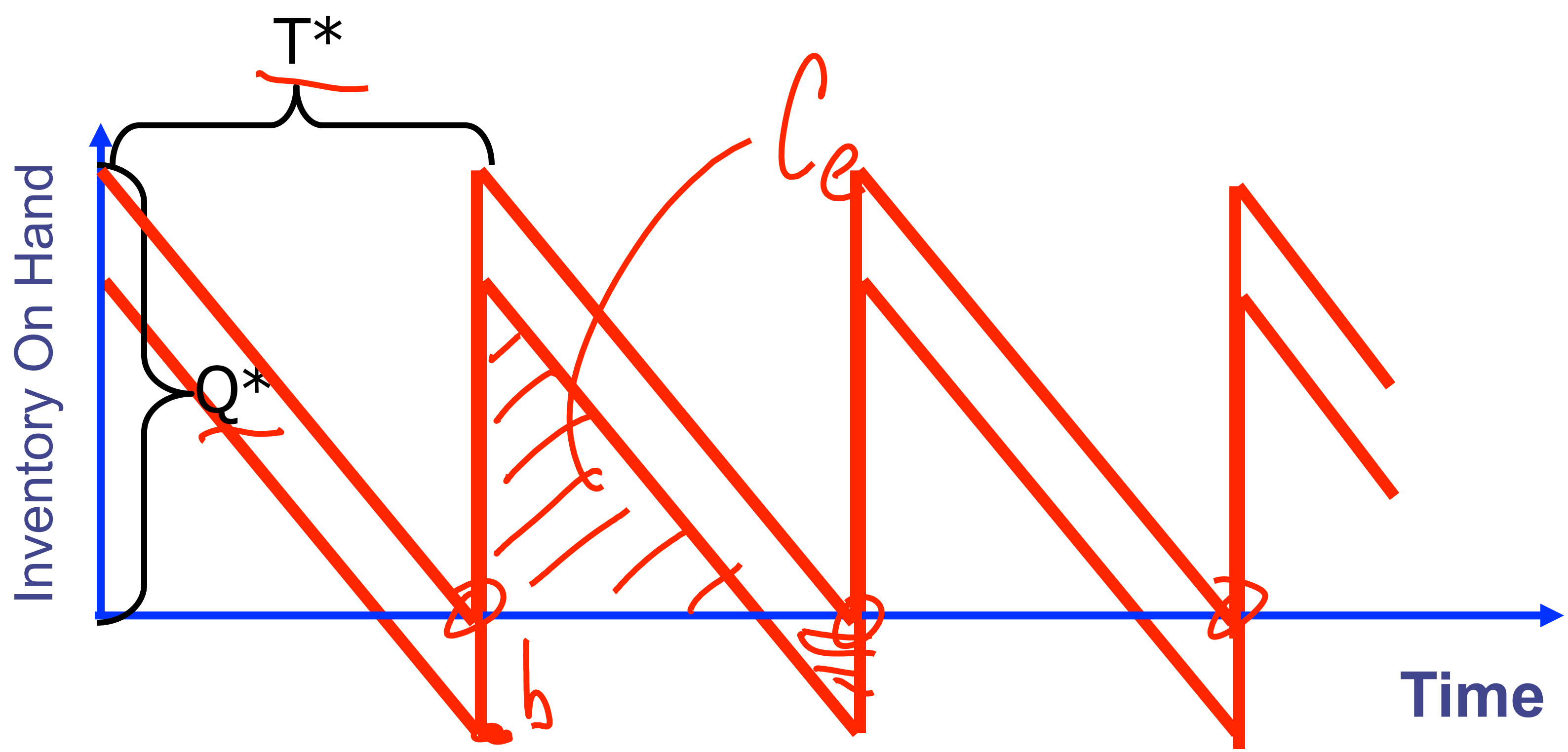
Single Period Inventory Models:
Allowing for Stockouts

MIT Center for
Transportation & Logistics

Assumptions: EOQ with Planned Backorders

- Demand
  - **Constant** vs Variable
  - **Known** vs Random
  - **Continuous** vs Discrete
- Lead Time
  - **Instantaneous**
  - Constant vs Variable
  - Deterministic vs Stochastic
  - Internally Replenished
- Dependence of Items
  - **Independent**
  - Correlated
  - Indentured
- Review Time
  - **Continuous** vs Periodic
- Number of Locations
  - **One** vs Multi vs Multi-Echelon
- Capacity / Resources
  - **Unlimited**
  - Limited / Constrained
- Discounts
  - **None**
  - All Units vs Incremental vs One Time
- Excess Demand
  - None
  - **All orders are backordered**
  - Lost orders
  - Substitution
- Perishability
  - **None**
  - Uniform with time
  - Non-linear with time
- Planning Horizon
  - Single Period
  - Finite Period
  - **Infinite**
- Number of Items
  - **One** vs Many
- Form of Product
  - **Single Stage**
  - Multi-Stage

EOQ with Planned Backorders



What will happen to Q* and T* if we allow for planned backorders at some cost (c_s)?

