we often would like to know, are these variables related in any way?

The hypothesis test for this Google experiment is really about assessing whether there is statistically significant evidence that the choice of the algorithm affects whether a user performs a second search. In other words, the goal is to check whether the the three search algorithms perform differently.

6.4.1 Expected counts in two-way tables

● **Example 6.38** From the experiment, we estimate the proportion of users who were satisfied with their initial search (no new search) as $7078/10000 = 0.7078$. If there really is no difference among the algorithms and 70.78% of people are satisfied with the search results, how many of the 5000 people in the "current algorithm" group would be expected to not perform a new search?

About 70.78% of the 5000 would be satisfied with the initial search:

$$0.7078 \times 5000 = 3539 \text{ users}$$

That is, if there was no difference between the three groups, then we would expect 3539 of the current algorithm users not to perform a new search.

⊙ **Guided Practice 6.39** Using the same rationale described in Example 6.38, about how many users in each test group would not perform a new search if the algorithms were equally helpful?[21]

We can compute the expected number of users who would perform a new search for each group using the same strategy employed in Example 6.38 and Guided Practice 6.39. These expected counts were used to construct Table 6.18, which is the same as Table 6.17, except now the expected counts have been added in parentheses.

The examples and exercises above provided some help in computing expected counts. In general, expected counts for a two-way table may be computed using the row totals, column totals, and the table total. For instance, if there was no difference between the

[21] We would expect $0.7078 * 2500 = 1769.5$. It is okay that this is a fraction.

Search algorithm	current		test 1		test 2		Total
No new search	3511	(3539)	1749	(1769.5)	1818	(1769.5)	7078
New search	1489	(1461)	751	(730.5)	682	(730.5)	2922
Total	5000		2500		2500		10000

Table 6.18: The observed counts and the **(expected counts)**.

groups, then about 70.78% of each column should be in the first row:

$$0.7078 \times (\text{column 1 total}) = 3539$$
$$0.7078 \times (\text{column 2 total}) = 1769.5$$
$$0.7078 \times (\text{column 3 total}) = 1769.5$$

Looking back to how the fraction 0.7078 was computed – as the fraction of users who did not perform a new search (7078/10000) – these three expected counts could have been computed as

$$\left(\frac{\text{row 1 total}}{\text{table total}}\right)(\text{column 1 total}) = 3539$$
$$\left(\frac{\text{row 1 total}}{\text{table total}}\right)(\text{column 2 total}) = 1769.5$$
$$\left(\frac{\text{row 1 total}}{\text{table total}}\right)(\text{column 3 total}) = 1769.5$$

This leads us to a general formula for computing expected counts in a two-way table when we would like to test whether there is strong evidence of an association between the column variable and row variable.

Computing expected counts in a two-way table

To identify the expected count for the i^{th} row and j^{th} column, compute

$$\text{Expected Count}_{\text{row } i,\text{ col } j} = \frac{(\text{row } i \text{ total}) \times (\text{column } j \text{ total})}{\text{table total}}$$

6.4.2 The chi-square test of homogeneity for two-way tables

The chi-square test statistic for a two-way table is found the same way it is found for a one-way table. For each table count, compute

$$\begin{array}{ll}
\text{General formula} & \dfrac{(\text{observed count} \ - \ \text{expected count})^2}{\text{expected count}} \\
\text{Row 1, Col 1} & \dfrac{(3511 - 3539)^2}{3539} = 0.222 \\
\text{Row 1, Col 2} & \dfrac{(1749 - 1769.5)^2}{1769.5} = 0.237 \\
\vdots & \vdots \\
\text{Row 2, Col 3} & \dfrac{(682 - 730.5)^2}{730.5} = 3.220
\end{array}$$

Adding the computed value for each cell gives the chi-square test statistic X^2:

$$X^2 = 0.222 + 0.237 + \cdots + 3.220 = 6.120$$

Just like before, this test statistic follows a chi-square distribution. However, the degrees of freedom are computed a little differently for a two-way table.[22] For two way tables, the degrees of freedom is equal to

$$df = (\text{number of rows - 1}) \times (\text{number of columns - 1})$$

In our example, the degrees of freedom parameter is

$$df = (2 - 1) \times (3 - 1) = 2$$

If the null hypothesis is true (i.e. the algorithms are equally useful), then the test statistic $X^2 = 6.12$ closely follows a chi-square distribution with 2 degrees of freedom. Using this information, we can compute the p-value for the test, which is depicted in Figure 6.19.

