Data Analysis: Statistical Modeling and Computation in Applications

Time Series Analysis: Statistical Models and Fitting

Overview

- Statistical models
- Time series regression
- Fitting statistical models
- Forecasting

Recap

- **Time series:** collection of observations $x_1, \ldots, x_n$ indexed by time (fixed time intervals)
- realizations of a collection of random variables $X_1, \ldots, X_n$

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(1) mean, variance independent of t: $\mu_X(t) = \mu$, $\text{var}_X(t) = \text{var}_X$
(2) autocovariance: $\gamma_X(s, t) = \gamma(|s - t|)$.

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- **Autocovariance / Autocorrelation:**

$$\gamma_X(s,t) = \text{cov}(X_s, X_t)$$

$$\rho_X(h) = \gamma_X(h)/\gamma_X(0) \qquad \text{for } -n < h < n$$

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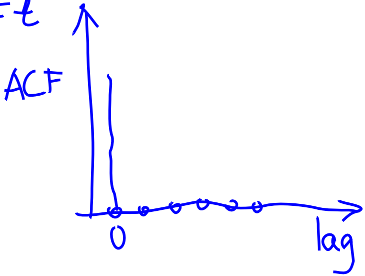
- often iid, e.g. Gaussian

$$\mathrm{cov}(X_s, X_t) = \begin{cases} 0 & s \neq t \\ \sigma^2 & s = t \end{cases}$$



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- stationary
- checking for white noise: for white noise, $\hat{\rho}(h)$ is approximately $\mathcal{N}(0, \frac{1}{n})$, under mild conditions.

Models 2: Autoregressive AR(p)

- **Autoregressive:** $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots \phi_p X_{t-p} + W_t$

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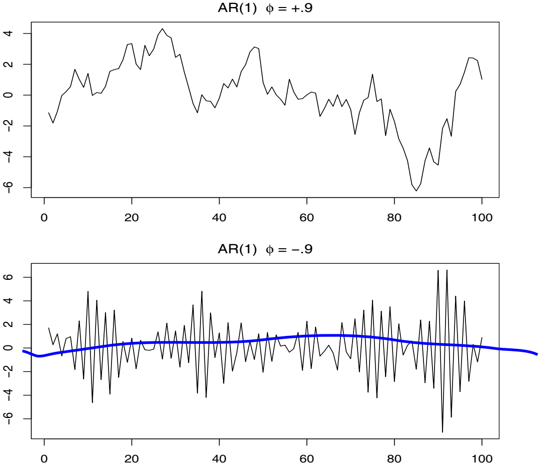


source: R.H. Shumway, D.S. Stoffer. Time Series Analysis and its Applications, with Examples in R. Springer, 2011.

Models 2: Autoregressive AR(p)

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots \phi_p X_{t-p} + W_t$$

- Can we model seasonality with this?

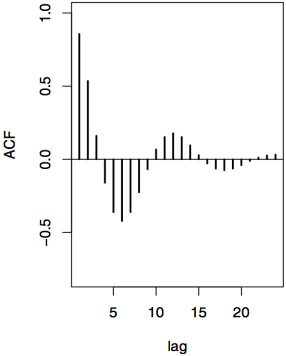
$$X_{june21} = X_{jun20} + W_{june21}$$

$$X_t = X_{t-12} + W_t$$

Models 2: Autoregressive AR(p)

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots \phi_p X_{t-p} + W_t$$

$$X_t = 1.5 X_{t-1} - 0.75 X_{t-2} + W_t$$



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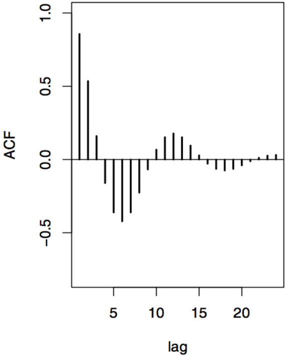
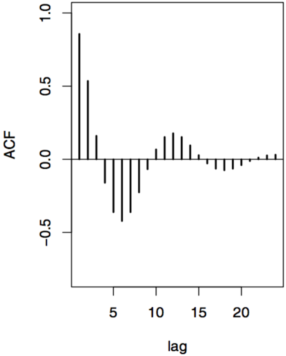


Models 2: Autoregressive AR(p)