EOQ with Planned Back Orders

Notation

D = Average Demand (units/time)

c = Variable (Purchase) Cost ($/unit)

c_t = Fixed Ordering Cost ($/order)

h = Carrying or Holding Charge ($/inventory $/time)

$c_e = c*h$ = Excess Holding Cost ($/unit/time)

c_s = Shortage Cost ($/unit/time)

Q = Replenishment Order Quantity (units/order)

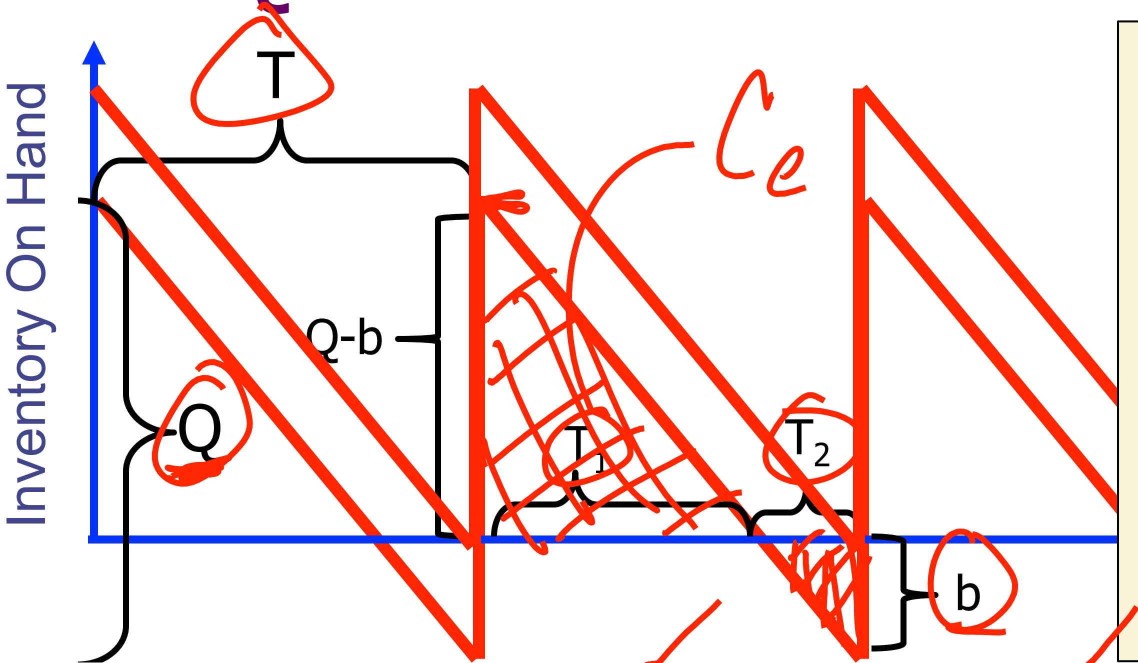
T = Order Cycle Time (time/order)

$N = 1/T$ = Orders per Time (order/time)

$TRC(Q)$ = Total Relevant Cost ($/time)

$TC(Q)$ = Total Cost ($/time)

EOQ with Planned Backorders



From similar triangles:

$$\frac{Q}{T} = \frac{(Q-b)}{T_1} = \frac{b}{T_2}$$

$$\frac{T_1}{T} = \frac{(Q-b)}{Q} \qquad \frac{T_2}{T} = \frac{b}{Q}$$

$$TRC(Q,b) = c_t\left(\frac{D}{Q}\right) + c_e\left(\frac{1}{2}\right)\left(\frac{T_1}{T}\right)(Q-b) + c_s\left(\frac{1}{2}\right)\left(\frac{T_2}{T}\right)(b)$$

$$TRC(Q,b) = c_t\left(\frac{D}{Q}\right) + c_e\left(\frac{1}{2}\right)\left(\frac{(Q-b)}{Q}\right)(Q-b) + c_s\left(\frac{1}{2}\right)\left(\frac{b}{Q}\right)(b)$$

$$TRC(Q,b) = c_t\left(\frac{D}{Q}\right) + c_e\left(\frac{(Q-b)^2}{2Q}\right) + c_s\left(\frac{b^2}{2Q}\right)$$

Planned Backorders - Solution

EOQ with Planned Backorders

$$TRC(Q,b) = c_t\left(\frac{D}{Q}\right) + c_e\left(\frac{(Q-b)^2}{2Q}\right) + c_s\left(\frac{b^2}{2Q}\right)$$

$$Q^*_{PBO} = \sqrt{\frac{2c_tD}{c_e}}\sqrt{\frac{(c_s+c_e)}{c_s}} = \boxed{Q^*\sqrt{\frac{(c_s+c_e)}{c_s}}}$$

$$b^* = \frac{c_eQ^*_{PBO}}{(c_s+c_e)} = \left(1-\frac{c_s}{(c_s+c_e)}\right)Q^*_{PBO}$$

