Introduction to Optimization

Week 3 - Dual Problems, Weak and Strong Duality, Relation with Game Theory

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Duality theory

The concept of duality

Every linear programming program has a partner called its *dual problem.* The original problem is called the *primal problem,* and the pair of them is called a *primal–dual pair.* (FYI, duality is a relative concept: The dual of the dual problem is the primal problem.)

Primal–dual pairs have several important implications in developing the theory of optimization problems and solution algorithms. It also connect the principles of linear programming with adjacent fields such as game theory.

There are various ways to define the dual problem that depend on your perspective. We will start by considering a simple example.

Getting lower/upper bounds

Let's consider a technique for determining an *upper bound* on Uncle Hong's profit in the following LP.

$$
\begin{array}{llllll}
\max & 3x_1 & +5x_2 & & \\
\text{sub.to} & x_1 & & \leq & 4, \\
& & x_2 & \leq & 6, \\
& 3x_1 & +2x_2 & \leq & 18, \\
& x_1 \geq 0, & x_2 \geq 0. & &
\end{array}
$$

If, for example, we multiply each inequality by 0, $3/2$, and 2 and add them up, we obtain the following set of inequalities: $3x_1 + 5x_2 \leq (1 \times 0 + 0 \times 3/2 + 3 \times 2)x_1 + (0 \times 0 + 1 \times 3/2 + 2 \times 2)x_2 \leq 4 \times 0 + 6 \times 3/2 + 18 \times 2 = 45$. This means that the objective function value cannot exceed 45. (If you compare this with the actual optimal value, you will note that this is not a particularly good upper bound.)

By picking different weights for the constraint equations, what is the *lowest* upper bound we can obtain? Let's denote these weights by y_1, y_2, and y_3. Then they must satisfy the following conditions:

- For the direction of the inequalities not to be reversed, we must have $y \geq 0$.
- When we add up the inequalities after multiplying by the weights, the coefficients must be larger than the coefficients from the original objective function. $x_1 \leftrightarrow 3 \leq 1y_1 + 0y_2 + 3y_3$, $x_2 \leftrightarrow 5 \leq 0y_1 + 1y_2 + 2y_3$.

The objective function is the smallest such lower bound: $\min 4y_1 + 6y_2 + 18y_3$.

Putting this together:

$$\begin{array}{rccccc} \min & 4y_1 & +6y_2 & +18y_3 & & \\ \text{sub.to} & 1y_1 & +0y_2 & +3y_3 & \geq & 3, \\ & 0y_1 & +1y_2 & +2y_3 & \geq & 5, \\ & y_1 \geq 0, & y_2 \geq 0, & y_3 \geq 0. & & \end{array} \tag{1.1}$$

This is an LP! It is called the dual problem, and by design, its optimal value is always greater than or equal to that of the primal problem. Similarly, if the primal problem is a minimization problem, then the dual problem provides a lower bound on its optimal objective value. This property is called the weak duality theorem.

Exercise 1.1

Write the linear program that determines the weights for the constraint equations of (1.1) *to produce the highest possible lower bound for its objective value.*

In general, the dual and primal problems can be determined from each other by following these rules. Each rule is independent of the others, so they can be applied in either direction.

Each constraint	$\longleftrightarrow$	Each decision variable
Minimization	$\longleftrightarrow$	Maximization
Coefficients of a constraint	$\longleftrightarrow$	Coefficients of corresp. variable
Objective coefficient	$\longleftrightarrow$	Right-hand constant
Minimization problem $\leq$ constraint	$\longleftrightarrow$	Corresponding variable ≤ 0
Minimization problem $=$ constraint	$\longleftrightarrow$	Corresponding variable free
Minimization problem $\geq$ constraint	$\longleftrightarrow$	Corresponding variable ≥ 0
Maximization problem $\leq$ constraint	$\longleftrightarrow$	Corresponding variable ≥ 0
Maximization problem $=$ constraint	$\longleftrightarrow$	Corresponding variable free
Maximization problem $\geq$ constraint	$\longleftrightarrow$	Corresponding variable ≤ 0

$$
\begin{array}{lrrrcl}
\min & 3x_1 & -x_2 & +2x_3 & & \\
\textsf{sub.to} & x_1 & -4x_2 & +x_3 & \leq & 3, \\
 & & +3x_2 & +2x_3 & = & 2, \\
\multicolumn{6}{c}{x_1 \leq 0, \quad x_2 \geq 0, \quad x_3 \text{ unr.}}
\end{array} \tag{1.2}
$$