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots \phi_p X_{t-p} + W_t$$

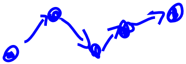
- autocorrelation decays exponentially (never zero)
- stationary only under conditions *(later)*

$$X_t = 1.5X_{t-1} - 0.75X_{t-2} + W_t$$



Models 3: Random Walk (with drift)

$$X_t = X_{t-1} + W_t + \delta$$



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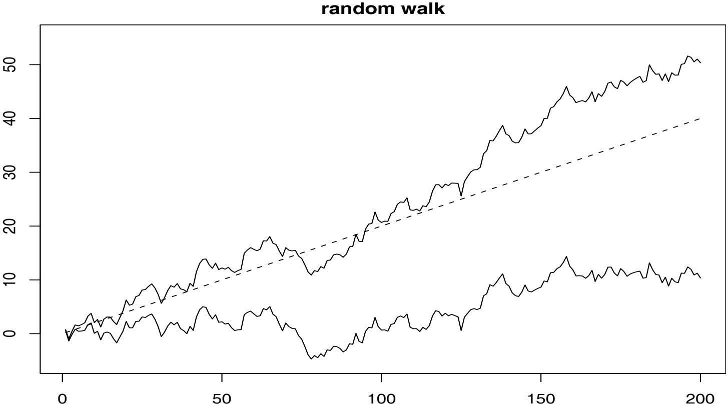

Random Walk: properties

$$X_t = X_{t-1} + W_t \quad + \delta$$

$$= X_{t-2} + W_{t-1} + W_t \quad + 2\delta$$

$$= X_0 + W_1 + W_2 + \dots + W_t \quad + t \cdot \delta$$

Random Walk: properties

- Random Walk:

$$\begin{aligned} X_t &= X_{t-1} + W_t \\ &= X_0 + W_1 + \ldots + W_t \end{aligned}$$

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  (assume $X_0 = 0$ fixed)
- Random Walk with drift: $X_t = \delta + X_{t-1} + W_t = t\delta + X_0 + \sum_{s=0}^{t} W_s$

Random Walk: properties

- Random Walk:

$$\begin{aligned} X_t &= X_{t-1} + W_t \\ &= X_0 + \underbrace{W_1 + \ldots + W_t} \end{aligned}$$

(assume $X_0 = 0$ fixed)

- Random Walk with drift: $X_t = \delta + X_{t-1} + W_t = t\delta + X_0 + \sum_{s=0}^{t} W_s$

- $\mathbb{E}[X_t] = X_0 + t \cdot \delta$

Random Walk: properties

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- $\mathbb{E}[X_t] = t\delta + X_0$
- without drift: $\mathrm{var}(X_t) = t\sigma_w^2$

Random Walk: properties

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- Random Walk:

$$
\begin{aligned}
X_t &= X_{t-1} + W_t \\
&= \underbrace{X_0}_{=0} + W_1 + \ldots + W_t
\end{aligned}
$$

$$\operatorname{cov}(X_s, X_t) = \operatorname{cov}\left(\sum_{i=1}^{s} W_i, \sum_{j=1}^{t} W_j\right)$$

$$= \sum_{i=1}^{s} \sum_{j=1}^{t} \operatorname{cov}(W_i, W_j)$$

$$= \min(s,t) \cdot \sigma^2$$

Random Walk: properties

- Random Walk:

$$\begin{aligned} X_t &= X_{t-1} + W_t \\ &= X_0 + W_1 + \ldots + W_t \end{aligned}$$

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$\nabla X_t = X_t - X_{t-1}$
$= W_t$

- $\mathbb{E}[X_t] = t\delta + X_0$
- without drift: $\text{var}(X_t) = t\sigma_w^2$
- without drift: $\gamma_X(s,t) = \min\{s,t\}\sigma_w^2$
- *not stationary*, but ∇X_t is stationary

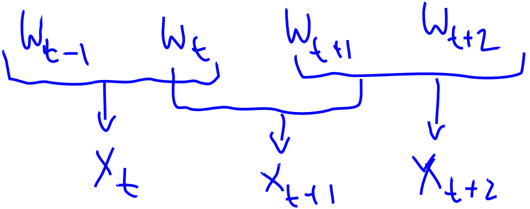
Overview

- Statistical models
  - White noise
  - Autoregressive and Random Walk
  - Moving Average
  - ARMA
- Time series regression
- Fitting statistical models
- Forecasting