Inventory Policy

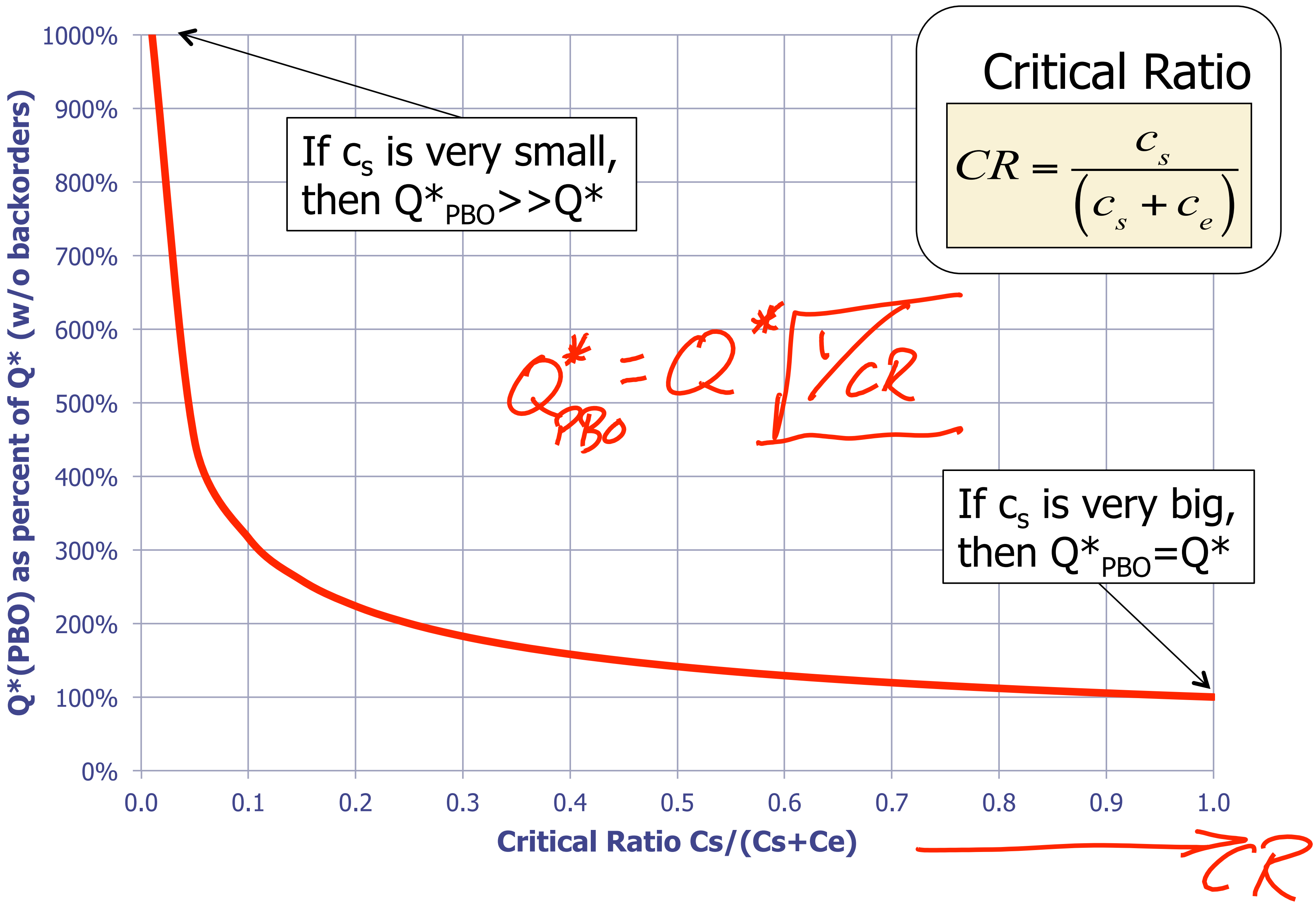
Order Q^*_{PBO} when IOH = - b*

Order Q^*_{PBO} every T^*_{PBO} time periods

Critical Ratio

$$CR = \frac{c_s}{(c_s+c_e)}$$

EOQ with Planned Backorders



Critical Ratio

$$CR = \frac{c_s}{\left(c_s + c_e\right)}$$

Probabilistic Demand: Single Period Models

Assumptions: Single Period Models

- Demand
  - Constant vs **Variable**
  - Known vs **Random**
  - **Continuous** vs Discrete
- Lead Time
  - **Instantaneous**
  - Constant vs Variable
  - Deterministic vs Stochastic
  - Internally Replenished
- Dependence of Items
  - **Independent**
  - Correlated
  - Indentured
- Review Time
  - **Continuous** vs Periodic
- Number of Locations
  - **One** vs Multi vs Multi-Echelon
- Capacity / Resources
  - **Unlimited**
  - Limited / Constrained
- Discounts
  - **None**
  - All Units vs Incremental vs One Time
- Excess Demand
  - None
  - All orders are backordered
  - **Lost orders**
  - Substitution
- Perishability
  - **None**
  - Uniform with time
  - Non-linear with time
- Planning Horizon
  - **Single Period**
  - Finite Period
  - Infinite
- Number of Items
  - **One** vs Many
- Form of Product
  - **Single Stage**
  - Multi-Stage

Example: NFL Replica Jerseys

- Situation:
  - In 2002 Reebok had sole rights to sell replica NFL football jerseys
  - Jerseys have unique names & numbers
  - Peak sales last about 8 weeks
  - Lead time from contract manufacturer is 12-16 weeks
- Main Issue:
  - Reebok had to commit to an order in advance while the actual demand was uncertain
- Question:
  - How many Jerseys of each player should they order?