In the Uncle Hong problem, we obtain the following primal–dual pair.

$$
\begin{array}{rllll}
\text{max} & 3x_1 & +5x_2 & & \\
\text{sub. to} & x_1 & & \leq 4 & \longleftrightarrow y_1 \\
& & x_2 & \leq 6 & \longleftrightarrow y_2 \\
& 3x_1 & +2x_2 & \leq 18 & \longleftrightarrow y_3 \\
& x_1 \geq 0, & x_2 \geq 0 & &
\end{array}
\tag{1.3}
$$

$$
\begin{array}{rlllll}
\text{min} & 4y_1 & +6y_2 & +18y_3 & & \\
\text{sub. to} & 1y_1 & +0y_2 & +3y_3 & \geq 3 & \longleftrightarrow x_1 \\
& 0y_1 & +1y_2 & +2y_3 & \geq 5 & \longleftrightarrow x_2 \\
& y_1 \geq 0, & y_2 \geq 0, & y_3 \geq 0 & &
\end{array}
$$

Theorem 1.2

Weak Duality *For every primal-dual pair of feasible solutions, the objective value of the minimization problem is greater than or equal to the objective value of the maximization problem.*

The weak duality we have shown for the primal-dual pair of Uncle Hong's problem is valid for the primal-dual pair in the form. $\max\{c^T x : Ax \leq b, x \geq 0\}$ and $\min\{b^T y : A^T y \geq c,\ y \geq 0\}$, e.g. the pair of problems (1.3). Essentially the same arguments work for other cases. Weak duality tells us the following useful facts.

Corollary 1.3

- *If the objective of one problem can be improved arbitrarily (minimized or maximized without a bound), the other problem is infeasible.*
- *If the objective values of a primal-dual feasible pair (x, y) are equal, x and y are optimal solutions of their problems.*

Let's consider another perspective. Since Uncle Hong is trying to maximize his profit using a limited amount of raw material, it is a kind of "resource allocation" problem.

$$
\begin{array}{rlllll}
\text{max} & 3x_1 & +5x_2 & & & \\
\text{sub. to} & x_1 & & \leq 4 & \text{pine} & \\
 & & x_2 & \leq 6 & \text{glass} & \qquad (1.4) \\
 & 3x_1 & +2x_2 & \leq 18 & \text{mulberry} & \\
 & x_1 \geq 0, & x_2 \geq 0 & & &
\end{array}
$$

Let's introduce Ms. Kim, who would like to purchase all of Uncle Hong's raw materials. Since she is trying to determine how much money to offer for each of his materials, hers is a "resource evaluation" problem.

- Decision variables: Amount of money to pay per unit of raw material y_1, y_2, and y_3.
- Objective function: Ms. Kim would like to spend as little money as possible. min $4y_1 + 6y_2 + 18y_3$.

What conditions must Ms. Kim satisfy?

- The prices must be nonnegative.
- Now Ms. Kim considers Uncle Hong's perspective. If, for example, Uncle Hong has 1 unit of pine and 3 units of mulberry, he can make 3 units ($30) of profit. Therefore, there is no incentive for him to sell these resources for less than that amount. This means that $y_1 + 3y_3 \geq 3$. Similarly, $y_2 + 2y_3 \geq 5$.