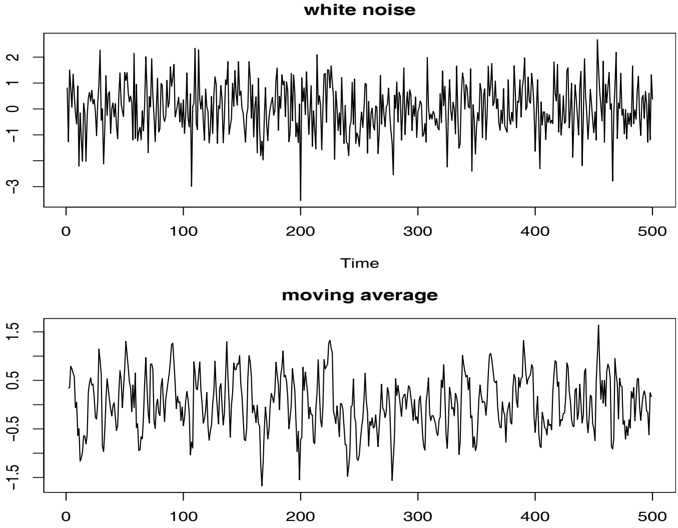
(1) $W_{t-1} \quad W_t \quad W_{t+1} \; \ldots$

(2) $X_t = W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} \ldots \theta_q W_{t-q}$

$q=1$



Models 4: Moving Average



Models 4: Moving Average MA(q)

$$X_t = W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \ldots + \theta_q W_{t-q}$$

- $\mathbb{E}[X_t] =$

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Models 4: Moving Average MA(q)

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- $\mathbb{E}[X_t] = 0$
- autocovariance?

Autocovariance for Moving Average MA(q)

- Example: $X_t = \frac{1}{3}(W_t + W_{t-1} + W_{t-2})$

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$$\mathrm{cov}(X_t, X_t) =$$

Autocovariance for Moving Average MA(q)

- Example: $X_t = \frac{1}{3}(W_t + W_{t-1} + W_{t-2})$

$$\text{cov}(X_t, X_t) = \frac{1}{9}\,\text{cov}(W_t + W_{t-1} + W_{t-2}, W_t + W_{t-1} + W_{t-2})$$

Autocovariance for Moving Average MA(q)

- Example: $X_t = \frac{1}{3}(W_t + W_{t-1} + W_{t-2})$

$$\begin{aligned}
\text{cov}(X_t, X_t) &= \frac{1}{9}\,\text{cov}(W_t + W_{t-1} + W_{t-2}, W_t + W_{t-1} + W_{t-2}) \\
&= \frac{3}{9}\sigma^2
\end{aligned}$$

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&= \tfrac{2}{9}\sigma^2
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\operatorname{cov}(X_t, X_{t-2}) &
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Autocovariance for Moving Average MA(q)

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&= \tfrac{1}{9}\sigma^2
\end{aligned}$$

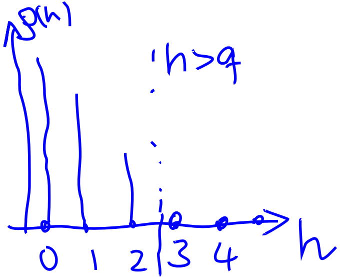
Autocovariance for Moving Average MA(q)

$q=2$

- Example: $X_t = \frac{1}{3}(W_t + W_{t-1} + W_{t-2})$

$$
\begin{aligned}
0 \quad \mathrm{cov}(X_t, X_t) &= \tfrac{1}{9}\,\mathrm{cov}(W_t + W_{t-1} + W_{t-2}, W_t + W_{t-1} + W_{t-2}) \\
&= \tfrac{3}{9}\sigma^2 \\
1 \quad \mathrm{cov}(X_t, X_{t-1}) &= \tfrac{1}{9}\,\mathrm{cov}(W_t + W_{t-1} + W_{t-2}, W_{t-1} + W_{t-2} + W_{t-3}) \\
&= \tfrac{2}{9}\sigma^2 \\
2 \quad \mathrm{cov}(X_t, X_{t-2}) &= \tfrac{1}{9}\,\mathrm{cov}(W_t + W_{t-1} + W_{t-2}, W_{t-2} + W_{t-3} + W_{t-4}) \\
&= \tfrac{1}{9}\sigma^2 \\
3 \quad \mathrm{cov}(X_t, X_{t-3}) &= 0
\end{aligned}
$$

