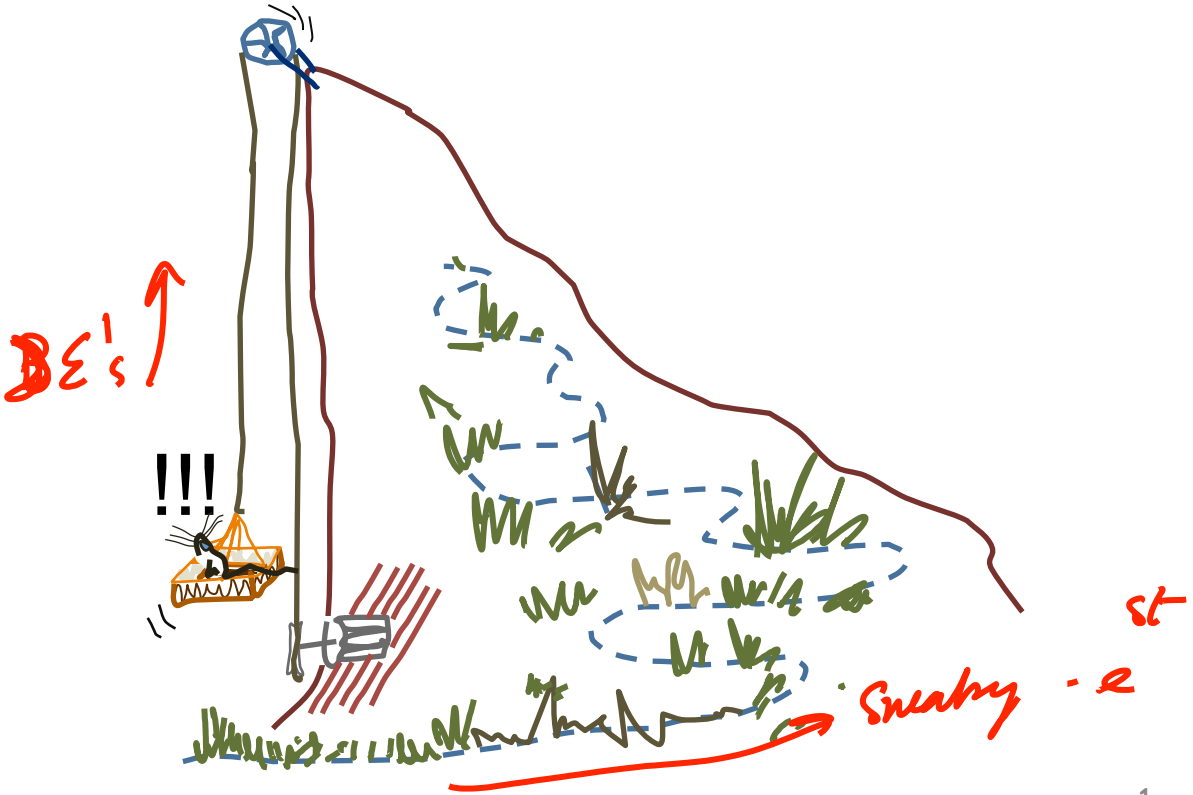
6.002x

CIRCUITS AND ELECTRONICS

The Impedance Model

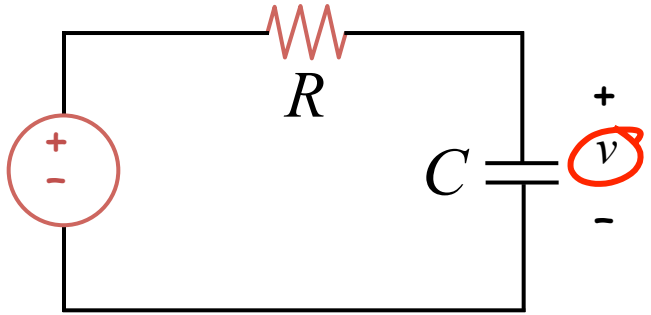


Reading: Section 13.3 from text

Review

- Sinusoidal Steady State (SSS)