Case adapted from Parsons, J. (2004) "Using A Newsvendor Model for Demand Planning of NFL Replica Jerseys," MIT Supply Chain Management Program Thesis.
Image Source: http://commons.wikimedia.org/wiki/File:Tom_Brady_%28cropped%29.jpg

Example: NFL Replica Jerseys

- Data:
  - Unit cost = c = 10.90 \$/jersey
  - Unit selling price = p = 24 \$/jersey
  - Forecast demand = 32,000 jerseys (σ = 11,000)
    - History showed demand to be Normally distributed

- Select Q* that maximizes profit where X = actual demand:

$$\text{Profit} = p\left(\text{MIN}\left[x, Q\right]\right) - cQ$$

- How do I determine the "best" policy?
  1. Data table
  2. Marginal analysis

Solving Single Period Model: Data Table

Sample spreadsheets in MS Excel and LibreOffice are available in this unit.

Data Table

$$\text{Profit} = pMIN(x,Q) - cQ$$

	A	B	C	D	E	F	G
1							
2	Mean	32.000					
3	StdDev	11.000					
4			Price=	$ 24.00	Cost=	$ 10.90	
5			Order	24	25	26	27
6	CumProb	**Demand**	**Prob**				
7	0.3%	2	0.3%	$ (214)	$ (225)	$ (235)	$ (246)
8	0.5%	4	0.2%	$ (166)	$ (177)	$ (187)	$ (198)
9	0.9%	6	0.4%	$ (118)	$ (129)	$ (139)	$ (150)
10	1.5%	8	0.6%	$ (70)	$ (81)	$ (91)	$ (102)
11	2.3%	10	0.8%	$ (22)	$ (33)	$ (43)	$ (54)
12	3.5%	12	1.2%	$ 26	[illegible]	[illegible]	[illegible]
13	5.1%	14	1.6%	$ 74	[illegible]	[illegible]	[illegible]
14	7.3%	16	2.2%	$ 122	$ 112	$ 101	$ 90
15	10.2%	18	2.9%	$ 170	$ 160	$ 149	$ 138
16	13.8%	20	3.6%	$ 218	$ 208	$ 197	$ 186
17	18.2%	22	4.4%	$ 266	$ 256	$ 245	$ 234
18	23.4%	24	5.2%	$ 31[illegible]	[illegible]	[illegible]	[illegible]

Potential order sizes (Q)

=$D$4*MIN($B8,E$5)−$F$4*E$5

Probability of demand P[x]

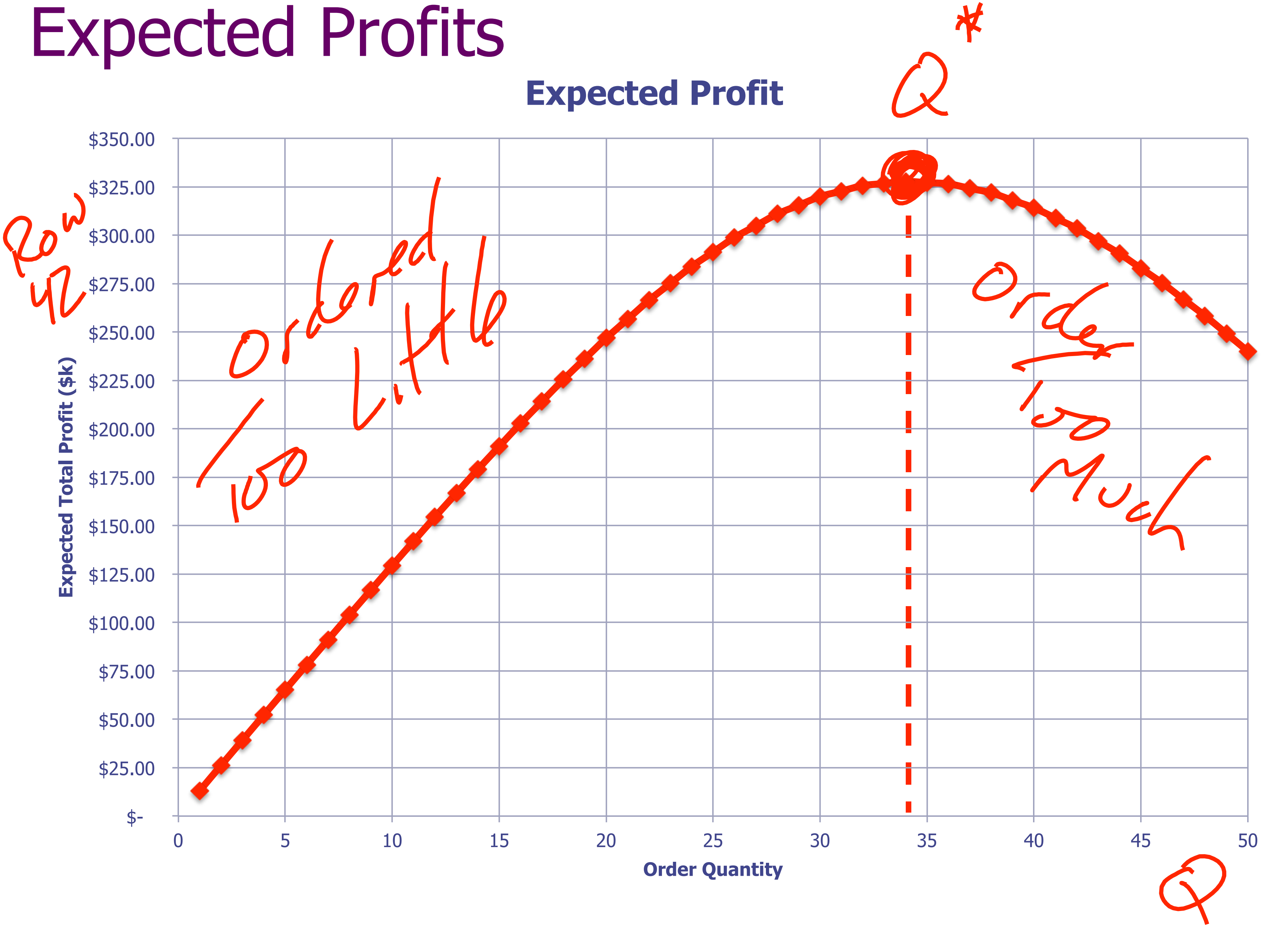
Potential demand (x)

