Single Period Inventory Models: Allowing for Stockouts
Assumptions: EOQ with Planned Backorders

- Demand
  - **Constant** vs Variable
  - **Known** vs Random
  - **Continuous** vs Discrete
- Lead Time
  - **Instantaneous**
  - Constant vs Variable
  - Deterministic vs Stochastic
  - Internally Replenished
- Dependence of Items
  - **Independent**
  - Correlated
  - Indentured
- Review Time
  - **Continuous** vs Periodic
- Number of Locations
  - **One** vs Multi vs Multi-Echelon
- Capacity / Resources
  - **Unlimited**
  - Limited / Constrained
- Discounts
  - **None**
  - All Units vs Incremental vs One Time
- Excess Demand
  - **None**
  - All orders are backordered
  - Lost orders
  - Substitution
- Perishability
  - **None**
  - Uniform with time
  - Non-linear with time
- Planning Horizon
  - Single Period
  - Finite Period
  - **Infinite**
- Number of Items
  - **One** vs Many
- Form of Product
  - **Single Stage**
  - Multi-Stage
What will happen to $Q^*$ and $T^*$ if we allow for planned backorders at some cost ($c_s$)?
EOQ with Planned Back Orders
Notation

\( D = \) Average Demand (units/time)
\( c = \) Variable (Purchase) Cost ($/unit)
\( c_t = \) Fixed Ordering Cost ($/order)
\( h = \) Carrying or Holding Charge ($/inventory $/time)
\( c_e = c \cdot h = \) Excess Holding Cost ($/unit/time)
\( c_s = \) Shortage Cost ($/unit/time)
\( Q = \) Replenishment Order Quantity (units/order)
\( T = \) Order Cycle Time (time/order)
\( N = 1/T = \) Orders per Time (order/time)

\( TRC(Q) = \) Total Relevant Cost ($/time)
\( TC(Q) = \) Total Cost ($/time)
EOQ with Planned Backorders

From similar triangles:
\[
\frac{Q}{T} = \frac{(Q - b)}{T_1} = \frac{b}{T_2}
\]

\[
T_1 = \frac{(Q - b)}{Q} \quad T_2 = \frac{b}{Q}
\]

\[
TRC(Q, b) = c_t \left( \frac{D}{Q} \right) + c_e \left( \frac{1}{2} \right) \left( \frac{T_1}{T} \right) (Q - b) + c_s \left( \frac{1}{2} \right) \left( \frac{T_2}{T} \right) (b)
\]

\[
TRC(Q, b) = c_t \left( \frac{D}{Q} \right) + c_e \left( \frac{1}{2} \right) \left( \frac{(Q - b)}{Q} \right) (Q - b) + c_s \left( \frac{1}{2} \right) \left( \frac{b}{Q} \right) (b)
\]

\[
TRC(Q, b) = c_t \left( \frac{D}{Q} \right) + c_e \left( \frac{(Q - b)^2}{2Q} \right) + c_s \left( \frac{b^2}{2Q} \right)
\]
Planned Backorders - Solution
EOQ with Planned Backorders

\[ TRC(Q, b) = c_t \left( \frac{D}{Q} \right) + c_e \left( \frac{(Q - b)^2}{2Q} \right) + c_s \left( \frac{b^2}{2Q} \right) \]

\[ Q_{PBO}^* = \sqrt{\frac{2c_tD}{c_e}} \sqrt{\left( \frac{c_s + c_e}{c_s} \right)} = Q^* \sqrt{\frac{c_s + c_e}{c_s}} \]

\[ b^* = \frac{c_e Q_{PBO}^*}{(c_s + c_e)} = \left( 1 - \frac{c_s}{(c_s + c_e)} \right) Q_{PBO}^* \]

Inventory Policy

Order \( Q_{PBO}^* \) when IOH = - \( b^* \)

Order \( Q_{PBO}^* \) every \( T_{PBO}^* \) time periods

Critical Ratio

\[ CR = \frac{c_s}{(c_s + c_e)} \]
EOQ with Planned Backorders

Critical Ratio

If \( c_s \) is very small, then \( Q^*_{PBO} \gg Q^* \)

\[
Q^*_{PBO} = Q^* \sqrt{\frac{1}{CR}}
\]