Putting it together, Ms. Kim's problem is as follows.

$$\begin{array}{rl} \text{min} & 4y_1 + 6y_2 + 18y_3 \\ \text{sub. to} & 1y_1 + 0y_2 + 3y_3 \geq 3 \\ & 0y_1 + 1y_2 + 2y_3 \geq 5 \\ & y_1 \geq 0, y_2 \geq 0, y_3 \geq 0 \end{array}$$

This is exactly the same as the dual problem we saw before! Therefore, Uncle Hong is guaranteed to make 36 units of profit either way. The optimal solution is $y_1 = 0$, $y_2 = 3$, $y_3 = 1$, and the objective value is 36.

This means that Ms. Kim must pay Uncle Hong at least the potential profit associated with converting his raw materials into finished goods in order to obtain the raw materials themselves.

- The optimal objective value of Ms. Kim's problem is 36, the same value as the optimal objective value of Uncle Hong's problem.
- It means if both sides do their utmost to strike a fair bargain, then Ms. Kim ends up paying Uncle Hong for his raw materials an amount equal to Uncle Hong's profit when he turns the raw materials into finished goods.
- This must be the case for every primal-dual pair of linear programs if one problem has optimal solution. And it is called the strong duality theorem.
- The negotiated unit price of mulberry is 1, which equals the shadow price of mulberry: The dual optimal solutions are the shadow prices of the constraints of the primal problem.

In general, a linear program can be written in the form of minimizing $c^T x$ in n variables subject to greater-than-or-equal-to constraints:

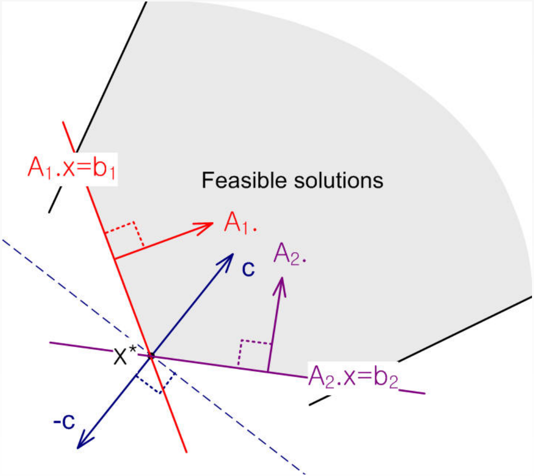
$$
\begin{array}{cccccc}
a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & \geq & b_1, \\
a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & \geq & b_2, \\
 & & \cdots & & & \\
a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & \geq & b_m.
\end{array}
$$

$$
\Leftrightarrow \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \ddots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_{n-1} \\ x_n \end{bmatrix} \geq \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix}.
$$

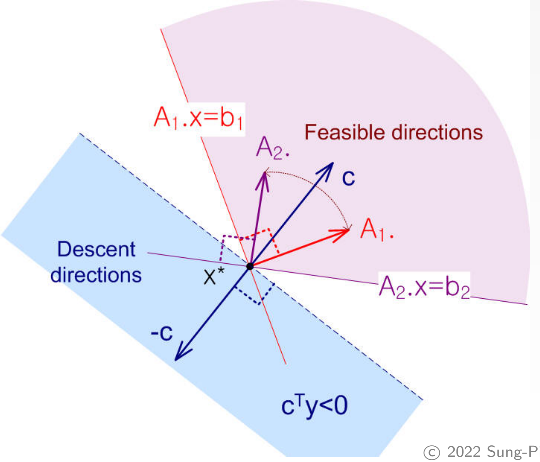
$$
\Leftrightarrow \quad \begin{array}{ll}
A_{1\cdot}x & \geq b_1, \\
A_{2\cdot}x & \geq b_2, \\
\cdots & \\
A_{m\cdot}x & \geq b_m.
\end{array}
$$

$$
\Leftrightarrow \quad Ax \geq b.
$$

Now let's consider a geometric interpretation of duality. Although it is not a formal proof of the strong duality theorem, it will give us an intuitive understanding. Let x^* be an optimum of $\min\{c^T x : Ax \geq b\}$ satisfying $A_1.x \geq b_1$ and $A_2.x \geq b_2$ with equality. We call these the *active constraints* of x^*.



