组合数学 Combinatorics

6 Inclusion-Exclusion theorem and pigeonhole principle

6-1 inclusion and exclusion

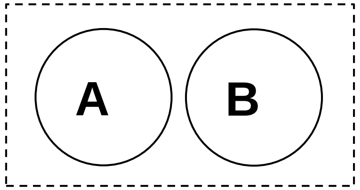
Yuchun Ma
Tsinghua University

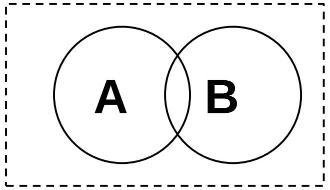
Speak of the Addition rule

[Addition Rule]Assume that event A can happen in m ways, event B can happen in n ways. Event A **or** B can happen in $m+n$ ways.

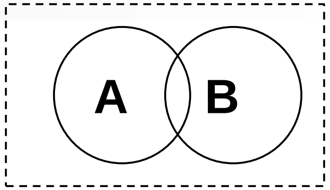
Language of set theory

If $|A| = m$, $|B| = n$, $A \cap B = \varnothing$, then $|A \cup B| = m + n$ 。





Inclusion-Exclusion Principle



- **Inclusion-Exclusion Principle**
  - Inclusion and Exclusion Principle
  - The idea of inclusion-exclusion is:
    - Ignore duplications at first and include all objects which are in it.
    - Then exclude the duplications.
    - Finally the result has no repetitions and no losses.

Inclusion-Exclusion principle

Eg The number of multiples of 2 or 3 in [1,20]

[Solution] Multiples of 2:

2， 4， 6， 8， 10， 12， 14， 16， 18， 20。

10 numbers

Multiples of 3:

3， 6， 9， 12， 15， 18。

6 numbers

The answer is not 10+6=16, as 6,12,18 appears in both classes. So we need to subtract them. The answer is: 16-3=13

Introduction to Inclusion-Exclusion Principle

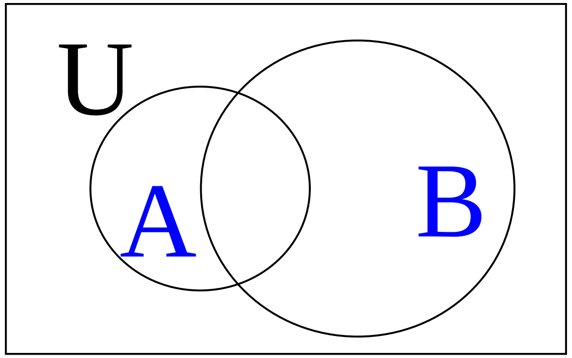
- **If both A and B are subsets of U, the *complement***

$$\overline{A} = \{x \mid x \in U \text{ 且 } x \notin A\}$$

- **[De Morgan's Law]**

(a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

(b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$



Introduction to Inclusion-Exclusion Principle

A generalization of De Mogan's law: Suppose $A_1, A_2, \ldots. A_n$ are subsets of U

$$So\ (a)\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cup \overline{A_n}$$

$$(b)\overline{A_1 \cap A_2 \cap \ldots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_n}$$

Proof: Use mathematical induction

$$\overline{A_1 \cup A_2 \cup \ldots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}$$ is correct

则

$$\overline{A_1 \cup A_2 \cup \ldots \cup A_n \cup A_{n+1}} = \overline{(A_1 \cup \ldots \cup A_n) \cup A_{n+1}}$$

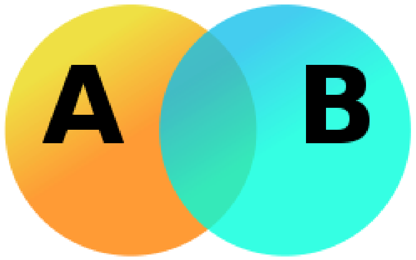
$$= \overline{(A_1 \cup A_2 \cup \ldots \cup A_n} \cap \overline{A_{n+1}}$$

$$= \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n} \cap \overline{A_{n+1}}$$

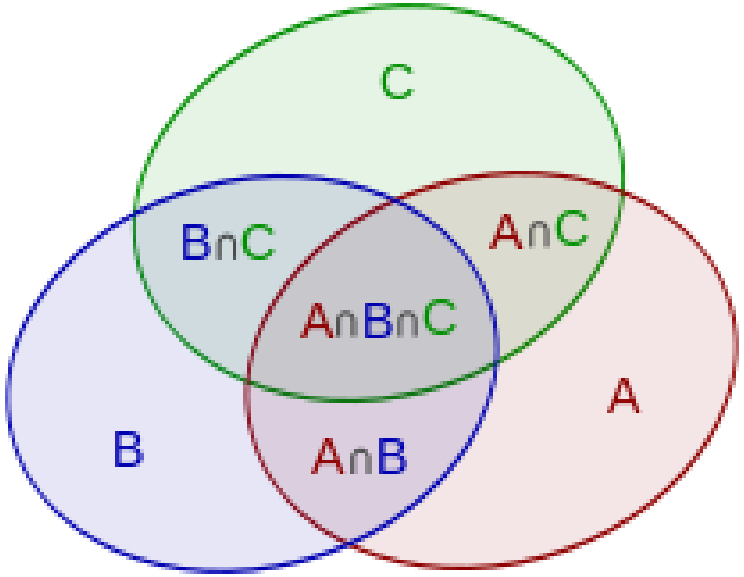
So the theorem is also correct for n+1

Inclusion-Exclusion principle

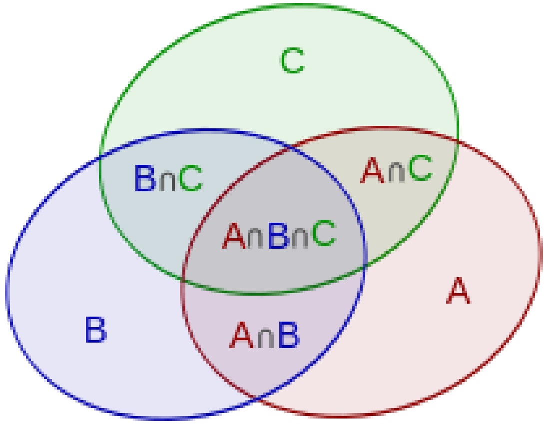
Calculate the number of elements in the union of finite set A and B. $|A \cup B| = |A| + |B| - |A \cap B|$



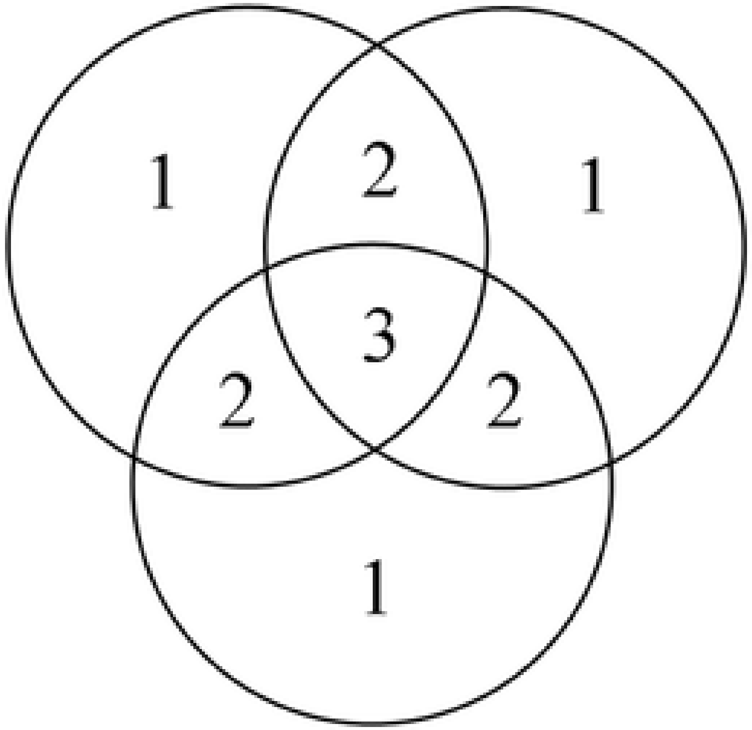
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