Computing degrees of freedom for a two-way table

When applying the chi-square test to a two-way table, we use

$$df = (R - 1) \times (C - 1)$$

where R is the number of rows in the table and C is the number of columns.

TIP: Use two-proportion methods for 2-by-2 contingency tables

When analyzing 2-by-2 contingency tables, use the two-proportion methods introduced in Section 6.2.

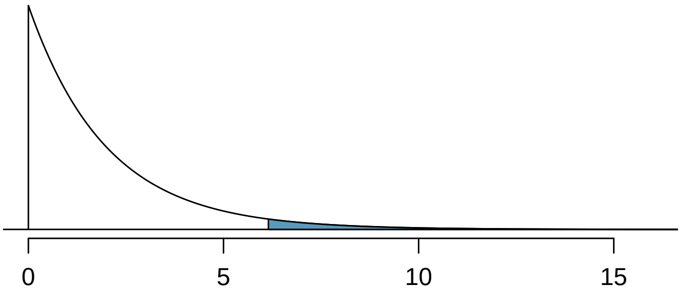


Figure 6.19: Computing the p-value for the Google hypothesis test.

[22] Recall: in the one-way table, the degrees of freedom was the number of cells minus 1.

		Congress		
	Obama	Democrats	Republicans	Total
Approve	842	736	541	2119
Disapprove	616	646	842	2104
Total	1458	1382	1383	4223

Table 6.20: Pew Research poll results of a March 2012 poll.

TIP: Conditions for the chi-square test of homeneity
There are two conditions that must be checked before performing a chi-square test of homogeneity. If these conditions are not met, this test should not be used.

Mutliple random samples or randomly allocated treatments. Data collected by multiple independent random samples or multiple randomlly allocated treatments. Data can then be organized into a two-way table.

All Expected Counts at least 5. All of the expected counts must be at least 5.

Example 6.40 Compute the p-value and draw a conclusion about whether the search algorithms have different performances.

Looking in Appendix B.3 on page 392, we examine the row corresponding to 2 degrees of freedom. The test statistic, $X^2 = 6.120$, falls between the fourth and fifth columns, which means the p-value is between 0.02 and 0.05. Because we typically test at a significance level of $\alpha = 0.05$ and the p-value is less than 0.05, the null hypothesis is rejected. That is, the data provide convincing evidence that there is some difference in performance among the algorithms.

6.4.3 The chi-square test of independence for two-way tables

The chi-square test of Independence proceeds exactly like the chi-square test of homogeneity, except that it applies when there is only one random sample (versus multiple random samples or an experiment with multiple randomly allocated treatments). The null claim is always that two variables are independent, while the alternate claim is that the variables are dependent.

Example 6.41 Table 6.20 summarizes the results of a Pew Research poll.[23] We would like to determine if three groups and approval ratings are associated. What are appropriate hypotheses for such a test?

H_0: The ratings are independent of the group. (There is no difference in approval ratings between the three groups.)

H_A: The ratings are dependent on the group. (There is some difference in approval ratings between the three groups, e.g. perhaps Obama's approval differs from Democrats in Congress.)

[23] See the Pew Research website: www.people-press.org/2012/03/14/romney-leads-gop-contest-trails-in-matchup-with-obama. The counts in Table 6.20 are approximate.