If $c^T y < 0$, the objective function decreases from x^* in the direction of y (blue region). Denote $A^\circ = \begin{bmatrix} A_{1\cdot} \\ A_{2\cdot} \end{bmatrix}$. Then for any $y : A^\circ y \geq 0$ we can move from x^* in the direction of y satisfying both $A_1.x \geq b_1$ and $A_2.x \geq b_2$. Since x^* satisfies other constraints with strict inequality, we can move in the direction y for a positive distance maintaining feasibility. We call such a y a *feasible direction* from x^* (red region).



Therefore, if x^* is optimal, there can be no y with $c^T y < 0$ such that $A^\circ y \geq 0$:

$$c^T y \geq 0, \ \forall y \text{ such that } A^\circ y \geq 0. \tag{1.5}$$

In other words, the blue and red regions should not intersect. From the figure it means the objective coefficient vector c lies between two vectors $A_{1\cdot}^T$ and $A_{2\cdot}^T$ inclusively:

$$\exists \lambda \in \mathbb{R}^2 : \ c = \lambda_1 A_{1\cdot}^T + \lambda_2 A_{2\cdot}^T = (A^\circ)^T \lambda, \ \lambda_1 \geq 0, \ \lambda_2 \geq 0. \tag{1.6}$$

Remark 1.4

We can show (1.5) implies (1.6) for general number of active constraints by using a theorem such as the separating hyperplane theorem.

Let's extend λ into $\hat{\lambda} \in \mathbb{R}^m$ by assigning zeroes to the components corresponding to inactive constraints of x^*.

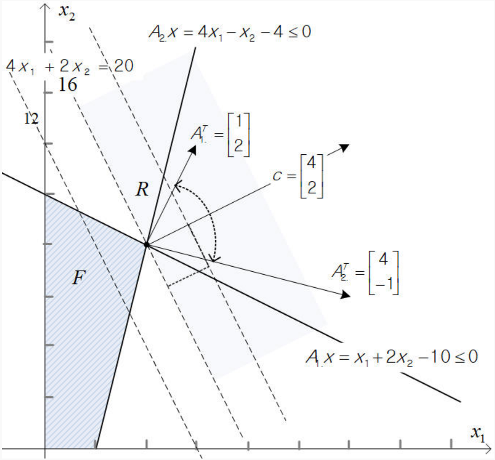
Then from (1.6), $\hat{\lambda}$ is a feasible solution of the dual problem $\max\{b^T y : A^T y = c,\ y \geq 0\}$ of $\min\{c^T x : Ax \geq b\}$. Furthermore, $\hat{\lambda}$ is an optimal solution of the dual problem. Let b° to be the right-hand constant vector of A°. Then $b^T \hat{\lambda} = (b^\circ)^T \lambda = (x^*)^T (A^\circ)^T \lambda = (x^*)^T c$. Hence by Corollary 1.3, $\hat{\lambda}$ is a dual optimal solution.

Exercise 1.5

Show that $\max\{b^T y : A^T y = c,\ y \geq 0\}$ *is the dual problem of* $\min\{c^T x : Ax \geq b\}$.

Example 1.6

$$
\begin{array}{rrrrl}
\max & c^T x = & 4x_1 & +2x_2 & \\
\text{sub. to} & A_1.x = & x_1 & +2x_2 & \leq 10 \\
& A_2.x = & 4x_1 & -x_2 & \leq 4 \\
& & x_1 \geq 0 & x_2 \geq 0 &
\end{array}
\qquad (1.7)
$$



From the feasible solution $(2, 4)$ of linear program (1.8), any y: $[4, 2]^T y > 0$ is an ascent direction. Also any y having a nonpositive inner product with $[1, 2]^T$, $[4, -1]^T$, which are the active constraint rows $A_{1.}$ and $A_{2.}$, is a feasible direction. Hence if $\bar{x}$ is optimal, the two direction sets should be disjoint.