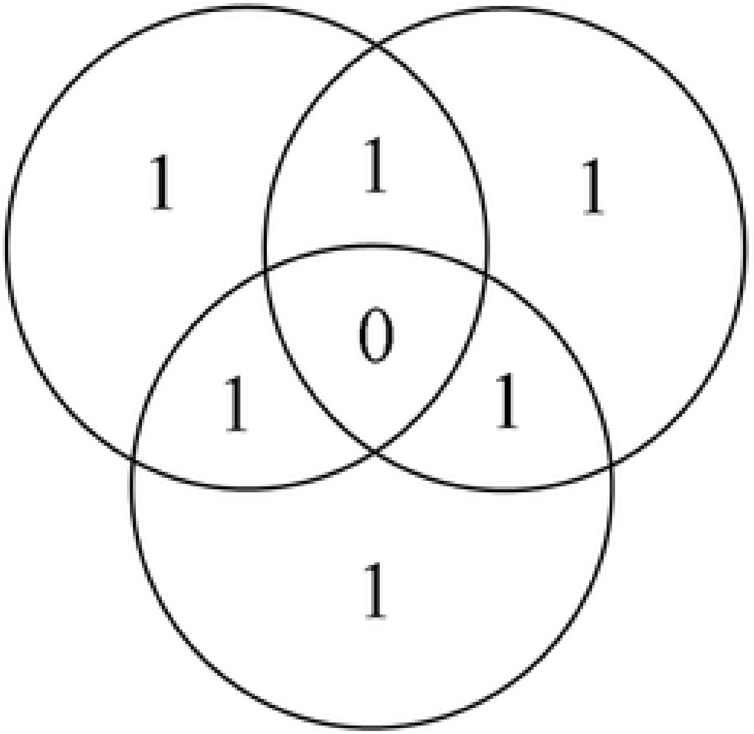
Inclusion-Exclusion Principl



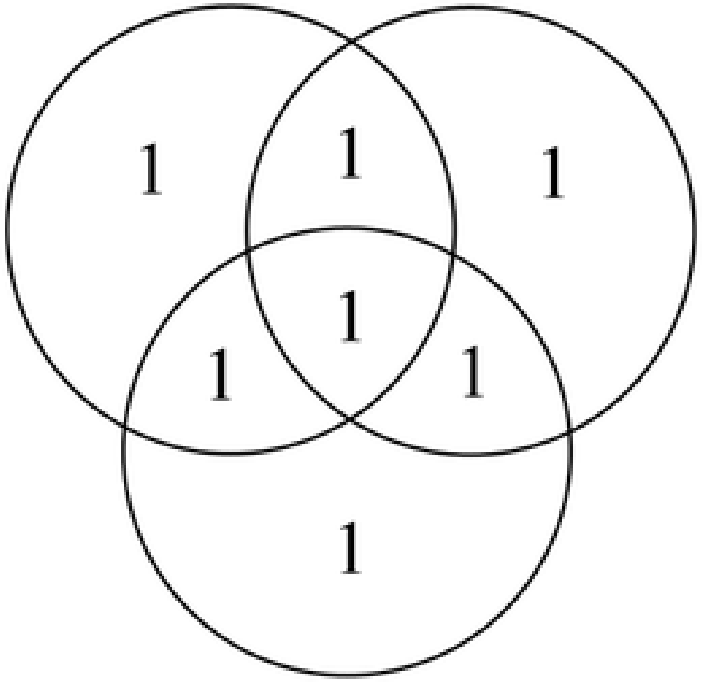
Theorem: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B|$
$- |\mathrm{A} \cap \mathrm{C}| - |B \cap C| + |A \cap B \cap C|$



$|A| + |B| + |C|$



$|A| + |B| + |C|$
$-(|A \cap B| + |A \cap C| + |B \cap C|)$



$|A| + |B| + |C|$
$-(|A \cap B| + |A \cap C| + |B \cap C|)$
$+|A \cap B \cap C|$

Inclusion-Exclusion Principle

For 4 sets:

$$
\begin{aligned}
|A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| - |A \cap B| \\
&- |A \cap C| - |B \cap C| - |A \cap D| + |A \cap B \cap C| \\
&+ |A \cap B \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|
\end{aligned}
$$

Inclusion-Exclusion principle

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i=1}^{n} \sum_{j>i} |A_i \cap A_j|$$

$$+ \sum_{i=1}^{n} \sum_{j>i} \sum_{k>j} |A_i \cap A_j \cap A_k| - \ldots$$

$$+ (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

$$|\overline{A}| = N - |A|,$$

$$|\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}| = N - |A_1 \cup A_2 \cup \ldots \cup A_{n-1} \cup A_n|$$

$$= N - \sum_{i=1}^{n} |A_i| + \sum_{i=1}^{n} \sum_{j>i} |A_i \cap A_j| - \sum_{i=1}^{n} \sum_{j>i} \sum_{k>j} |A_i \cap A_j \cap A_k| + \ldots$$

$$+ (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Inclusion-Exclusion Principle

$$|\overline{A_1} \cap \overline{A_2}| = |S| - |A_1| - |A_2| + |A_1 \cap A_2|$$

Calculate the number of elements that are in neither A_1 nor A_2.

If x is in neither A_1 nor A_2

1 1-0-0+0 = 1

If x is in A1 but not A2

0 1-1-0+0 = 0

If x is in A2 but not A1

0 1-0-1+0 = 0

If x is in A2 and A1

0 1-1-1+1 = 0

Two sides are equal

$(x+y)^m = C(m,0)x^m + C(m,1)x^{m-1}y + \ldots + C(m,m)y^m$

If $x=1$, $y=-1$: $0 = C(m,0) - C(m,1) + \ldots + (-1)^m C(m,m)$

$$\left|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_m}\right| = |S| - \sum|A_i| + \sum|A_i\cap A_j| - \sum|A_i\cap A_j\cap A_k| + \cdots + (-1)^m\sum|A_1\cap A_2\cap\cdots\cap A_m|$$

Calculate the number of elements that no property holds.

X doesn't satisfy any property

1 $\quad$ **1-0-0…+$(-1)^m$0 = 1**

X satisfies 1 property

0 $\quad$ **1-1-0 …+$(-1)^m$0 = 0**

………

X satisfies n properties

$n \le m$

0 $\quad$ **$C(n,0)-C(n,1)+C(n,2)+\ldots+(-1)^m C(n,m)$**

$= C(n,0)-C(n,1)+C(n,2)+\ldots+(-1)^n C(n,n) +0\ldots +0$

$=0$

Two sides are equal, calculate the number of elements which satisfies no property in the same way.

INCLUSION EXCLUSION

- **Inclusion–exclusion principle**
  - This concept is attributed to Abraham de Moivre (1718)
  - It first appears in a paper of Daniel da Silva (1854)
  - Later in a paper by J. J. Sylvester (1883)

INCLUSION EXCLUSION

"One of the most useful principles of enumeration in discrete probability and combinatorial theory is the celebrated principle of inclusion–exclusion.
When skillfully applied, this principle has yielded the solution to many a combinatorial problem."

组合数学 Combinatorics

6 Inclusion-Exclusion Principle and the Pigeonhole theorem

6-2 the elegancy of inclusion-exclusion principle

Yuchun Ma
Tsinghua University

Examples

Eg Calculate the number of multiples of 3 or 5 from 1 to 500.