  Reading 13.1, 13.2

$$v_I = V_i \cos \omega t$$

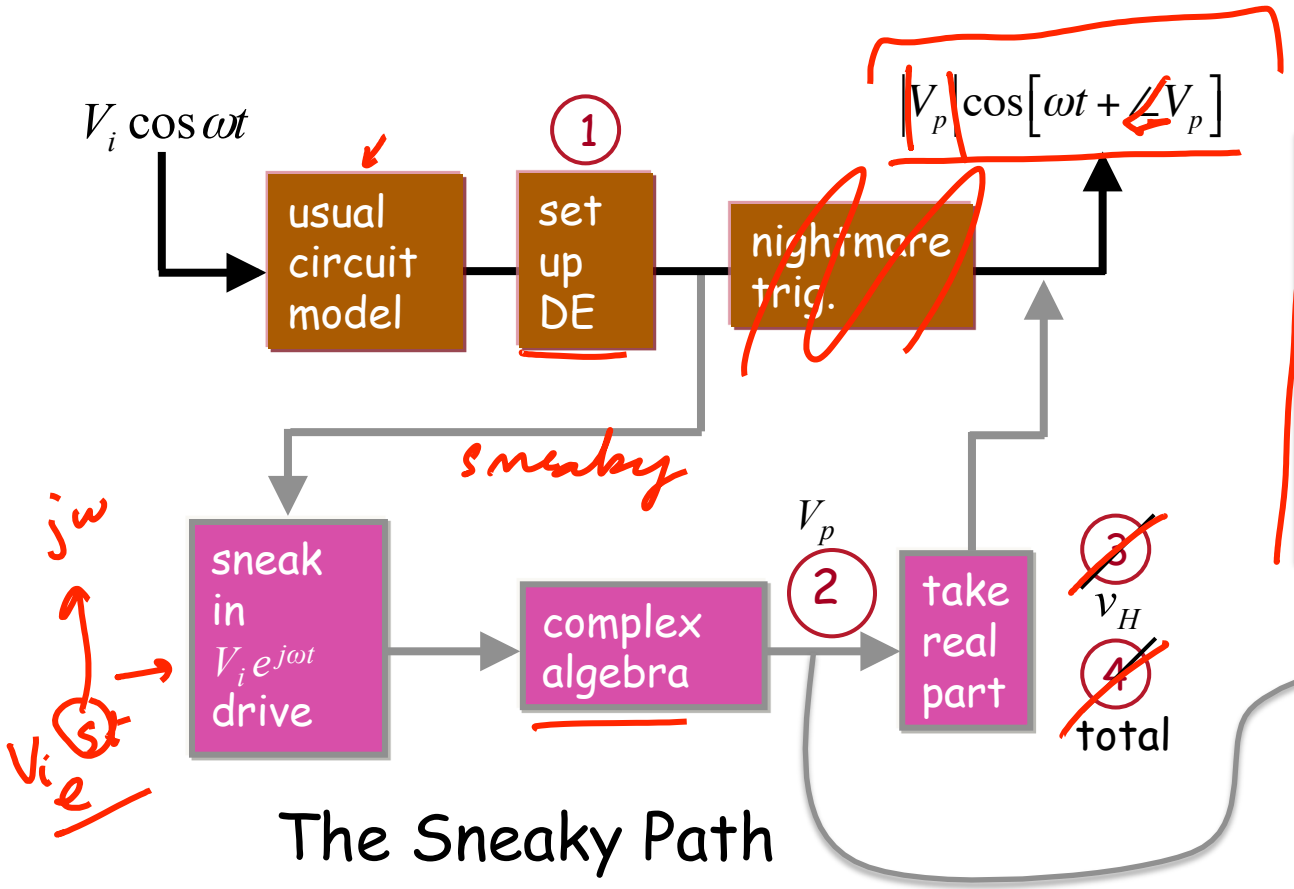


SSS

- Focus on sinusoids.
- Focus on steady state, only care about v_P as v_H dies away.