It implies $[4, 2]^T y \leq 0$ for every y such that $[1, 2]^T y \leq 0$, $[4, -1]^T y \leq 0\}$. It means there is $\lambda \geq 0$: $[4, 2]^T = [1, 2]^T \lambda_1 + [4, -1]^T \lambda_2$. In fact, $\lambda = (\frac{4}{3}, \frac{2}{3})$ satisfies it.

$$\begin{array}{cccc} \min & 10y_1 & +4y_2 & \\ & y_1 & +4y_2 & \geq 4 \\ & 2y_1 & -y_2 & \geq 2 \\ & y_1 \geq 0 & y_2 \geq 0 & \end{array} \tag{1.8}$$

Exercise 1.7

Show $\lambda = (\frac{4}{3}, \frac{2}{3})$ is an optimal solution of the dual problem (1.8).

Exercise 1.8

1. Sketch the feasible solution set and the level curves of objective function of the following linear programs.

$$
\begin{array}{rrrcl}
\textit{maximize} & 2x_1 & +x_2 & & \\
\textit{subject to} & x_1 & +x_2 & \geq & 1 \\
 & x_1 & & \leq & 4 \\
 & -2x_1 & +x_2 & \leq & 2 \\
 & x_1, & x_2, & \geq & 0
\end{array}
$$

$$
\begin{array}{rrrcr}
\textit{minimize} & & x_2 & & \\
\textit{subject to} & -2x_1 & +4x_2 & \leq & 0 \\
 & 3x_1 & +x_2 & \leq & 15 \\
 & x_1 & +x_2 & \geq & 0
\end{array}
$$

2. Find the dual optimal solutions of the linear programs.

Duality summarized

- Weak duality theorem: In a primal–dual pair, the objective value in the maximization problem is always less than or equal to the objective value in the minimization problem.
- Strong duality theorem: In a primal–dual pair, if one problem has an optimal solution, then so does the other, and their objective values are equal.

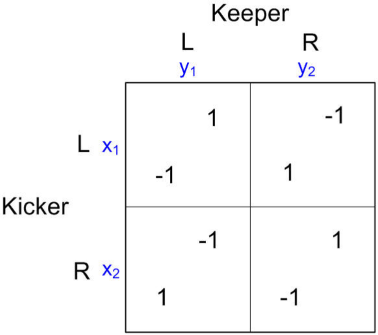
Duality and zero-sum games

- The kicker tries to kick the ball into the goal and the keeper tries to block it. Each player has a choice of two strategies: kick (or block) on the left, or kick (or block) on the right. If both players choose different directions, then the kicker scores a goal; if not, then the keeper blocks him.
- If the kicker scores a goal, then he wins 1 point and the keeper loses 1 point. If the keeper blocks the kicker, then the kicker loses 1 point and the keeper wins 1 point. Then we can summarize this game in a table like that on the right.

		Keeper	
		L	R
Kicker	L	1 / -1	-1 / 1
	R	-1 / 1	1 / -1

- Since the sum of the points awarded to each player is always 0, this is a *zero-sum game.*

- Suppose that the kicker aims to left and right randomly, with probabilities x_1 and x_2. Likewise, the keeper blocks to the left and right with probabilities y_1 and y_2.
- From each player's perspective, what is the best way to set $x = (x_1, x_2)$ and $y = (y_1, y_2)$? That is, are these values x and y such that if it is publicly known that the keeper is using strategy y, then the best counterstrategy for the kicker is x, and vice-versa?
- Such an x–y pair is called a *Nash equilibrium*. We call the kicker's expected payoff v in a Nash equilibrium the *game value*.



Let A be the payoff matrix for the kicker: $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. If kicker is known to use x, then the keeper's expected payoff from the jth alternative is $-x^T A_{\cdot j}$, where $A_{\cdot j}$ is the jth column of A.

Then the keeper will choose y to minimize the payoff his opponent $x^T Ay$. Then it is the same as $\min\{x^T A_{\cdot j} : 1 \le j \le 2\}$. Therefore, the best the kicker can hope to do is to choose the value of x that maximizes this minimal payoff.