=NORMDIST(B10,$B$2,$B$3,1)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1																
2	Mean	32.000														
3	StdDev	11.000														
4			Price=	$ 24.00	Cost=	$ 10.90										
5			Order	24	25	26	27	28	29	30	31	32	33	34	35	36
6	CumProb	Demand	Prob													
7	0.3%	2	0.3%	$ (214)	$ (225)	$ (235)	$ (246)	$ (257)	$ (268)	$ (279)	$ (290)	$ (301)	$ (312)	$ (323)	$ (334)	$ (34
8	0.5%	4	0.2%	$ (166)	$ (177)	$ (187)	$ (198)	$ (209)	$ (220)	$ (231)	$ (242)	$ (253)	$ (264)	$ (275)	$ (286)	$ (29
9	0.9%	6	0.4%	$ (118)	$ (129)	$ (139)	$ (150)	$ (161)	$ (172)	$ (183)	$ (194)	$ (205)	$ (216)	$ (227)	$ (238)	$ (24
10	1.5%	8	0.6%	$ (70)	$ (81)	$ (91)	$ (102)	$ (113)	$ (124)	$ (135)	$ (146)	$ (157)	$ (168)	$ (179)	$ (190)	$ (20
11	2.3%	10	0.8%	$ (22)	$ (33)	$ (43)	$ (54)	$ (65)	$ (76)	$ (87)	$ (98)	$ (109)	$ (120)	$ (131)	$ (142)	$ (15
12	3.5%	12	1.2%	$ 26	$ 16	$ 5	$ (6)	$ (17)	$ (28)	$ (39)	$ (50)	$ (61)	$ (72)	$ (83)	$ (94)	$ (10
13	5.1%	14	1.6%	$ 74	$ 64	$ 53	$ 42	$ 31	$ 20	$ 9	$ (2)	$ (13)	$ (24)	$ (35)	$ (46)	$ (5
14	7.3%	16	2.2%	$ 122	$ 112	$ 101	$ 90	$ 79	$ 68	$ 57	$ 46	$ 35	$ 24	$ 13	$ 3	$ (
15	10.2%	18	2.9%	$ 170	$ 160	$ 149	$ 138	$ 127	$ 116	$ 105	$ 94	$ 83	$ 72	$ 61	$ 51	$ 4
16	13.8%	20	3.6%	$ 218	$ 208	$ 197	$ 186	$ 175	$ 164	$ 153	$ 142	$ 131	$ 120	$ 109	$ 99	$ 8
17	18.2%	22	4.4%	$ 266	$ 256	$ 245	$ 234	$ 223	$ 212	$ 201	$ 190	$ 179	$ 168	$ 157	$ 147	$ 13
18	23.4%	24	5.2%	$ 314	$ 304	$ 293	$ 282	$ 271	$ 260	$ 249	$ 238	$ 227	$ 216	$ 205	$ 195	$ 18
19	29.3%	26	5.9%	$ 314	$ 328	$ 341	$ 330	$ 319	$ 308	$ 297	$ 286	$ 275	$ 264	$ 253	$ 243	$ 23
20	35.8%	28	6.5%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 356	$ 345	$ 334	$ 323	$ 312	$ 301	$ 291	$ 28
21	42.8%	30	7.0%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 382	$ 371	$ 360	$ 349	$ 339	$ 32
22	50.0%	32	7.2%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 408	$ 397	$ 387	$ 37
23	57.2%	34	7.2%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 435	$ 42
24	64.2%	36	7.0%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 459	$ 47
25	70.7%	38	6.5%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 459	$ 47
26	76.6%	40	5.9%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 459	$ 47
27	81.8%	42	5.2%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 459	$ 47
28	86.2%	44	4.4%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 459	$ 47
29	89.8%	46	3.6%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 459	$ 47
30	92.7%	48	2.9%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 459	$ 47
31	94.9%	50	2.2%	$ 314	$ 328	$ 341	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$ 47
32	96.5%	52	1.6%	$ 314	$ 328	$ 341	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$ 47
33	97.7%	54	1.2%	$ 314	$ 328	$ 341	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$ 47
34	98.5%	56	0.8%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 459	$ 47
35	99.1%	58	0.6%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 459	$ 47
36	99.5%	60	0.4%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 459	$ 47
37	99.7%	62	0.2%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 459	$ 47
38	99.8%	64	0.1%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 459	$ 47
39	99.9%	66	0.1%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 459	$ 47
40	99.9%	68	0.0%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 459	$ 47
41	100.0%	70	0.0%	$ 314	$ 328	$ 341	$ 354	$ 367	$ 380	$ 393	$ 406	$ 419	$ 432	$ 445	$ 459	$ 47
42			99.97%	$283.87	$291.36	$298.86	$304.93	$311.00	$315.50	$320.00	$322.83	$325.66	$326.7	$327.85	$327.22	$326.5