Review

Sinusoidal Steady State SSS Approach



V_p contains all the information we need:

- $|V_p|$ Amplitude of output cosine
- $\angle V_p$ Phase

$$V_p e^{j\omega t}$$

$$\frac{V_i}{1 + j\omega RC}$$

$s = j\omega$

complex amplitude

Review

Frequency response - magnitude

$$\frac{V_p}{V_i} = \frac{1}{1 + j\omega RC} = H(j\omega)$$

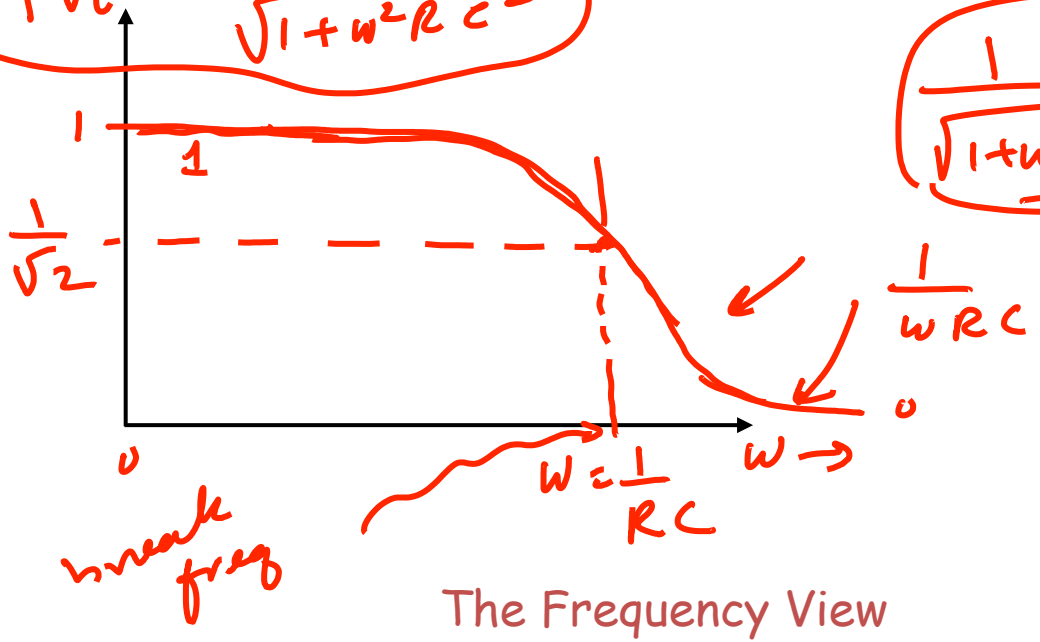
transfer function

$$V_p = \frac{V_i}{1 + j\omega RC}$$

$$v = |V_p| \cos(\omega t + \angle V_p)$$

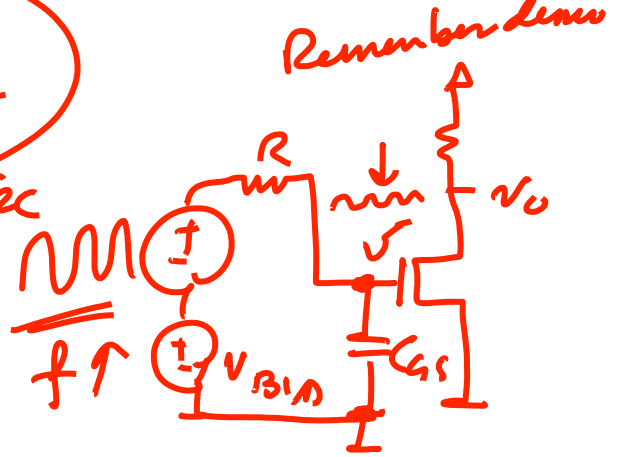
mag phase

$$\left|\frac{V_p}{V_i}\right| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$