Solution： Let A be the set of multiples of 3 from 1 to 500.

B is the set of multiples of 5 from 1 to 500.

$$|A| = \left\lfloor \frac{500}{3} \right\rfloor = 166, |B| = \left\lfloor \frac{500}{5} \right\rfloor = 100;$$

$$|A \cap B| = \left\lfloor \frac{500}{15} \right\rfloor = 33$$

The number of multiples of 3 or 5 is:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 166 + 100 - 33 = 233$$

Examples

Eg Calculate the number of permutations of *a*,*b*,*c*,*d*,*e*,*f*,*g* which don't contain *ace* and *df*.

Solution：Permutations of 6 letters: |S| = 6!

Assume A is the permutation set in which ace is an element: |A|=4!，

B is the permutation set in which df is an element:|B|=5!，

A∩B is the number of permutations contain ace and df: |A∩B |=3!。

$$|\overline{A}\cap\overline{B}|=|\overline{A\cup B}|=S-|A|-|B|+|A\cap B|$$

$$=6!-4!-5!+3!=582$$

Example

Eg Calculate the number of n-strings with alphabet {*a*,*b*,*c*,*d*} such that a,b,c show up at least once.

Solution： Let A、B、C be sets in which a, b or c are not in n-strings separately. As every digit in n-strings could be any of a, b, c, d, so the number of n-strings which contains no a is 3^n, which is：

$$|A| = |B| = |C| = 3^n \quad |A \cap B \cap C| = 1$$

$$|A \cap B| = |A \cap C| = |C \cap B| = 2^n$$

the set of n-strings in which a, b, c appears at least once is:

$$|\overline{A} \cap \overline{B} \cap \overline{C}| = 4^n - (|A| + |B| + |C|) + (|A \cap B|$$

$$+ |A \cap C| + |C \cap B|) - |A \cap B \cap C|$$

$$= 4^n - 3 \bullet 3^n + 3 \bullet 2^n - 1$$

Examples

Eg Calculate the number of primes which are ≤ 120.

As $11^2=121$, non-primes ≤ 120 must be multiples of 2, 3, 5 or 7, and their factors couldn't be larger than 11.

Suppose A_i is the set of i-multiples ≤ 120, i=2，3，5，7。

$$|A_2| = \left\lfloor \frac{120}{2} \right\rfloor = 60, |A_3| = \left\lfloor \frac{120}{3} \right\rfloor = 40,$$

$$|A_5| = \left\lfloor \frac{120}{5} \right\rfloor = 24, |A_7| = \left\lfloor \frac{120}{7} \right\rfloor = 17,$$

$$|A_2 \cap A_3| = \left\lfloor \frac{120}{2\times 3} \right\rfloor = 20, |A_2 \cap A_5| = \left\lfloor \frac{120}{10} \right\rfloor = 12,$$

$$|A_2 \cap A_7| = \left\lfloor \frac{120}{14} \right\rfloor = 8, |A_3 \cap A_5| = \left\lfloor \frac{120}{15} \right\rfloor = 8,$$

$$|A_3 \cap A_7| = \left\lfloor \frac{120}{21} \right\rfloor = 5, |A_5 \cap A_7| = \left\lfloor \frac{120}{35} \right\rfloor = 3,$$

$$|A_2 \cap A_3 \cap A_5| = \left\lfloor \frac{120}{2 \times 3 \times 5} \right\rfloor = 4,$$

$$|A_2 \cap A_3 \cap A_7| = \left\lfloor \frac{120}{2 \times 3 \times 7} \right\rfloor = 2,$$

$$|A_2 \cap A_5 \cap A_7| = \left\lfloor \frac{120}{2 \times 5 \times 7} \right\rfloor = 1,$$

$$\begin{aligned}
|\overline{A_2} \cap \overline{A_3} \cap \overline{A_5} \cap \overline{A_7}| &= 120 - |A_2| - |A_3| - |A_5| \\
&\quad - |A_7| + |A_2 \cap A_3| + |A_2 \cap A_5| + |A_2 \cap A_7| \\
&\quad + |A_3 \cap A_5| + |A_3 \cap A_7| + |A_5 \cap A_7| \\
&\quad - |A_2 \cap A_3 \cap A_5| - |A_2 \cap A_3 \cap A_7| \\
&\quad - |A_2 \cap A_5 \cap A_7| - |A_3 \cap A_5 \cap A_7| \\
&\quad + |A_2 \cap A_3 \cap A_5 \cap A_7| \\
&= 120 - (60 + 40 + 24 + 17) + (20 + 12 + 8 \\
&\quad + 8 + 5 + 3) - (4 + 2 + 1 + 1) \\
&= 27.
\end{aligned}$$

NOTE: as 2, 3, 5, and 7 are excluded in these 27 numbers while 1 is included, the number of primes ≤ 120 are:

27+4-1=30

Integers which are larger than 1 and has no other factors except itself and 1 is called primes

Eg Euler Function $\Phi(n)$ is the number of integers which are smaller than n and relatively prime to n.

$\Phi(8)=4$

$8 = 2^{3},\ \leq 8$ and relatively prime to 8: 1 3 5 7

The RSA key algorithm, which is widely used in the IT, is also based on Euler Function.

Euler: the incarnation of analysis

Euler calculated without striking a blowing, it seems to be simply breathing or an eagle hovering in the wind.

The most fertile mathematician in the history.

- Petersburg College cost 47 years to reorganize his publications.
- He could usually finish a paper during the 30 minutes between 2 diner calls.
- Euler is the mathematician who published second most papers in the history with a total of 75 volumes. This record was only broken by Paul Erdős in the 20th century.

Twists and turns: blind, fire

Sept. 18th 1783, when Euler is 77 years old, he wrote his calculation of the orbit of Uranus. When he was drinking tea and playing with kids, he got a stroke. His pipe fell down and he said “I’m dying”, then his life and calculations ended.

Examples

Eg Euler Function $\Phi(n)$ is the number of integers which is smaller than n and relatively prime to n.

Solution: If n is a product of different primes

$$n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$$

Assume that the set of multiples of p_i from *1* to n is A_i.

For $p_i \neq p_j$, $A_i \cap A_j$ is both p_i's multiples and p_j's multiples.

$$|A_i| = \frac{n}{p_i}, i = 1,2\dots k$$

$$|A_i \cap A_j| = \frac{n}{p_i p_j}, i, j = 1,2\dots, k, i \neq j$$

Example

Eg Euler Function $\Phi(n)$ is to calculate the number of integers which are smaller than *n* and relatively prime to *n*.

$$\Phi(\mathrm{n}) = |\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}|$$

Solution:

$$= n - (\frac{n}{p_1} + \frac{n}{p_2} + \cdots + \frac{n}{p_k}) + (\frac{n}{p_1 p_2} + \frac{n}{p_2 p_3} + \cdots + \frac{n}{p_1 p_k}) \ldots \pm (\frac{n}{p_1 p_2 \ldots p_k})$$

$$\Phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_k}\right)$$

$$\Phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\dots\left(1-\frac{1}{p_k}\right)$$

- **Eg** Euler Function $\Phi(n)$ is to calculate the number of integers which are smaller than *n* and relatively prime to *n*.