Guided Practice 6.42 A chi-square test for a two-way table may be used to test the hypotheses in Example 6.41. As a first step, compute the expected values for each of the six table cells.[24]

Guided Practice 6.43 Compute the chi-square test statistic.[25]

Guided Practice 6.44 Because there are 2 rows and 3 columns, the degrees of freedom for the test is $df = (2 - 1) \times (3 - 1) = 2$. Use $X^2 = 106.4$, $df = 2$, and the chi-square table on page 392 to evaluate whether to reject the null hypothesis.[26]

TIP: Conditions for the chi-square test of independence

There are two conditions that must be checked before performing a chi-square test of independence. If these conditions are not met, this test should not be used.

One simple random sample with two variables/questions. The data must be arrived at by taking a simple random sample. After the data is collected, it is separated and categorized according to two variables and can be organized into a two-way table.

All Expected Counts at least 5 All of the expected counts must be at least 5.

[24] The expected count for row one / column one is found by multiplying the row one total (2119) and column one total (1458), then dividing by the table total (4223): $\frac{2119 \times 1458}{3902} = 731.6$. Similarly for the first column and the second row: $\frac{2104 \times 1458}{4223} = 726.4$. Column 2: 693.5 and 688.5. Column 3: 694.0 and 689.0

[25] For each cell, compute $\frac{(\text{obs} - \text{exp})^2}{exp}$. For instance, the first row and first column: $\frac{(842 - 731.6)^2}{731.6} = 16.7$. Adding the results of each cell gives the chi-square test statistic: $X^2 = 16.7 + \cdots + 34.0 = 106.4$.

[26] The test statistic is larger than the right-most column of the $df = 2$ row of the chi-square table, meaning the p-value is less than 0.001. That is, we reject the null hypothesis because the p-value is less than 0.05, and we conclude that Americans' approval has differences among Democrats in Congress, Republicans in Congress, and the president.

6.4.4 Summarizing the chi-square tests for two-way tables

X^2 test of homogeneity

1. State the name of the test being used: X^2 test of homogeneity.
2. Verify conditions: multiple random samples or treatments and all expected counts ≥ 5 (calculate and recorded expected counts).
3. Write the hypotheses in plain language. No mathematical notation is needed for this test.
   - H$_0$: distribution of [variable 1] matches the distribution of [variable 2].
   - H$_A$: distribution of [variable 1] does not match the distribution of [variable 2].
4. Identify the significance level α.
5. Calculate the test statistic and degrees of freedom.

$$X^2 = \sum \frac{(\text{observed counts - expected counts})^2}{\text{expected counts}}$$
$$df = (\#\text{ of categories} - 1)$$

6. Find the p-value and compare it to α to determine whether to reject or not reject H_0.
7. Write the conclusion in the context of the question.

X^2 test of independence

1. State the name of the test being used: X^2 test of independence.
2. Verify conditions: a random sample and all expected counts ≥ 5 (calculate and record expected counts).
3. Write the hypotheses in plain language. No mathematical notation is needed for this test.
   - H$_0$: [variable 1] and [variable 2] are independent.
   - H$_A$: [variable 1] and [variable 2] are dependent.
4. Identify the significance level α.
5. Calculate the test statistic and degrees of freedom.

$$X^2 = \sum \frac{(\text{observed counts - expected counts})^2}{\text{expected counts}}$$
$$df = (\#\text{ of categories} - 1)$$

6. Find the p-value and compare it to α to determine whether to reject or not reject H_0.
7. Write the conclusion in the context of the question.

Example 6.45 A 2011 survey asked 806 randomly sampled adult Facebook users about their Facebook privacy settings. One of the questions on the survey was, "Do you know how to adjust your Facebook privacy settings to control what people can and cannot see?" The responses are cross-tabulated based on gender.[27]

		Gender		
		Male	Female	Total
Response	Yes	288	378	666
	No	61	62	123
	Not sure	10	7	17
	Total	359	447	806

Carry out an appropriate test at the 0.10 significance level to see if there is an association between gender and knowing how to adjust Facebook privacy settings to control what people can and cannot see.