$$\frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

ωRC

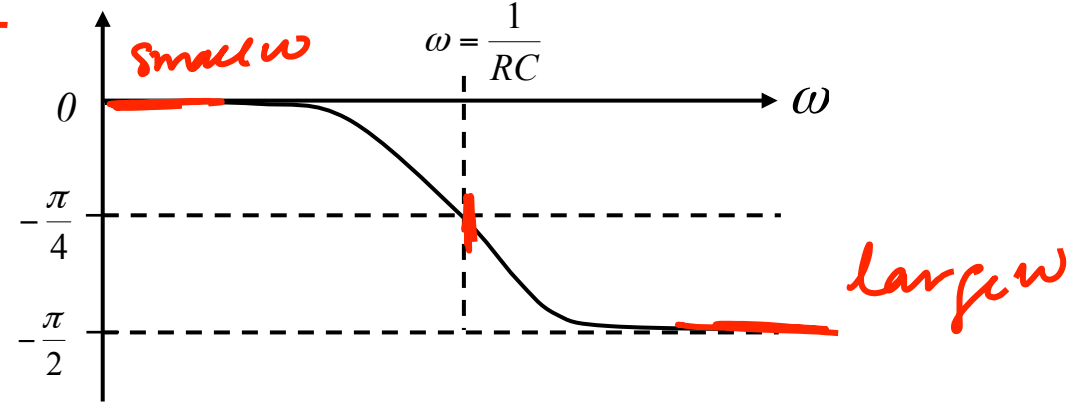


Complex algebra appendix in text

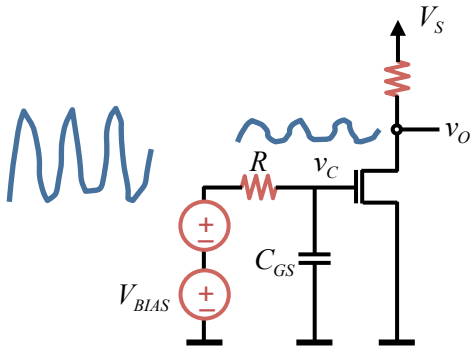
Review

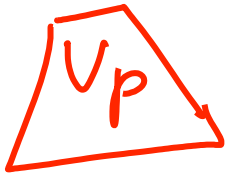
Frequency response - phase

$$\angle \frac{V_p}{V_i} = tan^{-1}\left(\frac{-\omega RC}{1}\right)$$



The Frequency View



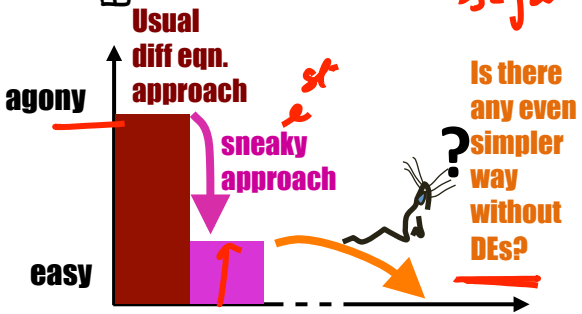


Is there an even simpler way to get V_p ?

$$V_p = \frac{V_i}{1 + j\omega RC} = \frac{V_i}{1 + sRC}$$

$$V_p = \frac{V_i}{1 + j\omega RC}$$

$s = j\omega$



Divide numerator and denominator by sC

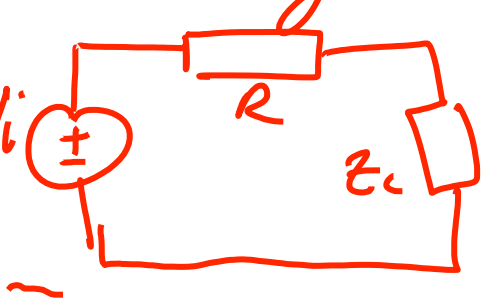
$$V_p = V_i \frac{\frac{1}{sC}}{\frac{1}{sC} + R}$$

$\frac{1}{sC} \rightarrow z_c$

$\frac{1}{sC} + R \rightarrow z_c$

Hmmm.... looks like a voltage divider relation.

$$V_p = V_i \frac{z_c}{z_c + R}$$



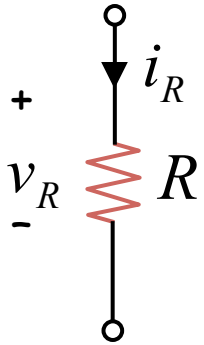
Let's explore this further.

The Impedance Model

Is there an even simpler way to get V_p?

Consider resistor:

Resistor



When $s = j\omega$ shorthand as usual

$i_R = I_r e^{st}$ (complex ampl.)