$\Phi(8)=8(1-1/2) = 4$

$8 = 2^3$, integers < 8 and relatively prime to 8: 1 3 5 7

$$For\ example\ n = 60 = 2^2 \times 3 \times 5, then$$

$$\psi(60) = 60\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right) = 16$$

So there are 16 integers smaller than 60 and relatively prime to 60：

7，11，13，17，19，23，29，31，37，41，43，47，49，53，59， and 1。

组合数学 Combinatorics

6 Inclusion-Exclusion theorem and pigeonhole principle

6-3 New solutions to old problems

Yuchun Ma
Tsinghua University

Review

Calculate the number of non-negative integers of equation

$x_1+x_2+x_3=15$

$C(n+b\text{-}1,b) = C(3+15\text{-}1,15)$

If the limitations are $0 \le x_1 \le 5$, $0 \le x_2 \le 6$, $0 \le x_3 \le 7$,

Eg Calculate the number of non-negative integer roots of $x_1+x_2+x_3=15$
Limitation is : $0\le x_1 \le 5$, $0\le x_2 \le 6$, $0\le x_3\le 7$。

Solution: The number of non-negative integer root of $x_1+x_2+\ldots+x_n=b$ is C($n+b$-1,b)

The number of non-negative roots of $x_1+x_2+x_3=15$ without limitation is C(15+3-1,15)= C(17,2)

Assume A1 is the solution when $x_1\ge 6$, $y_1+6+x_2+x_3=15$

$|A1|= C(9+3-1,9)= C(11,2)$

Assume $A2$ is the solution when $x_2\ge 7$, $x_1+y_2+7+x_3=15$

$|A2|= C(8+3-1,8)= C(10,2)$

Assume $A3$ is the solution when $x_3\ge 8$, $x_1+x_2+ y_3+8=15$

$|A3|= C(7+3-1,7)= C(9,2)$

Eg Calculate the number of non-negative solutions of $x_1+x_2+x_3=15$

Limitation is: $0\le x_1 \le 5$, $0\le x_2 \le 6$; $0\le x_3\le 7$

Solution: Without limitation, $x_1+x_2+x_3=15$ has $C(15+3-1,15)=$ $C(17,2)$ non-negative solutions.

$|A1|= C(9+3-1,9)= C(11,2)$

$|A2|= C(8+3-1,8)= C(10,2)$

$|A3|= C(7+3-1,7)= C(9,2)$

$A1\cap A2$: $y_1+6+y_2+7+x_3=15$ $|A1\cap A2| = C(2+3-1,2) = C(4,2)$

$A1\cap A3$: $y_1+6+x_2+y_3+8=15$ $|A1\cap A3| = C(1+3-1,1)= C(3,1)$

$A2\cap A3$: $x_1+y_2+7+y_3+8=15$ $|A2\cap A3| = 1$

$\mathrm{A1}\cap\mathrm{A2}\cap\mathrm{A3}$: $y_1+6+y_2+7+y_3+8=15$; $|\mathrm{A1}\cap\mathrm{A2}\cap\mathrm{A3}|= 0$

$|\overline{\mathrm{A1}}\cap\overline{\mathrm{A2}}\cap\overline{\mathrm{A3}}|=C(17,2) - C(11,2) - C(10,2) - C(9,2)$

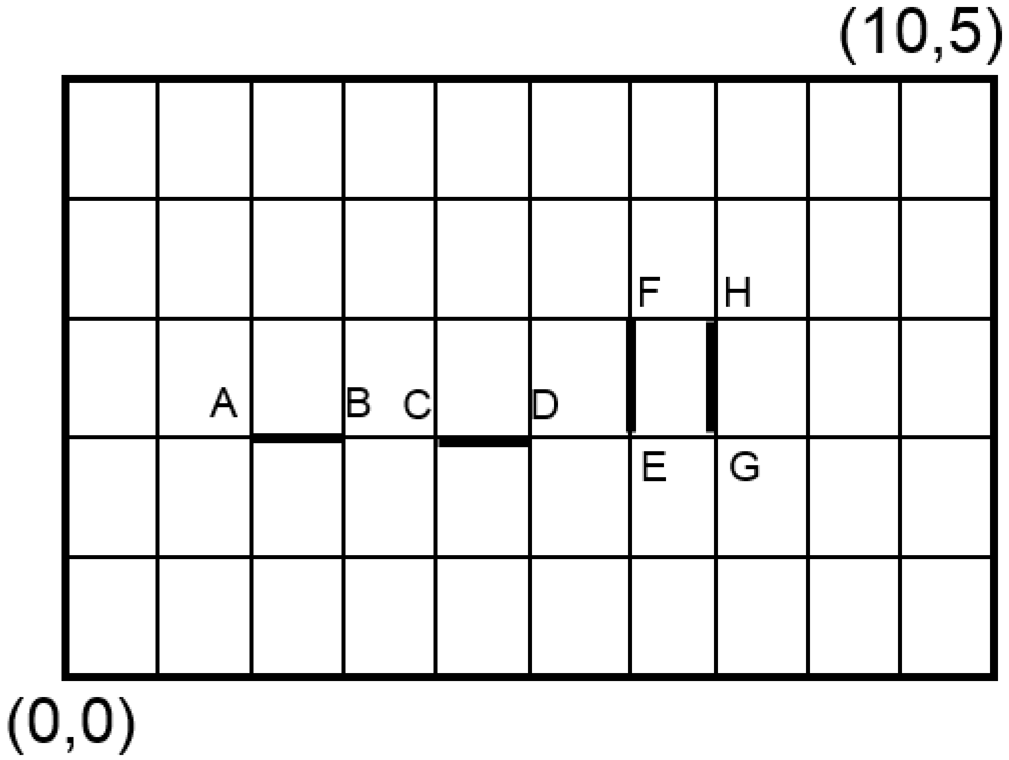
$+C(4,2) + C(3,1)+1=10$

Applications of Inclusion-Exclusion principals

Eg Lattice Path with barriers:

How many paths go from (0,0) to (10,5) without passing AB, CD, EF, GH?

The coordinates of the points are A(2,2),B(3,2),C(4,2),D(5,2),E(6,2),F(6,3),G(7,2),H(7,3)



Solution: Total number of paths:C(15,5);

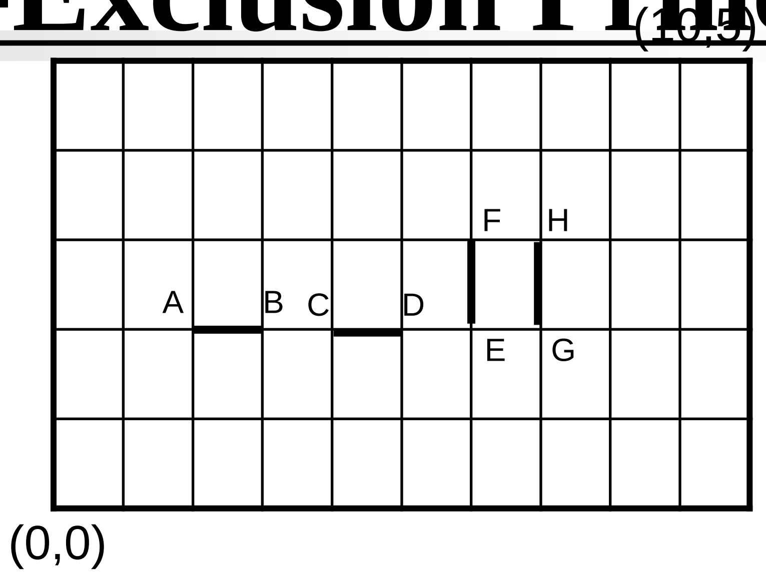
Paths that pass AB: $|A_1|$=C(2+2,2)C(7+3,3);

Paths that pass CD: $|A_2|$=C(4+2,2)C(5+3,3);

Paths that pass EF: $|A_3|$=C(8,2)C(6,2);

Paths that pass GH: $|A_4|$=C(9,2)C(5,2);

Applications of Inclusion-Exclusion Principle



Paths that pass AB, CD:

$|A_1 \cap A_2| = C(4,2)C(8,3)$;

Paths that pass AB, EF:

$|A_1 \cap A_3| = C(4,2)C(6,2)$;

Paths that pass AB, HG:

$|A_1 \cap A_4| = C(4,2)C(5,2)$;

Paths that pass CD, EF:

$|A_2 \cap A_3| = C(6,2)C(6,2)$;

Paths that pass CD, HG:

$|A_2 \cap A_4| = C(6,2)C(5,2)$;

Paths that pass EF, HG:

$|A_3 \cap A_4| = 0$;

Paths that pass AB, CD, EF:

$|A_1 \cap A_2 \cap A_3| = C(4,2)C(6,2)$;

Paths that pass AB, CD, HG:

$|A_1 \cap A_2 \cap A_4| = C(4,2)C(5,2)$;

$|A_2 \cap A_3 \cap A_4| = 0$

$|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = 2049$$

Application of Inclusion-Exclusion Principle

- **Eg Expansion of Stirling Number of the second kind**

$$S(n,m)=\frac{1}{m!}\sum_{k=1}^{m}C(m,k)(-1)^k(m-k)^n$$

- The meaning of $S(n,m)$: The number of arrangements to put n labeled balls into m **indifferent** boxes without leaving any empty box.
- Analyze the situation of putting n labeled balls to **different** boxes at first.