=SUMPRODUCT($C$7:$C$48,E7:E48)

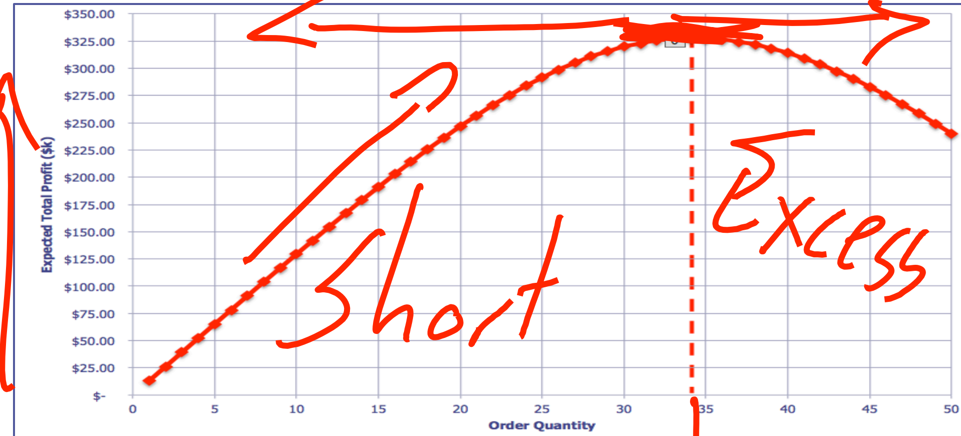
Expected Profits



Solving Single Period Model: Marginal Analysis

Marginal Analysis

P



Q*   Q

For single-period problems we have two costs:

- c_e = Excess cost when $D<Q$ (\$/unit) i.e. having too much product
- c_s = Shortage cost when $D>Q$ (\$/unit) i.e. having too little product

Assuming a continuous distribution of demand , we get

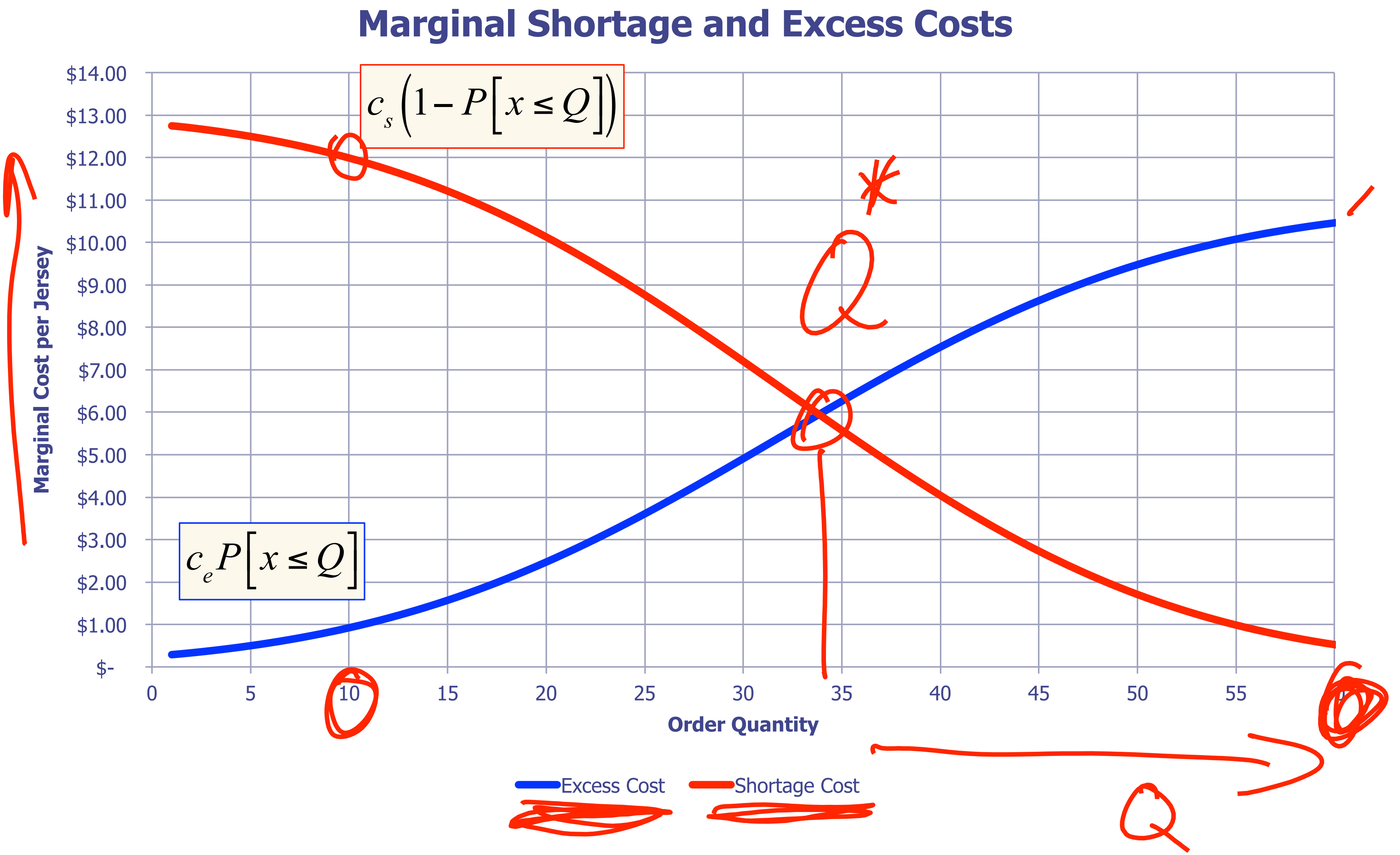
- $c_e\, P[X \le Q]$ = expected excess cost of the Qth unit ordered
- $c_s\, (1-P[X \le Q])$ = expected shortage cost of the Qth unit ordered