According to the problem, there was one random sample taken. Two variables were recorded on the respondents: gender and response to the question regarding privacy settings. Because there was one random sample rather than two independent random samples, we carry out a X^2 test of independence.
H_0: Gender and knowing how to adjust Facebook privacy settings are independent.
H_A: Gender and knowing how to adjust Facebook privacy settings are dependent.
$\alpha = 0.1$

Table of expected counts:

296.64	369.36
54.785	68.215
7.572	9.428

All expected counts are ≥ 5. $X^2 = 3.13$; $df = 2$ p-value$= 0.209 > \alpha$ We do not reject H_0. We do not have evidence that gender and knowing how to adjust Facebook privacy settings are dependent.

Appendix A

End of chapter exercise solutions

6.21 The margin of error, which is computed as $z^\star SE$, must be smaller than 0.01 for a 90% confidence level. We use $z^\star = 1.65$ for a 90% confidence level, and we can use the point estimate $\hat{p} = 0.52$ in the formula for SE. $1.65\sqrt{0.52(1-0.52)/n} \leq 0.01$. Therefore, the sample size n must be at least 6,796.

6.23 This is not a randomized experiment, and it is unclear whether people would be affected by the behavior of their peers. That is, independence may not hold. Additionally, there are only 5 interventions under the provocative scenario, so the success-failure condition does not hold. Even if we consider a hypothesis test where we pool the proportions, the success-failure condition will not be satisfied. Since one condition is questionable and the other is not satisfied, the difference in sample proportions will not follow a nearly normal distribution.

6.25 (a) False. The entire confidence interval is above 0. (b) True. (c) True. (d) True. (e) False. It is simply the negated and reordered values: (-0.06,-0.02).

6.27 (a) (0.23, 0.33). We are 95% confident that the proportion of Democrats who support the plan is 23% to 33% higher than the proportion of Independents who do. (b) True.

6.29 (a) College grads: 23.7%. Non-college grads: 33.7%. (b) Let p_{CG} and p_{NCG} represent the proportion of college graduates and non-college graduates who responded "do not know". $H_0 : p_{CG} = p_{NCG}$. $H_A : p_{CG} \neq p_{NCG}$. Independence is satisfied (random sample, $<$ 10% of the population), and the success-failure condition, which we would check using the pooled proportion ($\hat{p} = 235/827 = 0.284$), is also satisfied. $Z = -3.18 \to$ p-value $= 0.0014$. Since the p-value is very small, we reject H_0. The data provide strong evidence that the proportion of college graduates who do not have an opinion on this issue is different than that of non-college graduates. The data also indicate that fewer college grads say they "do not know" than non-college grads (i.e. the data indicate the direction after we reject H_0).

6.31 (a) College grads: 35.2%. Non-college grads: 33.9%. (b) Let p_{CG} and p_{NCG} represent the proportion of college graduates and non-college grads who support offshore drilling. $H_0 : p_{CG} = p_{NCG}$. $H_A : p_{CG} \neq p_{NCG}$. Independence is satisfied (random sample, $<$ 10% of the population), and the success-failure condition, which we would check using the pooled proportion ($\hat{p} = 286/827 = 0.346$), is also satisfied. $Z = 0.39 \to$ p-value $= 0.6966$. Since the p-value $> \alpha$ (0.05), we fail to reject H_0. The data do not provide strong evidence of a difference between the proportions of college graduates and non-college graduates who support offshore drilling in California.

6.33 Subscript $_C$ means control group. Subscript $_T$ means truck drivers. (a) $H_0 : p_C = p_T$. $H_A : p_C \neq p_T$. Independence is satisfied (random samples, < 10% of the population), as is the success-failure condition, which we would check using the pooled proportion ($\hat{p} = 70/495 = 0.141$). $Z = -1.58 \rightarrow$ p-value $= 0.1164$. Since the p-value is high, we fail to reject H_0. The data do not provide strong evidence that the rates of sleep deprivation are different for non-transportation workers and truck drivers.