If \( c_s \) is very big, then \( Q^*_{PBO} = Q^* \)

\[
CR = \frac{c_s}{c_s + c_e}
\]
Probabilistic Demand:
Single Period Models
Assumptions: Single Period Models

- Demand
  - Constant vs Variable
  - Known vs Random
  - Continuous vs Discrete

- Lead Time
  - Instantaneous
  - Constant vs Variable
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- Planning Horizon
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- Number of Items
  - One vs Many

- Form of Product
  - Single Stage
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Example: NFL Replica Jerseys

- **Situation:**
  - In 2002 Reebok had sole rights to sell replica NFL football jerseys
  - Jerseys have unique names & numbers
  - Peak sales last about 8 weeks
  - Lead time from contract manufacturer is 12-16 weeks

- **Main Issue:**
  - Reebok had to commit to an order in advance while the actual demand was uncertain

- **Question:**
  - How many Jerseys of each player should they order?

Image Source: http://commons.wikimedia.org/wiki/File:Tom_Brady_%28cropped%29.jpg
Example: NFL Replica Jerseys

- **Data:**
  - Unit cost = \( c = 10.90 \) $/jersey
  - Unit selling price = \( p = 24 \) $/jersey
  - Forecast demand = 32,000 jerseys (\( \sigma = 11,000 \))
    - History showed demand to be Normally distributed

- Select \( Q^* \) that maximizes profit where \( X = \) actual demand:

\[
\text{Profit} = p \left( \text{MIN}\left[ x, Q \right] \right) - cQ
\]

- How do I determine the “best” policy?
  1. Data table
  2. Marginal analysis
Solving Single Period Model: Data Table
Sample spreadsheets in MS Excel and LibreOffice are available in this unit.

Data Table

<table>
<thead>
<tr>
<th>Demand</th>
<th>Prob</th>
<th>Price</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
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<td>0.3%</td>
<td>$214</td>
<td>$240</td>
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<tr>
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<tr>
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<td>$314</td>
<td>$324</td>
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</table>

Profit = \( p \text{MIN}(x, Q) - cQ \)

Profit calculation: 

- Potential order sizes (Q)
- Potential demand (x)
- Probability of demand \( P[x] \)
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<th>Prob</th>
<th>Order</th>
<th>Price</th>
<th>StdDev</th>
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<td></td>
<td>$(40)$</td>
<td>$(363)$</td>
</tr>
</tbody>
</table>

Formula: \(=\text{SUMPRODUCT}(\text{C7:C48}, E7:E48)\)
Expected Profits

Expected Total Profit ($k)

Order Quantity

Expected Profit

Row 42

Ordered Too Little

Order Too Much

Q *
Solving Single Period Model: Marginal Analysis
For single-period problems we have two costs:

\[ c_e = \text{Excess cost when } D<Q \text{ (}/\text{unit}) \text{ i.e. having too much product} \]

\[ c_s = \text{Shortage cost when } D>Q \text{ (}/\text{unit}) \text{ i.e. having too little product} \]

Assuming a continuous distribution of demand, we get

\[ c_e P[X\leq Q] = \text{expected excess cost of the Qth unit ordered} \]

\[ c_s (1-P[X\leq Q]) = \text{expected shortage cost of the Qth unit ordered} \]

If \( E[\text{Excess Cost}] < E[\text{Shortage Cost}] \) then increase \( Q \)

We are at \( Q^* \) when \( E[\text{Shortage Cost}] = E[\text{Excess Cost}] \)
Marginal Analysis

Marginal Shortage and Excess Costs

\[ c_s (1 - P[x \leq Q]) \]

\[ c_e P[x \leq Q] \]

Order Quantity

Marginal Cost per Jersey

$\sim N(32, 11)$
Marginal Analysis

\[ c_e P[x \leq Q] = c_s \left(1 - P[x \leq Q]\right) \]

\[ c_e P[x \leq Q] = c_s - c_s P[x \leq Q] \]

\[ c_e P[x \leq Q] + c_s P[x \leq Q] = c_s \]

\[ P[x \leq Q] (c_e + c_s) = c_s \]

