Key Concepts: Week 6 Lesson 1: Single Period Inventory Models

Learning Objectives

- Understand the trade-offs between excess and shortage contained within the Critical Ratio
- Ability to use the Critical Ratio to determine the optimal order quantity to maximize expected profits
- Ability to established inventory policies for EOQ with planned back orders as well as single period models

Lesson Summary:

The single period inventory model is second only to the economic order quantity in its wide spread use and influence. Also referred to as the Newsvendor or (for less politically correct folks, the Newsboy) model, the single period model differs from the EOQ in three main ways. First, while the EOQ assumes uniform and deterministic demand, the single-period model allows demand to be variable and stochastic (random). Second, while the EOQ assumes a steady state condition (stable demand with essentially an infinite time horizon), the single-period model assumes a single period of time. All inventories must be ordered prior to the start of the time period and it cannot be replenished during the time period. Any inventory left over at the end of the time period is scrapped and cannot be used at a later time. If there is extra demand that is not satisfied during the period, it too is lost. Third, for EOQ we are minimizing the expected costs while for the single period model we are actually maximizing the expected profitability.

We start the lesson, however, by extending the EOQ model by allowing planned backorders. A planned backorder is where we stock out on purpose knowing that customers will wait, but we do incur a penalty cost, \( c_s \), for stocking out. From this, we develop the idea of the critical ratio (CR), which is the ratio of the \( c_s \) (the cost of shortage or having too little product) to the ratio of the sum of \( c_s \) and \( c_e \) (the cost of having too much or an excess of product). The critical ratio, by definition, ranges between 0 and 1 and is good metric of level of service. A high CR indicates a desire to stockout less frequently. The EOQ with planned backorders is essentially the generalized form where \( c_s \) is essentially infinity, meaning you will never ever stock out. As \( c_s \) gets smaller, the \( Q*_{PBO} \) gets larger and a larger percentage is allowed to be backordered – since the penalty for stocking out gets reduced.

The critical ratio applies directly to the single period model as well. We show that the optimal order quantity, \( Q^* \), occurs when the probability that the demand is less than \( Q^* = \text{the Critical Ratio} \). In other words, the Critical Ratio tells me how much of the demand probability that should be covered in order to maximize the expected profits.
Key Concepts:

Marginal Analysis: Single Period Model

Two costs are associated with single period problems
- Excess cost ($c_e$) when $D<Q$ ($$/unit$) i.e. too much product
- Shortage cost ($c_s$) when $D>Q$ ($$/unit$) i.e. too little product

If we assume continuous distribution of demand
- $c_e P[X\le Q] = \text{expected excess cost of the Qth unit ordered}$
- $c_s (1-P[X\le Q]) = \text{expected shortage cost of the Qth unit ordered}$

This implies that if $E[\text{Excess Cost}] < E[\text{Shortage Cost}]$ then increase $Q$ and that we are at $Q^*$ when $E[\text{Shortage Cost}] = E[\text{Excess Cost}]$. Solving this gives us: $P[x \le Q] = \frac{c_s}{c_e+c_s}$

In words, this means that the percentage of the demand distribution covered by $Q$ should be equal to the Critical Ratio in order to maximize expected profits.

Notation:

- $B$: Penalty for not satisfying demand beyond lost profit ($$/unit$)
- $b$: Backorder Demand (units)
- $b^*$: Optimal units on backorder when placing an order (unit)
- $c$: Purchase cost ($$/unit$)
- $c_I$: Ordering Costs ($$/order$)
- $c_e$: Excess holding Costs ($$/unit/time$); Equal to $ch$
- $c_s$: Shortage Costs ($$/unit$)
- $D$: Average Demand (units/time)
- $g$: Salvage value for excess inventory ($$/unit$)
- $h$: Carrying or holding cost ($$/inventory$/time)
- $L$: Replenishment Lead Time (time)
- $Q$: Replenishment Order Quantity (units/order)
- $Q_{PBO}$: Optimal Order Quantity with Planned backorders
- $T$: Order Cycle Time (time/order)
- $TRC(Q)$: Total Relevant Cost ($$/time$$)
- $TC(Q)$: Total Cost ($$/time$$)
Formulas:

EOQ with Planned Backorders

This is an extension of the standard EOQ with the ability to allow for backorders at a penalty of $c_s$.

\[
TRC(Q, b) = c_t \left( \frac{D}{Q} \right) + c_e \left( \frac{(Q - b)^2}{2Q} \right) + c_s \left( \frac{b^2}{2Q} \right)
\]

\[
Q_{PBO}^* = \sqrt{\frac{2c_tD}{c_e}} \sqrt{\frac{c_sc_e}{c_s}} Q^* \sqrt{\frac{c_s + c_e}{c_s}} = Q^* \sqrt{\frac{1}{CR}}
\]

\[
b^* = \frac{c_e Q_{PBO}^*}{(c_s + c_e)} = \left( 1 - \frac{c_s}{(c_s + c_e)} \right) Q_{PBO}^*
\]

\[
T_{PBO}^* = \frac{D}{Q_{PBO}^*}
\]

Order $Q_{PBO}^*$ when IOH = -b*; Order $Q_{PBO}^*$ every $T_{PBO}^*$ time periods

Single Period (Newsvendor) Model

We found that to maximize expected profitability, we need to order sufficient inventory, Q, such that the probability that the demand is less than or equal to this amount is equal to the Critical Ratio. Thus, the probability of stocking out is equal to $1 - CR$.

\[
P[x \leq Q] = \frac{c_s}{(c_e + c_s)}
\]

For the simplest case where there is neither salvage value nor extra penalty of stocking out, these become:

$c_s = p - c$, that is the lost margin of missing a potential sale and
$c_e = c$, that is, the cost of purchasing one unit

The Critical Ratio becomes: $CR = \frac{c_s}{c_s + c_e} = \frac{(p-c)}{(p-c+c)} = \frac{p-c}{p}$ which is simply the margin divided by the price!

When we consider also salvage value (g) and shortage penalty (B), these become:

$c_s = p - c + B$, that is the lost margin of missing a potential sale plus a penalty per item short and
$c_e = c - g$, that is, the cost of purchasing one unit minus the salvage value I can gain back.

Now the critical ratio becomes

\[
CR = \frac{c_s}{c_s + c_e} = \frac{(p-c+B)}{(p-c+B+c-g)} = \frac{(p-c+B)}{(p+B-g)}
\]
**Additional References:**

Single Period Inventory models are covered in Nahmias Chpt 5 and Silver, Pyke & Peterson Chpt 10, and Ballou Chpt 9.