Solution: The set of events of putting n labeled balls to m different boxes is S, $|S|=m^n$

A_i means that the i^{th} box is empty, $i=1,2\ldots m$;

$|A_i| = (m-1)^n$ A total of C(m,1) ones

$|A_i \cap A_j| = (m-2)^n$ A total of C(m,2) ones

………………..

Calculate the number of arrangements without empty boxes

Examples of I-E Principle

$|A_i| = (m\text{-}1)^n$ C(m,1) ones

$|A_i \cap A_j| = (m\text{-}2)^n$ C(m,2) ones

………………..

Arrangements for m different boxes, no empty boxes

$$N = |\overline{A1} \cap \overline{A2} \ldots\ldots \cap \overline{An}|$$

$$= m^n - C(m,1)(m-1)^n + C(m,2)(m-2)^n + \ldots + (-1)^m C(m,m)(m-m)^n$$

$$= \sum_{k=0}^{m} (-1)^k C(m,k)(m-k)^n$$

The Stirling Number of the second kind requires the boxes to be indifferent, so

$$S(n,m) = \frac{1}{m!} \sum_{k=0}^{m} (-1)^k C(m,k)(m-k)^n$$

Corollary: As S(m,m) = 1,

$$m! = \sum_{k=0}^{m} (-1)^k C(m,k)(m-k)^m$$

Example

Eg Derangement: Label n elements as 1, 2, …, n. In a derangement of the n elements, every element is not in its original position.

Assume A_i is the set of all permutations with i in the i^{th} position, i=1, 2, …, n. As i is not movable, we have : $|A_i|=(n-1)!$

$$|A_i \cap A_j|=(n-2)!$$

The number of derangements is:

$$\left|\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}\right| = n! - C(n,1)(n-1)!$$
$$+ C(n,2)(n-2)! - \cdots - \pm C(n,n)1!$$

$$C(n,i)(n-i)! = \frac{n!}{(n-i)!\,i!}(n-i)! = \frac{n!}{i!}$$

6 Inclusion-Exclusion theorem and pigeonhole principle

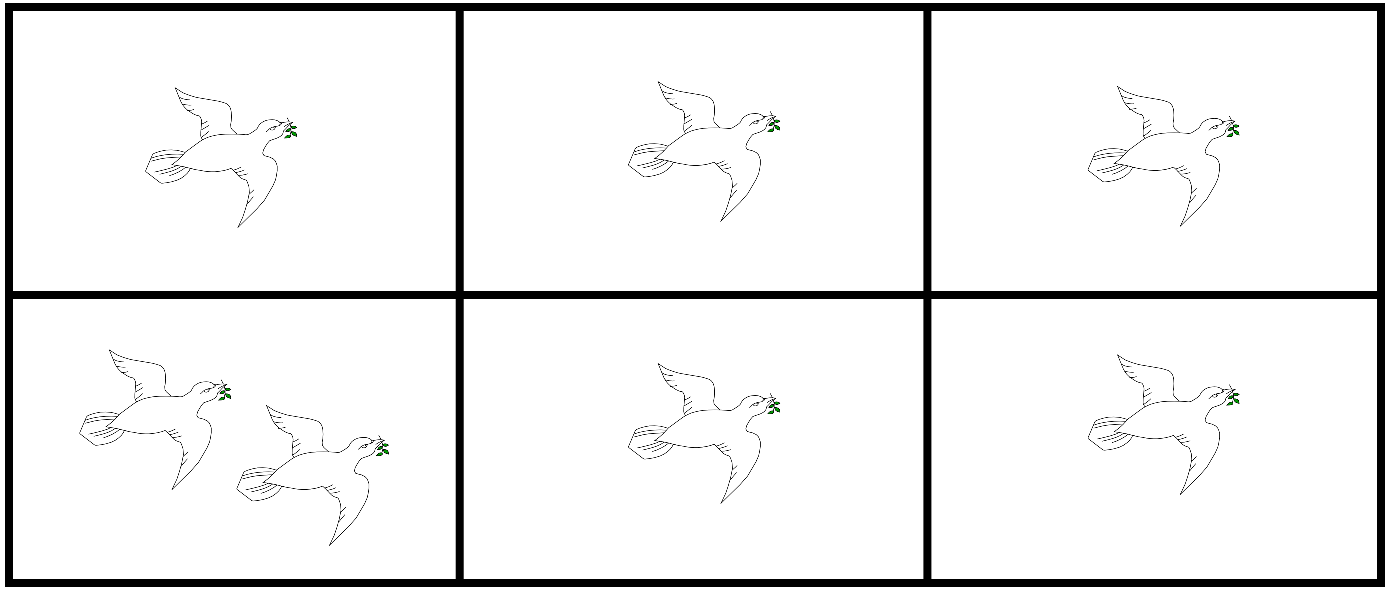
6-4 pigeonhole principle

组合数学 Combinatorics

Yuchun Ma
Tsinghua University

Pigeonhole Principle

pigeonhole principle is the simplest and most fundamental principle in combinatorics. It’s also called the drawer principle. It is “If there’s n-1 pigeonholes, n pigeons, so there’s at least one pigeonhole which has at least 2 pigeons”



Dirichlet

- Dirichlet（1805～1859）
- German mathematician, founder of analytical number theory and he defined the modern function concept.
- Dirichlet is famous for proving the *n*=5 situation of Fermat’s Last Theorem in 1825. In 1834, he raised pigeonhole principle, which is called the drawer principle at that time.
- Gauss died in 1855, University of Goettingen hired Dirichlet to inherit the position of Gauss.
- Berlin University starts its golden age from Dirichlet .
- Taken from: http://episte.math.ntu.edu.tw/people/p_dirichlet/

Pigeonhole Principle

Eg There are 30 people in a class, at least 3 are born in the same month.

Eg Conflicts in Hash tables are unavoidable as the number of keys are always larger than the number of indices.

quiz

Pigeonhole Principle

Eg 10 pairs of blue socks, 12 pairs of white socks, how many times do we need to pick randomly to get a pair.

What are the pigeon and pigeonholes here?

Pigeonhole Principle

- We know n+1 positive integers, all of them are $\le 2n$, prove that at least 2 of them are relatively prime.
- Famous Hungarian mathematician Paul Erdos (1913-1996) asked 11-year-old Louis Pósa this problem. Pósa answered it in half minute.
- （Hint ….）
- Pósa thought: take *n* boxes, put 1 and 2 in the first one, 3 and 4 in the second one, 5 and 6 in the third one, so forth, 2n-1 and 2n in the n^{th} one.
- Now we take n+1 numbers from n boxes, so at least one box would be emptied. So there must be a pair of adjacent numbers among these n+1 ones, and they are relatively prime.