$\gamma(h)$



Autocovariance for Moving Average MA(q)

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$$
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\end{aligned}
$$

- autocovariance γ depends only on $|s - t|$

Autocovariance for Moving Average MA(q)

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\operatorname{cov}(X_t, X_{t-3}) &= 0
\end{aligned}
$$

- autocovariance γ depends only on $|s-t|$
- reflects order: $\gamma(s,t) = 0$ if $\underbrace{|s-t|}_{h} > q$

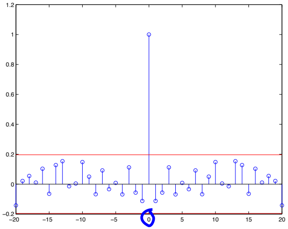
Models 4: Moving Average MA(q)

$$X_t = W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \ldots + \theta_q W_{t-q}$$

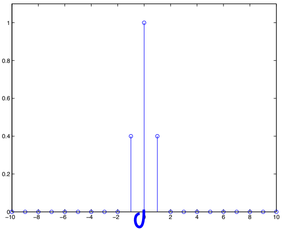
- $\mathbb{E}[X_t] = 0$

- autocovariance γ depends only on $|s-t| \Rightarrow$ *stationary*
- ACF reflects order: $\gamma(s,t) = 0$ if $|s-t| > q$
- ACF distinguishes MA and AR models!

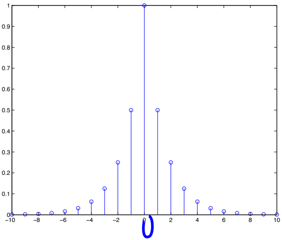
Autocorrelation to distinguish models



White Noise



MA



AR

Models 5: ARMA

- **Autoregressive:** $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots \phi_p X_{t-p} + W_t$
- **Moving Average:** $X_t = W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \ldots + \theta_q W_{t-q}$

Models 5: ARMA

- **Autoregressive:** $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots \phi_p X_{t-p} + W_t$
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- **ARMA(p,q)**: autoregressive moving average:

$$\begin{aligned} X_t =& \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots \phi_p X_{t-p} \\ &+ W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \dots + \theta_q W_{t-q} \end{aligned}$$

Models 5: ARMA

- **Autoregressive:** $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots \phi_p X_{t-p} + W_t$
- **Moving Average:** $X_t = W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \ldots + \theta_q W_{t-q}$

- **ARMA(p,q)**: autoregressive moving average:

$$\begin{aligned} X_t =& \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots \phi_p X_{t-p} \\ & + W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \ldots + \theta_q W_{t-q} \end{aligned}$$

- **ARIMA**: ARMA after differencing

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  - Moving Average
  - ARMA
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- Fitting statistical models
- Forecasting

Regression & Time Series

- critical: linearity, stationarity, homogeneity of variances over time

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- *Which external variables? Order of the model?*

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Also PACF (coming soon).

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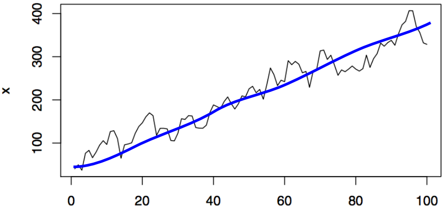

$X_t \sim Y$ $\quad X_t - \hat{T}_t$

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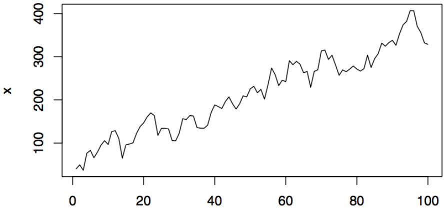
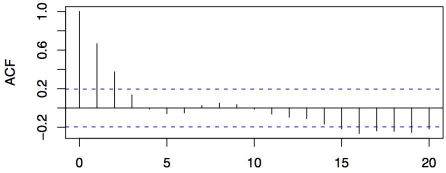
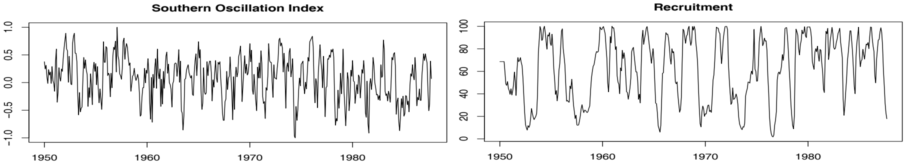




left: series; right: autocorrelation of residuals after fitting only a linear trend.