$v_R = V_r e^{st}$

We know from last sequence that response to e^{st} is also of this form e^{st}

$\rightarrow \boxed{v_R = R\, i_R}$

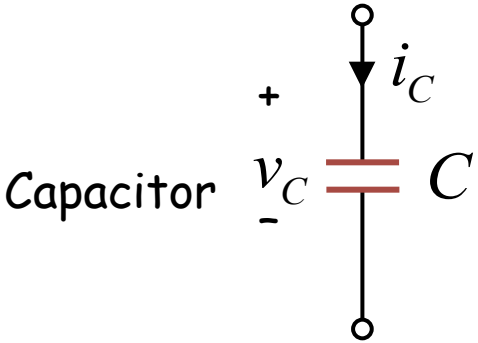
$V_r e^{st} = R\, I_r e^{st}$

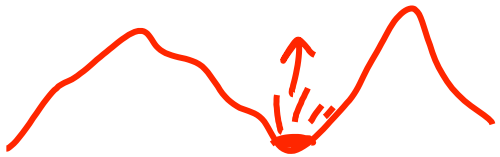
$\underline{V_r = R\, I_r}$

In other words, if $v_R = R\, i_R$, then complex amplitudes related as $V_r = R\, I_r$

The Impedance Model -- Capacitor

Consider capacitor:





complex ampl

$i_C = I_C e^{st}$

$v_C = V_C e^{st}$

$i_C = C \frac{dv_C}{dt}$ ← $V_C e^{st}$

$I_C e^{st} = C V_C s e^{st}$

$I_C = C V_C s$

$V_C = \frac{1}{sC} \cdot I_C$

Z_C

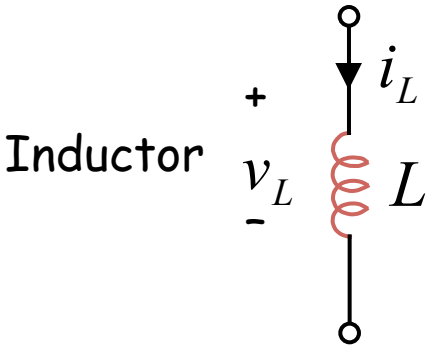
impedance

$V_C = Z_C I_C$

In other words, complex amplitudes related as "$V = IR$" !!!!

The Impedance Model -- Inductor

Consider Inductor:



$i_L = I_e e^{st}$

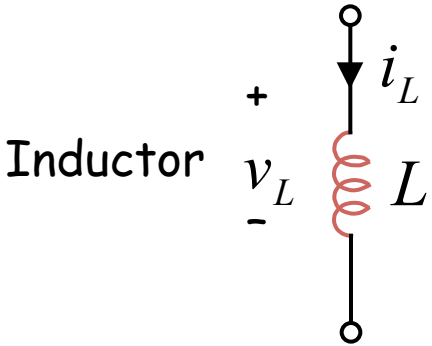
$v_L = V_e e^{st}$

$v_L = L \frac{di_L}{dt}$

$V_e = ?$

The Impedance Model -- Inductor

Consider Inductor:



$$i_L = I_l e^{st}$$

$$v_L = V_l e^{st}$$

$$v_L = L\frac{di_L}{dt}$$

$I_l e^{st}$

$V_l e^{st} = L \cdot I_l \cdot s e^{st}$

$V_l = Ls\, I_l$

Ls → Z_L (impedance)

$s = jw$

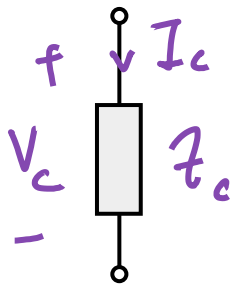
$V_l = Z_L I_l$

$Z_L = Ls$ or Ljw

$V_l = jwL\, I_l$

The Impedance Model -- Summary

capacitor

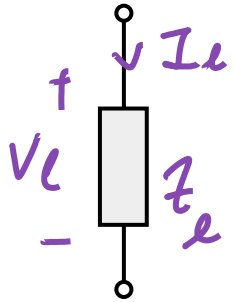


$V_c = Z_c I_c$

$Z_c = \frac{1}{sC}$

"impedance"