6.35 (a) Summary of the study:

		Virol. failure		
		Yes	No	Total
Treatment	Nevaripine	26	94	120
	Lopinavir	10	110	120
	Total	36	204	240

(b) $H_0 : p_N = p_L$. There is no difference in virologic failure rates between the Nevaripine and Lopinavir groups. $H_A : p_N \neq p_L$. There is some difference in virologic failure rates between the Nevaripine and Lopinavir groups. (c) Random assignment was used, so the observations in each group are independent. If the patients in the study are representative of those in the general population (something impossible to check with the given information), then we can also confidently generalize the findings to the population. The success-failure condition, which we would check using the pooled proportion ($\hat{p} = 36/240 = 0.15$), is satisfied. $Z = 3.04 \rightarrow$ p-value $= 0.0024$. Since the p-value is low, we reject H_0. There is strong evidence of a difference in virologic failure rates between the Nevaripine and Lopinavir groups do not appear to be independent.

6.37 (a) False. The chi-square distribution has one parameter called degrees of freedom. (b) True. (c) True. (d) False. As the degrees of freedom increases, the shape of the chi-square distribution becomes more symmetric.

6.39 (a) H_0: The distribution of the format of the book used by the students follows the professor's predictions. H_A: The distribution of the format of the book used by the students does not follow the professor's predictions. (b) $E_{hard\ copy} = 126 \times 0.60 = 75.6$. $E_{print} = 126 \times 0.25 = 31.5$. $E_{online} = 126 \times 0.15 = 18.9$. (c) Independence: The sample is not random. However, if the professor has reason to believe that the proportions are stable from one term to the next and students are not affecting each other's study habits, independence is probably reasonable. Sample size: All expected counts are at least 5. Degrees of freedom: $df = k - 1 = 3 - 1 = 2$ is more than 1. (d) $X^2 = 2.32$, $df = 2$, p-value > 0.3. (e) Since the p-value is large, we fail to reject H_0. The data do not provide strong evidence indicating the professor's predictions were statistically inaccurate.

6.41 (a). Two-way table:

	Quit		
Treatment	Yes	No	Total
Patch + support group	40	110	150
Only patch	30	120	150
Total	70	230	300

(b-i) $E_{row_1,col_1} = \frac{(row\ 1\ total)\times(col\ 1\ total)}{table\ total} = \frac{150\times70}{300} = 35$. This is lower than the observed value. (b-ii) $E_{row_2,col_2} = \frac{(row\ 2\ total)\times(col\ 2\ total)}{table\ total} = \frac{150\times230}{300} = 115$. This is lower than the observed value.

6.43 H_0: The opinion of college grads and non-grads is not different on the topic of drilling for oil and natural gas off the coast of California. H_A: Opinions regarding the drilling for oil and natural gas off the coast of California has an association with earning a college degree.

$$\begin{array}{ll} E_{row\ 1,col\ 1} = 151.5 & E_{row\ 1,col\ 2} = 134.5 \\ E_{row\ 2,col\ 1} = 162.1 & E_{row\ 2,col\ 2} = 143.9 \\ E_{row\ 3,col\ 1} = 124.5 & E_{row\ 3,col\ 2} = 110.5 \end{array}$$

Independence: The samples are both random, unrelated, and from less than 10% of the population, so independence between observations is reasonable. Sample size: All expected counts are at least 5. Degrees of freedom: $df = (R - 1) \times (C - 1) = (3 - 1) \times (2 - 1) = 2$, which is greater than 1. $X^2 = 11.47$, $df = 2$ $\rightarrow 0.001 <$ p-value < 0.005. Since the p-value $< \alpha$, we reject H_0. There is strong evidence that there is an association between support for off-shore drilling and having a college degree.

6.45 (a) H_0: The age of Los Angeles residents is independent of shipping carrier preference variable. H_A: The age of Los Angeles residents is associated with the shipping carrier preference variable. (b) The conditions are not satisfied since some expected counts are below 5.