Pigeonhole Principle

Eg Take any n+1 integers from 1 to 2n, among them there's at least one pair such that one is the multiple of the other.

Proof Assume the n+1 numbers are $a_1, a_2, \cdots, a_{n+1}$.
Dividing 2's until all of them becomes odd numbers. Then it construct a sequence $r_1, r_{2,}, \cdots, r_{n+1}$.
These n+1 numbers are still in [1,2n] and they are all odd.
While there are only n odd numbers in [1,2n].
So There must be $r_i = r_j = r$, then $a_i = 2^{ki} r$, $a_j = 2^{kj} r$
If $a_i > a_j$, a_i is a multiple of a_j.

Pigeonhole Principle

Eg Assume $a_1, a_2, \cdots, a_{100}$ is a sequence consists of 1 and 2. And any subsequence of 10 consecutive in it has a sum that is ≤ 16:

$$a_i + a_{i+1} + \ldots + a_{i+9} \le 16, \quad 1 \le i \le 91$$

So ∃ h and k such that $k > h$ and

$$a_h + a_{h+1} + \ldots + a_k = 39$$

Proof Let $S_j = \sum_{i=1}^{j} a_i, \quad j = 1, 2, \ldots, 100$

$S_1 < S_2 < \ldots < S_{100}$,

And $S_{100} = (a_1 + \ldots + a_{10})$

$+ (a_{11} + \ldots + a_{20}) + \ldots + (a_{91} + \ldots + a_{100})$

§ 3 .7 Pigeonhole Principle

According to assumption $a_i + a_{i+1} + \ldots + a_{i+9} \leq 16$, $1 \leq i \leq 91$

We have $S_{100} \leq 10 \times 16 = 160$

Create sequence $S_1, S_2, \ldots, S_{100}, S_1 + 39, \ldots, S_{100} + 39$.

With 200 terms. The largest term $S_{100} + 39 \leq 160 + 39 = 199$

By pigeonhole principle, there must be two equal terms.

And it must be a term in the first part and a term in the second part. Assume

$S_k = S_h + 39$, $k > h$ $S_k - S_h = 39$ So

$a_h + a_{h+1} + \ldots + a_k = 39$

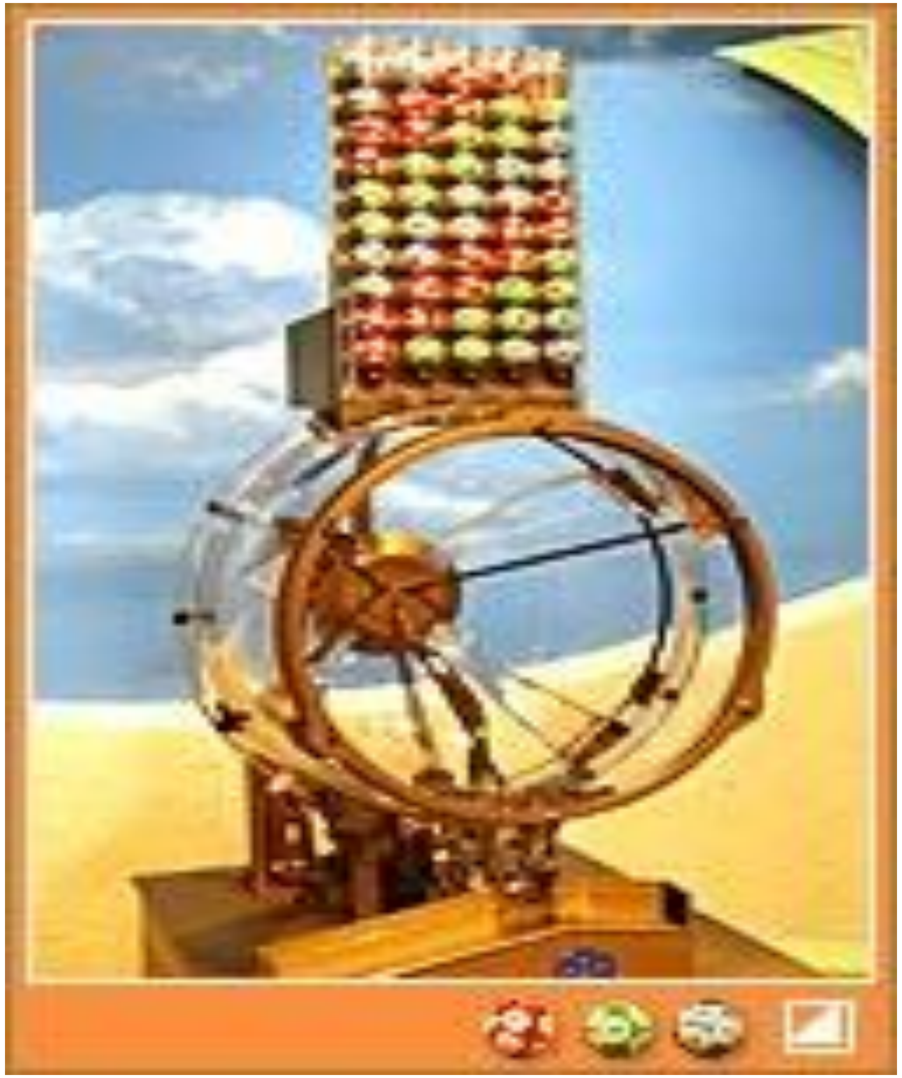
6 Inclusion-Exclusion theorem and pigeonhole principle

6-5 Pigeonholes

组合数学 Combinatorics

Yuchun Ma
Tsinghua University

Pigeonhole Principle



Mark Six (a lottery game)

49 labeled balls (1 to 49),
Draw 6 balls randomly and then the 7th as a special number

Pigeonhole Principle

Every time there must be two numbers among the 6 such that the first digit is the same. (Assume 1=01, 2=02, 3=03, 4=04).

Date	Draw Number	Draw Results							
20/12/2002	02/110	13	18	23	24	26	33	+	15
17/12/2002	02/109	6	18	39	40	41	42	+	9
12/12/2002	02/108	7	15	16	23	31	35	+	8
10/12/2002	02/107	5	36	37	38	46	49	+	17
05/12/2002	02/106	11	21	27	31	37	44	+	1
03/12/2002	02/105	9	11	14	17	24	28	+	46
28/11/2002	02/104	17	19	26	31	37	43	+	38
26/11/2002	02/103	19	21	40	42	46	47	+	33
21/11/2002	02/102	4	16	18	25	29	41	+	21
19/11/2002	02/101	3	15	22	23	42	47	+	18

Pigeonhole Principle

- Pick 6 master number every time.
- For every number, there are {0,1,2,3,4} 5 choices for the first digit;
- By pigeonhole principle, 6 pigeons are flying to 5 pigeonholes. So there’s at least one pigeonhole with 2 pigeons. This means that at least 2 numbers share their first-digits.

Pigeonhole Principle (2)

- There are 20 shirts in a drawer, in which 4 are blue, 7 are grey, 9 are red. How many do we need to pick to ensure that 4 shirts are the same color?
- How many do we need to pick to ensure 5, 6, 7, 8, or 9 same-colored shirts?？
- Pigeonhole Principle (2): n pigeonholes, kn+1 pigeons, at least 1 pigeonhole has k+1 pigeons.
- Solution: 3 colors, 3 pigeonholes, so k+1=4.
- K=3, kn+1=10, we need to pick at least 10 shirts？

Pigeonhole Principle (2)

- There are 20 shirts in a drawer, in which there are **4** blue ones,7 are grey, 9 are red. How many do we need to pick to ensure 5, 6, 7, 8, 9 same-colored shirts?
- Solution:（for 5 same-colored shirts）: If we pick 4 blue ones at first, then choosing from red and grey ones: $n=2,k+1=5$
- So we need to take $4+4\times2+1=13$ shirts to have 5 with the same color
- ……

Hanxin counts soldiers

- Chu and Han fought during the last years of Qing Dynasty.
- Chu armies comes, Hanxin counts soldiers to face it.
  - 3 per line, 2 are remained
  - 5 per line, 3 are remained
  - 7 per line, 2 are remained

$$\begin{cases} x = 3a + 2 \\ x = 5b + 3 \\ x = 7c + 2 \end{cases}$$

  - We have 1073 soldiers, there are fewer than 500 enemies, we occupies a commanding position. So we must be able to win.