If E[Excess Cost] < E[Shortage Cost] then increase Q

We are at Q* when E[Shortage Cost] = E[Excess Cost]

Marginal Analysis

~N(32, 11)



Marginal Analysis

$$c_e P\left[x \le Q\right] = c_s\left(1 - P\left[x \le Q\right]\right)$$

$$c_e P\left[x \le Q\right] = c_s - c_s P\left[x \le Q\right]$$

$$c_e P\left[x \le Q\right] + c_s P\left[x \le Q\right] = c_s$$

$$P\left[x \le Q\right]\left(c_e + c_s\right) = c_s$$

$$\boxed{P\left[x \le Q\right] = \frac{c_s}{\left(c_e + c_s\right)}}$$

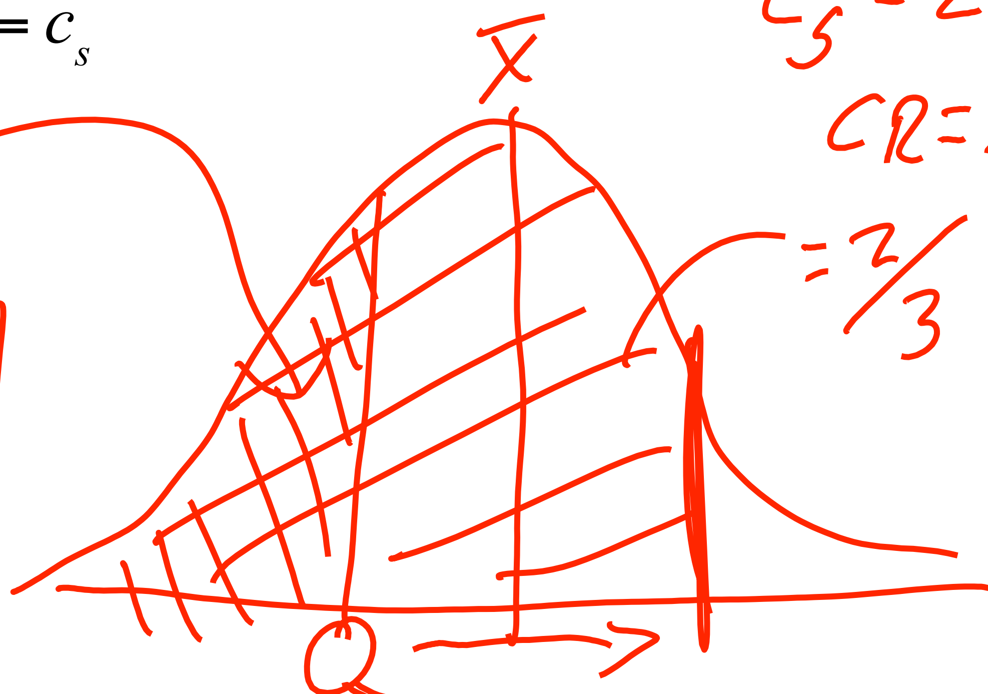
The Critical Ratio

$CR = C_s / (C_e + C_s)$

$C_e = C_s$

$C_s = 2C_e$

$CR = 2/3$



NFL Jersey Example - solved

Example: NFL Replica Jerseys

- Data:
  - Total cost = c = 10.90 \$/jersey
  - Selling price = p = 24 \$/jersey
  - Forecast demand ~N(32000, 11000)
- Solution:
  - $c_s = p - c = 24 - 10.90 = \$13.10$
  - $c_e = c = \$10.90$
  - CR = (13.10)/(10.9 + 13.10) =0.546
  - Select Q where P[x≤Q] = 0.546
    - Normal Table or use spreadsheet:

$$CR = \frac{p-c}{c+p-c} = \frac{p-c}{p}$$

Case adapted from Parsons, J. (2004) "Using A Newsvendor Model for Demand Planning of NFL Replica Jerseys," MIT Supply Chain Management Program Thesis.
Image Source: http://commons.wikimedia.org/wiki/File:Tom_Brady_%28cropped%29.jpg