Example: external regressors - fish and SOI



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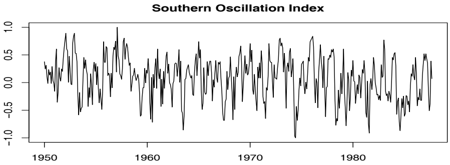

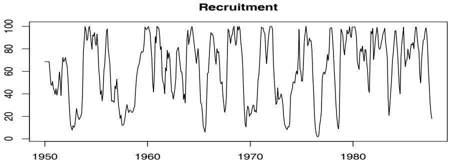

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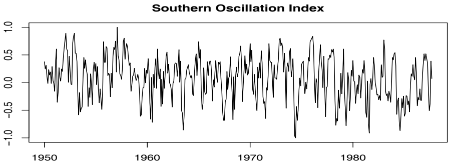

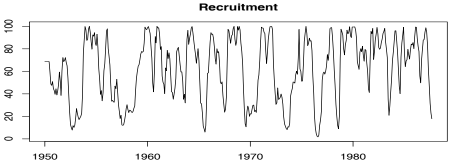

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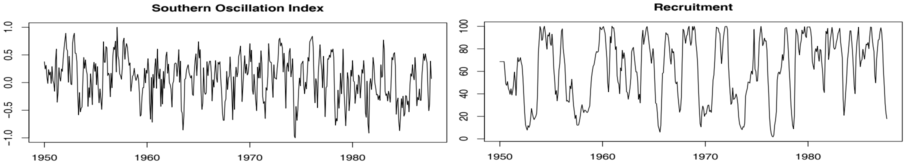

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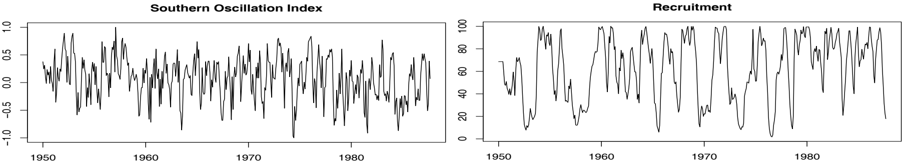

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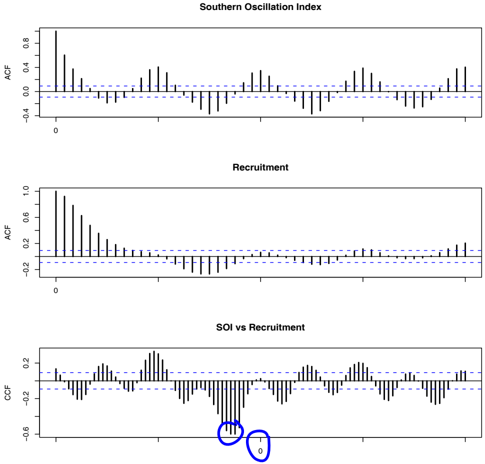

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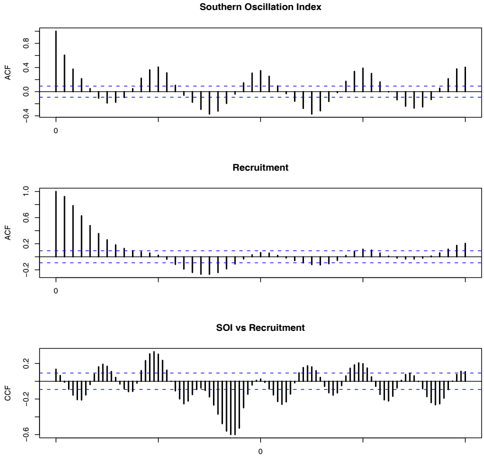

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CCF $\rho_{XY}(h)$ peaks negatively at $h = -6$
(SOI X_t 6 months before).