- The Mathematical Classic of Sunzi
  - There are something but we don't know the number. Count by 3, 2 are remained, count by five, 3 are remained, count by 7, 2 are remained. How many things are there? Find a number *x* such that $x \div 3 \equiv 2; x \div 5 \equiv 3; x \div 7 \equiv 2$.

$$\begin{cases} x = 3a + 2 \\ x = 5b + 3 \\ x = 7c + 2 \end{cases}$$

- Find the smallest integer to satisfy this:
- List numbers such that $x \div 3 \equiv 2$：
  - 2， 5， 8， 11， 14， 17， 20， 23， 26…
- List numbers such that $x \div 5 \equiv 3$：
  - 3， 8， 13， 18， 23， 28 …
- The first common number is 8. The least common multiple of 3 and 5 is 15.
  - Combine those 2 requirements, we need $8 + 15 \times integer$:
  - 8， 23， 38， …，
- Then list numbers such that $x \div 7 \equiv 2$:
  - 2， 9， 16， 23， 30…

- Find the smallest positive integer root which satisfies the following equations:

$$\begin{cases} x = 3a + 2 \\ x = 5b + 3 \\ x = 7c + 2 \end{cases}$$

- Construction method
  - S is a common multiple of 5, 7, $S \div 3 \equiv 1$.s = 70
  - S*2=140 is a common multiple of 5, 7, $2S \div 3 \equiv 2$.
  - T is a common multiple of 3, 7. $t \div 5 \equiv 1$, t = 21
  - t*3=63 is a common multiple of 3, 7, $63 \div 5 \equiv 3$.
  - H is a common multiple of 3, 5. $H \div 7 \equiv 1$. H = 15
  - h*2=30 is a common multiple of 3, 5, $h \div 7 \equiv 2$.
  - 2s + 3t + 2h = 233 satisfies the previous equations
  - To find the smallest positive integer solution, 105 is the LCM of 3,5 and 7.
    - 233-105=128>105, 128-105=23.
    - 「Which number x satisfies $x \div 3 \equiv 2$, $x \div 5 \equiv 3$, $x \div 7 \equiv 2$?」 The answer is： 「23」

Chinese Remainder Theorem

$$\begin{cases} x = 3a + 2 \\ x = 5b + 3 \\ x = 7c + 2 \end{cases}$$

- Chinese Remainder Theorem: m and n are 2 relatively prime positive integers. For any non-negative integer a and b ($a<m$, $b<n$), there must be a positive integer that makes this equation group solvable.

$$\begin{cases} x = pm + a \\ x = qn + b \end{cases}$$

p,q are non-negative integers

- Proof: consider n integers: a, $m+a$, $2m+a$, … $(n-1)m+a$
  - Every k among them have: $k \div m \equiv a$.
- Consider the remainders of these n numbers divided by n.
  - If the remainders are distinct, we could find $x=qn+b$ for any b
  - If two of them are equal, for some b we could not find $x=qn+b$

$$\begin{cases} im + a = q_i n + r \\ jm + a = q_j n + r \end{cases}$$

Eg. If $im+a$ and $jm+a$ have the same remainder

when divided by n. ($0 \le i \le j \le n-1$). Subtract: $(j-i)m=(q_j-q_i)n$

Is n a factor of $(j-i)m$？？ However n and m are relatively prime, and $0 \le i \le j \le n-1$ so n ***must not be*** a factor of $(j-i)m$

$$\begin{cases} x = 3a + 2 \\ x = 5b + 3 \\ x = 7c + 2 \end{cases}$$

Chinese Remainder Theorem

- Chinese Remainder Theorem: *m* and *n* are relatively prime, for any non-negative integer *a* and *b*(*a*<*m*, *b*<*n*), there must be positive integer *x* which makes the equations solvable.

$$\begin{cases} x = pm + a \\ x = qn + b \end{cases}$$

p,*q* are non-negative integers

- Proof: Consider *n* integers： *a*, *m*+*a*, 2*m*+*a*, …..(*n*-1)*m*+*a*

  There's no command remainders for the *n* numbers divided by *n*. [0,1,2…,*n*-1], a total of *n* ones.

So for *b*(*b*<*n*), there must exist a number in the sequence which satisfies *x*=*qn*+*b*

Chinese Remainder Theorem

- （**Chinese Remainder Theorem, RT**）Assume $m_1, m_2, ..., m_k$ are relative prime, so $gcd(m_i, m_j) = 1$, $i \neq j$, $i, j = 1, 2, ..., k$, and the congruence equations：

  $x \equiv b_1 \bmod m_1$

  $x \equiv b_2 \bmod m_2$

  ...

  $x \equiv b_k \bmod m_k$

Mod $[m_1, m_2, ..., m_k]$ has solutions, this means with $[m_1, m_2, ..., m_k]$ there exists x which satisfies $x \equiv b_i \bmod [m_1, m_2, ..., m_k]$, $i = 1, 2, ..., k$

6 Inclusion-Exclusion principle and pigeonhole principle

6-6 6 people and Ramsey

组合数学 Combinatorics

Yuchun Ma
Tsinghua University

Ramsey

- Frank Plumpton Ramsey（1903-1930）
- Philosopher, mathematician and economist
- Member of Cambridge Apostles, alumni of Winchester College and Trinity College, son of the president of Magdalene College.
- Died at 26 and it was a huge loss for economics although his main interests are in philosophy and mathematical
- The Decision Analysis Society annually awards the Frank P. Ramsey Medal to recognise substantial contributions to decision theory and its application to important classes of real decision problems.
- In combinatorics there’s a famous example of Ramsey’s Theory.
  - Among any 6 people in the world, there must be 3 mutual friends or 3 mutual strangers.

http://en.wikipedia.org/wiki/Frank_P._Ramsey
http://en.wikipedia.org/wiki/Ramsey%27s_theorem

Ramsey Number

- Famous mathematicial G. C. Rota once said: if the others ask us what’s the most elegant thing in combinatorics. Most combinatorists would say Reamsey Number problem.
- R.L.Graham, former chairman of American Mathematics Society, chose RAMSEY as his car number.
- There exists more than 2000 academic papers about Ramsey Numbers.
- Only 10 Ramsey numbers are known. All others are still unknown.
- R.L.Graham once said that it’s impossible to decide Ramsey Number R(5,5) in 100 years.