Such a kicker's strategy can be found using the following linear program.

$$\begin{array}{lllll} \max & & s & & \\ & e^T x & & = 1, & & \leftrightarrow s \\ & -A_{\cdot j}^T x & +s & \le 0, & 1 \le j \le 2, & \leftrightarrow y_j \\ & x \ge 0. & & & & \end{array} \tag{2.9}$$

If the kicker uses the strategy (x^*, s^*) given by the linear program (2.9), then no matter what, his expected payoff $\underline{v}$ will be at least as good as the objective value s^*. And given x^*, the keeper will choose the counterstrategy that *minimizes* the kicker's payoff, meaning that $\underline{v}$ cannot be larger than s^*. Hence $\underline{v} = s^*$.

Similarly, the keeper's strategy for minimizing the keeper's payoff can be formulated as the following linear program, which is left to the readers. Note that we state the keeper's strategy in terms of the kicker's payoff.

$$
\begin{array}{llllll}
\min & & t & & & \\
& e^T y & & = 1, & & \leftrightarrow s \\
& -A_{i\cdot} y & +t & \geq 0, & 1 \leq i \leq 2, & \leftrightarrow x_i \\
& y \geq 0. & & & &
\end{array}
\tag{2.10}
$$

Just as before, when (y^*, t^*) is the optimal solution to the LP above, if the keeper uses the strategy y^*, then the expected payoff $\bar{v}$ of the kicker is the same as t^*.

We call the strategy of the two players the *minimax strategy*.

Exercise 2.1

Show that (2.9) and (2.10) are a primal–dual pair. Argue that any of them can not be unbounded.

Theorem 2.2

Minimax theorem *The minimax strategy of the two-person zero-sum game constitutes a Nash equilibrium in which* $\underline{v} = \bar{v}$.

Proof: Clearly, the kicker's minimax strategy x^* produces a guaranteed payoff of at least s^*. From Exercise 2.1 and the weak duality theorem, any minimax strategy y of the keeper, a feasible solution of LP (2.10), cannot make his opponent's payoff t smaller than s^*. But since $s^* = t^*$ by the strong duality theorem, y^* restricts the kicker's payoff to s^*. Thus y^* is an optimal counterstrategy to x^*.

Similarly, we can argue x^* is an optimal counterstrategy to y^*. Therefore, (x^*, y^*) is the Nash equilibrium inducing the same expected payoffs $\underline{v} = \bar{v}$.

□

We have established the minimax theorem by using LP duality. But we can go even further: It turns out that the converse is also true. Since $\underline{v} = s^*$ and $\bar{v} = t^*$, the minimax theorem implies the existence of the feasible solution pair (x^*, y^*) of the primal-dual pair (2.9) and (2.10) with the same objective value, namely the strong duality between two problems.

Remark 2.3

In fact, any primal-dual pair of linear programs can be posed as the minimax strategy problems of the players in a two-person zero-sum game. This means the minimax theorem is equivalent to the strong duality theorem of LP.

Remark 2.4

We have seen that the user optimization problem of kicker and keeper can be solved by linear programming, a system optimization problem. Unfortunately, such cases are limited: the two person zero-sum game is one of them and we will see another case later.

Exercise 2.5

Alice and Bob are playing a modified version of "rock, paper, scissors." If you beat scissors with rock, you get 2 points and your partner loses 2 points. Similarly, if you beat paper with scissors, you get 3 points and your partner loses 3, and if you beat rock with paper, you win 1 point and your partner loses 1. If Alice and Bob play the same symbol, both receive 0 payoff. (2.11) is the Alice's payoff matrix for this game.

$$\begin{array}{ccc} & & \text{Bob} \\ & & \begin{array}{ccc} \textit{Scissors} & \textit{Rock} & \textit{Paper} \end{array} \\ \textit{Alice} & \begin{array}{c} \textit{Scissors} \\ \textit{Rock} \\ \textit{Paper} \end{array} & \begin{bmatrix} 0 & -2 & +3 \\ +2 & 0 & -1 \\ -3 & +1 & 0 \end{bmatrix} \end{array} \tag{2.11}$$

1. What LP can be used to determine the optimal strategy for Bob and Alice?

2. Use the Excel solver to find the Nash equilibrium for this game.