The Critical Ratio

\[ P[x \leq Q] = \frac{c_s}{c_e + c_s} \]

\[ CR = \frac{c_s}{c_e + c_s} \]

\[ c_e = c_s \]

\[ c_s = 2c_e \]

\[ CR = \frac{2}{3} \]

\[ \frac{7}{3} \]
NFL Jersey Example - solved
Example: NFL Replica Jerseys

• Data:
  - Total cost = $c = 10.90 \$/jersey
  - Selling price = $p = 24 \$/jersey
  - Forecast demand ~N(32000, 11000)

• Solution:
  - $c_s = p - c = 24 - 10.90 = 13.10 \$
  - $c_e = c = 10.90 \$
  - $CR = (13.10) / (10.90 + 13.10) = 0.546$
  - Select Q where $P[x \leq Q] = 0.546$
    - Normal Table or use spreadsheet:

Image Source: http://commons.wikimedia.org/wiki/File:Tom_Brady_%28cropped%29.jpg
Standard Normal Table

\[ P(x \leq Q) = 0.546 \]

Find \( k = 0.115 \)

Recall \( k = (Q - \mu) / \sigma \)

So, \( Q = \mu + k \sigma \)

\[ Q = 32000 + (0.115)(11000) \]

\[ Q = 33,267 \text{ units} \]
Example: NFL Replica Jerseys

Data:
- Total cost = $c = 10.90$/jersey
- Selling price = $p = 24$/jersey
- Forecast demand ~N(32000, 11000)

Solution:
- $c_s = p - c = 24 - 10.90 = 13.10$
- $c_e = c = 10.90$
- $CR = \frac{13.10}{10.90 + 13.10} = 0.546$
- Select $Q$ where $P[x \leq Q] = 0.546$
  - Normal Table or use spreadsheet:
    - $=\text{NORMINV}(CR, \text{Mean}, \text{StdDev})$
    - $=\text{NORMINV}(0.546, 32000, 11000)$
- $Q^* = 33,267$ - the profit maximizing quantity

But what if I can sell the left overs at a discount?

Image Source: http://commons.wikimedia.org/wiki/File:Tom_Brady_%28cropped%29.jpg
Considering Other Costs

- **Other costs:**
  - \( g = \) salvage value, $/unit
  - \( B = \) Penalty for not satisfying demand (beyond lost profit), $/unit

- The excess and shortage costs change:
  - \( c_s = p - c + B \)
  - \( c_e = c - g \)
  - Critical Ratio: \[ \frac{c_s}{c_s + c_e} = \frac{p - c + B}{p - c + B + c - g} = \frac{(p - c + B)}{(p + B - g)} \]
Example: NFL Replica Jerseys

- **Data:**
  - Total cost = \( c = 10.90 \) $/jersey
  - Selling price = \( p = 24 \) $/jersey
  - Forecast demand \( \sim N(32000, 11000) \)
  - Salvage value = \( g = 7 \) $/jersey

- **Solution:**
  - \( c_s = p - c = 24 - 10.90 = $13.10 \)
  - \( c_e = c - g = 10.90 - 7.00 = $3.90 \)
  - \( CR = \frac{13.10}{3.9 + 13.10} = 0.771 \)
  - Select \( Q \) where \( P[x \leq Q] = 0.771 \)
    - Normal Table or use spreadsheet:
      - \( =\text{NORMINV}(CR, \text{Mean}, \text{StdDev})=\text{NORMINV}(0.771, 32000, 11000) \)
  - \( Q^* = 40,149 \) - the profit maximizing quantity

But, how do I determine the profitability?
Key Points from Lesson
Key Points

• Newsvendor problems are everywhere
  • Fashion items, perishable goods, fleet sizing, contracting, space missions, etc.
  • Whenever you have to make a firm bet in the face of uncertain demand in a single period

• Classic trade off between:
  • Having too much (excess cost $c_e$)
  • Having too little (shortage cost $c_s$)

• Critical Ratio captures this trade-off
  • $CR = C_s / (C_s + C_e)$
  • $CR = Pct$ of demand distribution to cover
    $= P[x \leq Q]$
Questions, Comments, Suggestions?
Use the Discussion!