Ramsey Problem

Ramsey Problem

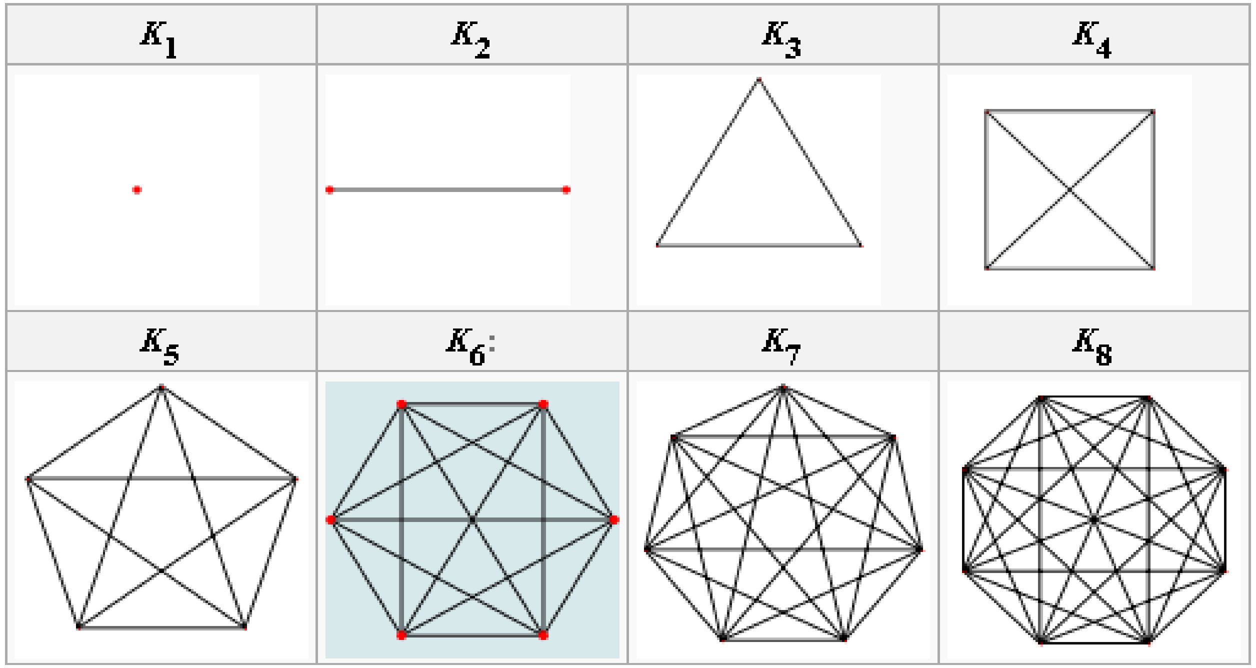
Ramsey Problem could be regarded as an extension to Pigeonhole's Principle. This is an example:

Eg There must be 3 mutual friends or 3 mutual strangers among 6 people.

It could be denoted as a coloring problem of complete graphs. Coloring the edges of K_6(6 vertices and there's an edge between each pair of vertices) with red and blue, prove that there's a unicolored triangle.

§ 3 .9 Ramsey Problem

- **A Complete Graph** is a simple graph such that there's exactly one edge between each pair of vertices. A complete graph with *n* vertices has *n* vertices and $\frac{n(n-1)}{2}$ edges. We use K_n to denote it.



Ramsey Problem

Suppose the vertex set of K_6 is $\{v_1, v_2, \cdots, v_6\}$,

$d_r(v)$ denotes the number of red edges incident to vertex v.

$d_b(v)$ denotes the number of blue edges incident to vertex v.

In K_6, we have $d_r(v) + d_b(v) = 5$, by pigeonhole principle, （5 pigeons to 2 pigeonholes） there must be 3 edges with the same color.

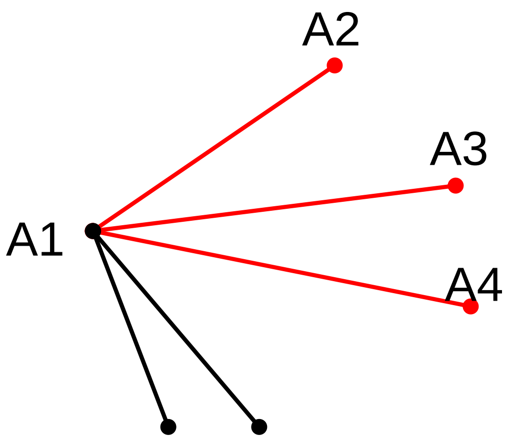
Let's assume that v is vertex A_1, suppose A_1A_2, A_1A_3, A_1A_4 are all red.

If there's a red edge in triangle $A_2A_3A_4$

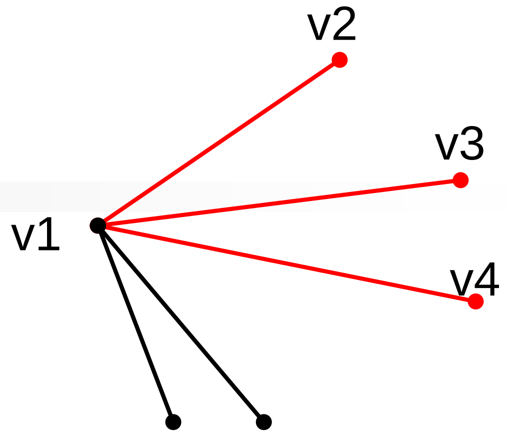
that edge construct a red triangle with A_1.

If all the three edges are blue

there's a blue triangle. ■



Ramsey Problem



Denote the proof process by the decision tree：

$d_r(v_1) \geq 3$?

N

Y

$d_b(v_1) \geq 3$

Suppose (v_1v_2) (v_1v_3) (v_1v_4) is red

Assume (v_1v_2) (v_1v_3) (v_1v_4) is blue

$\triangle v_2v_3v_4$ is blue $\triangle$?

N

Y

$\triangle v_2v_3v_4$ is red $\triangle$?

Assume (v_2v_3) is red √

N

Y

Assume (v_2v_3) is blue √

$\triangle v_1v_2v_3$ is red $\triangle$

$\triangle v_1v_2v_3$ is blue $\triangle$

Ramsey Number

Ramsey number： means bi-coloring here

R(3 , 3)＝6， R(3 , 4)＝9，

R(4 , 4)＝18

R(5,5) =?

- The complexity for searching all possible graphs is $O(2^{(n-1)(n-2)/2})$ for an upper bound of *n* nodes.
- Even the exact value of *R*(5,5) is unknown, although it is known to lie between 43 (Geoffrey Exoo) and 49 (Brendan McKay， etc) (inclusive); barring a breakthrough in theory, it is probable that the exact value of *R*(6,6) will remain unknown forever.
- "Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of R(5, 5) or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for *R*(6, 6). In that case, he believes, we should attempt to destroy the aliens."—Joel Spencer

Ramsey Number

- **Values of Ramsey Numbers（from Mathworld）**

q / p	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 43	46 51	52 59	59 69	66 78	73 88
4		18	25	35 41	49 61	56 84	69 115	80 149	96 191	128 238	133 291	141 349	153 417
5			43 49	58 87	80 143	95 216	121 316	141 442	153	181	193	221	242
6				102 165	111 298	127 495	153 780	177 1171	253	262	278	292	374
7					205 540	216 1031	7 1713	7 2826	322	416	511		
8						282 1870	8 3583	316 6090			635		703
9							565 6588	580 12677					
10								798 23556					

Inclusion-Exclusion Principle

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i=1}^{n} \sum_{j>i} |A_i \cap A_j|$$

$$+ \sum_{i=1}^{n} \sum_{j>i} \sum_{k>j} |A_i \cap A_j \cap A_k| - \ldots$$

$$+(-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

$$|\overline{A}| = N - |A|,$$

$$|\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}| = N - |A_1 \cup A_2 \cup \ldots \cup A_{n-1} \cup A_n|$$

$$= N - \sum_{i=1}^{n} |A_i| + \sum_{i=1}^{n} \sum_{j>i} |A_i \cap A_j| \quad - \sum_{i=1}^{n} \sum_{j>i} \sum_{k>j} |A_i \cap A_j \cap A_k| + \ldots$$

$$+ (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Pigeonhole Principle

pigeonhole principle is the simplest and most fundamental theorem in combinatorics. It's also called the drawer principle.

"If there are n-1 pigeonholes and n pigeons, there must be at least one pigeonhole with 2 pigeons."

When $a_k + a_{k+1} + \cdots + a_l$ appears, we often need to construct partial sums..

When parity, divisibility appear, we often need to use mod.

Proof by contradiction

Use pigeonholes for multiple times