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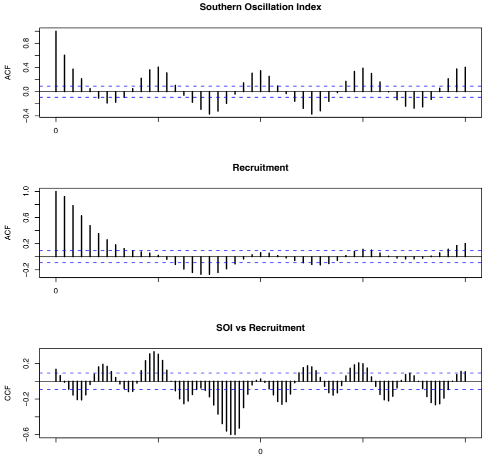

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Possible regression model:
$Y_t = \beta_1 + \beta_2 X_{t-6} + W_t$

Overview

- Statistical models
  - White noise
  - Autoregressive and Random Walk
  - Moving Average
  - ARMA
- Time series regression
- Fitting statistical models
- Forecasting

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Fitting AR(p):

1. compute PACF to get order
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Fitting ARMA(p,q):

1. attempt to fit an AR model, compute residuals
2. attempt to fit an MA model to residuals (or orig. data)
3. fit ARMA(p,q) using p, q determined in Steps 1,2 (max. likelihood) (won't cover, see Shumway & Stoffer)
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$$\Leftrightarrow \quad \mathbb{E}\big[(\hat{\phi}_1 X_{t-1} + \hat{\phi}_2 X_{t-2} - X_t)X_{t-2}\big] = 0$$

Where do Yule-Walker equations come from? Example

- **Equations:** $\mathbb{E}[(\hat{X}_t - X_t)X_{t-k}] = 0$ for $k = 1, \dots, p$

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$$\Leftrightarrow \quad \mathbb{E}\hat{\phi}_1 X_{t-1}X_{t-2} + \mathbb{E}\hat{\phi}_2 X_{t-2}X_{t-2} - \mathbb{E}X_t X_{t-2} = 0$$

$$\hat{\phi}_1 \gamma(1) + \hat{\phi}_2 \gamma(0) - \gamma(2) = 0$$

$$\Rightarrow \gamma(2) = \hat{\phi}_1 \gamma(2-1) + \hat{\phi}_2 (2-2)$$

$$\gamma(h) = \hat{\phi}_1 \gamma(h-1) + \hat{\phi}_2 \gamma(h-2)$$

$p=2$

Overview

- Statistical models
  - White noise
  - Autoregressive and Random Walk
  - Moving Average
  - ARMA
- Time series regression
- Fitting statistical models
- Forecasting

- predict X_{n+m} based on observed x_n, x_{n-1}, $\ldots x_1$.
- estimate coefficients $\hat{\phi}_1, \ldots, \hat{\phi}_p$ and plug in

$$\hat{x}_{n+1} = \hat{\phi}_1 x_n + \hat{\phi}_2 x_{n-1} + \ldots + \hat{\phi}_p x_{n-p+1}$$

$$\hat{x}_{n+2} = \underbrace{\hat{\phi}_1 \hat{x}_{n+1}}_{\phi_2} + \hat{\phi}_2 x_n + \ldots \hat{\phi}_p x_{n-p+2}$$

Forecasting with AR(p)

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- **Caution:** only for short horizons m. For long horizons, converges to mean.

Forecasting with AR(p)

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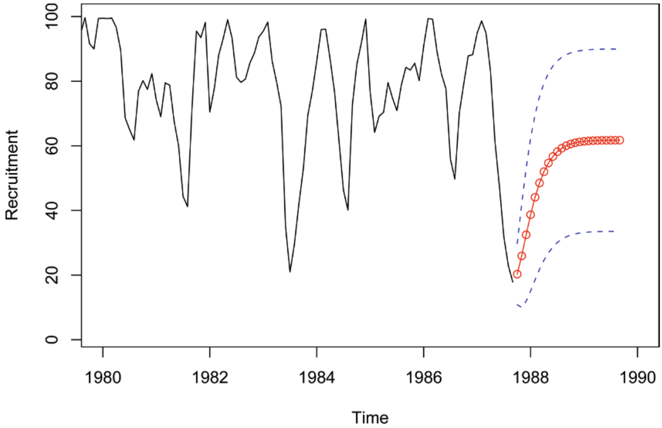
Caution: Fine for short term, but converges to mean for long horizons



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- Forecasting: mostly for short-term

References and Further Reading

- Paul S.P. Cowpertwait and Andrew V. Metcalfe. Introductory Time Series with R. Springer, 2009.
  Chapters 4–6
- R.H. Shumway, D.S. Stoffer. Time Series Analysis and its Applications, with Examples in R. Springer, 2011.
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