Standard Normal Table

$P[x \le Q] = 0.546$

Find k= 0.115

Recall $k=(Q-\mu)/\sigma$

So, $Q = \mu + k\sigma$
$= 32000+(0.115)(11000)$

Q = 33,267 units

k	P[x≤k]	G(k)	k	P[x≤k]	G(k)	k
0.00	0.5000	0.3989	0.50	0.6915	0.1978	1.00
0.01	0.5040	0.3940	0.51	0.6950	0.1947	1.01
0.02	0.5080	0.3890	0.52	0.6985	0.1917	1.02
0.03	0.5120	0.3841	0.53	0.7019	0.1887	1.03
0.04	0.5160	0.3793	0.54	0.7054	0.1857	1.04
0.05	0.5199	0.3744	0.55	0.7088	0.1828	1.05
0.06	0.5239	0.3697	0.56	0.7123	0.1799	1.06
0.07	0.5279	0.3649	0.57	0.7157	0.1771	1.07
0.08	0.5319	0.3602	0.58	0.7190	0.1742	1.08
0.09	0.5359	0.3556	0.59	0.7224	0.1714	1.09
0.10	0.5398	0.3509	0.60	0.7257	0.1687	1.10
0.11	0.5438	0.3464	0.61	0.7291	0.1659	1.11
0.12	0.5478	0.3418	0.62	0.7324	0.1633	1.12
0.13	0.5517	0.3373	0.63	0.7357	0.1606	1.13
0.14	0.5557	0.3328	0.64	0.7389	0.1580	1.14
0.15	0.5596	0.3284	0.65	0.7422	0.1554	1.15
0.16	0.5636	0.3240	0.66	0.7454	0.1528	1.16
0.17	0.5675	0.3197	0.67	0.7486	0.1503	1.17
0.18	0.5714	0.3154	0.68	0.7517	0.1478	1.18

Example: NFL Replica Jerseys

- Data:
  - Total cost = c = 10.90 $/jersey
  - Selling price = p = 24 $/jersey
  - Forecast demand ~N(32000, 11000)
- Solution:
  - c_s = p – c = 24 -10.90 = $13.10
  - c_e= c = $10.90
  - CR = (13.10)/(10.9 + 13.10) =0.546
  - Select Q where P[x≤Q] = 0.546
    - Normal Table or use spreadsheet:
    - =NORMINV(CR, Mean, StdDev)
    - =NORMINV(0.546, 32000, 11000)
  - Q* = 33,267 - the profit maximizing quantity

But what if I can sell the left overs at a discount?

Case adapted from Parsons, J. (2004) "Using A Newsvendor Model for Demand Planning of NFL Replica Jerseys," MIT Supply Chain Management Program Thesis.
Image Source: http://commons.wikimedia.org/wiki/File:Tom_Brady_%28cropped%29.jpg

Considering Other Costs

- Other costs:
  - g = salvage value, \$/unit
  - B = Penalty for not satisfying demand (beyond lost profit), \$/unit

- The excess and shortage costs change:
  - $c_s = p - c + B$
  - $c_e = c - g$
  - Critical Ratio $= c_s/(c_s+c_e)$

$$= (p-c+B)/(p-c+B+c-g)$$

$$\boxed{=(p-c+B)/(p+B-g)}$$

Example: NFL Replica Jerseys

- Data:
  - Total cost = c = 10.90 \$/jersey
  - Selling price = p = 24 \$/jersey
  - Forecast demand ~N(32000, 11000)
  - Salvage value = g = 7 \$/jersey
- Solution:
  - $c_s = p - c = 24 - 10.90 = \$13.10$
  - $c_e = c - g = 10.90 - 7.00 = \$3.90$
  - CR = (13.10)/(3.9 + 13.10) =0.771
  - Select Q where P[x≤Q] = 0.771
    - Normal Table or use spreadsheet:
    - =NORMINV(CR, Mean, StdDev)=NORMINV(.771, 32000, 11000)
  - Q* = 40,149- the profit maximizing quantity

But, how do I determine the profitability?

Key Points from Lesson

Key Points

- Newsvendor problems are everywhere
  - Fashion items, perishable goods, fleet sizing, contracting, space missions, etc.
  - Whenever you have to make a firm bet in the face of uncertain demand in a single period
- Classic trade off between:
  - Having too much (excess cost c_e)
  - Having too little (shortage cost c_s)
- Critical Ratio captures this trade-off
  - $CR = C_s / (C_s + C_e)$
  - CR = Pct of demand distribution to cover $= P[x \le Q]$

Questions, Comments, Suggestions? Use the Discussion!

MIT Center for Transportation & Logistics

caplice@mit.edu