inductor

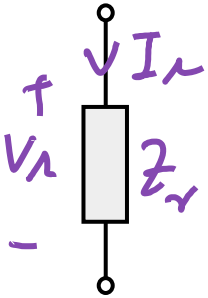


$V_l = Z_l I_l$

$Z_l = sL$

"$V = IR$" $j\omega$

resistor



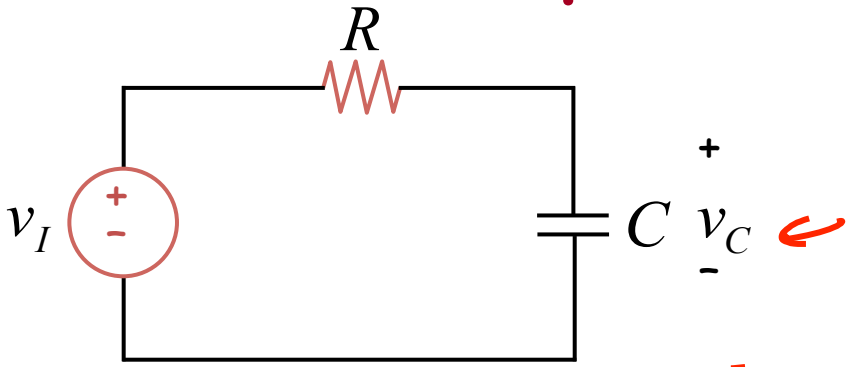
$V_R = Z_R I_R$

$Z_R = R$

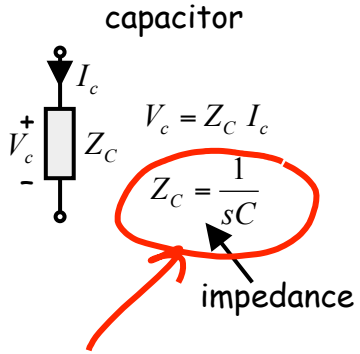
□ → box

For a drive of the form $V_i e^{st}$, the complex amplitude of V_c is related to the complex amplitude I_c algebraically by a generalization of Ohm's Law

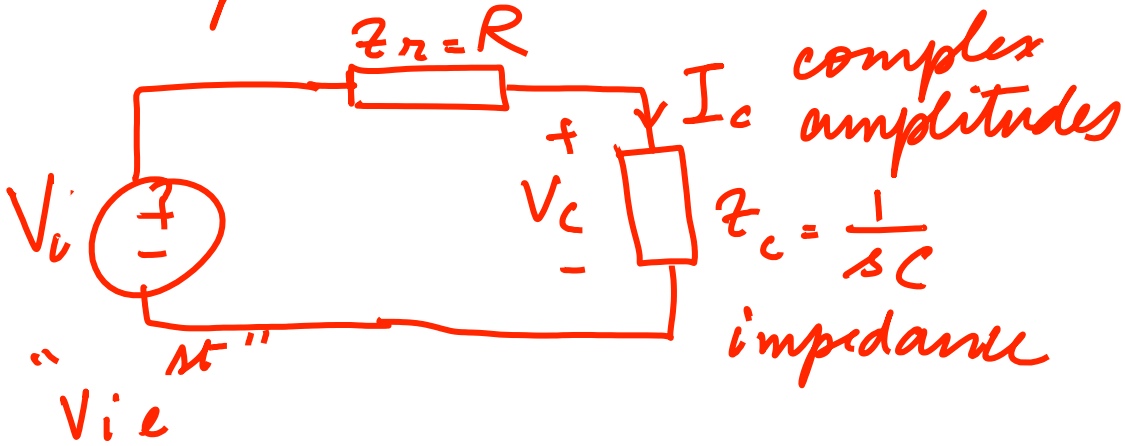
Back to RC example…



Is there an even simpler way to get V_p ?

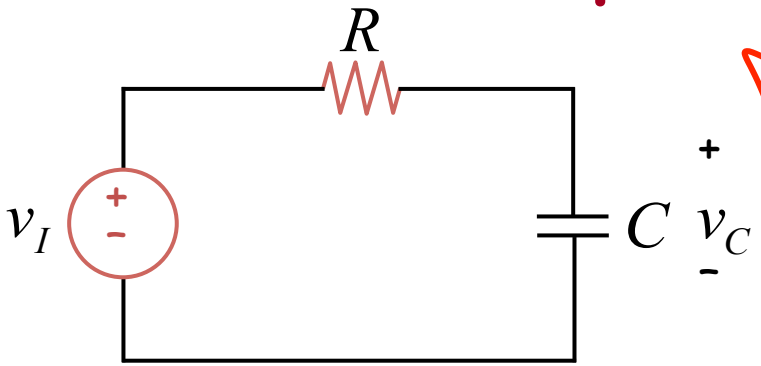


Impedance circuit model:

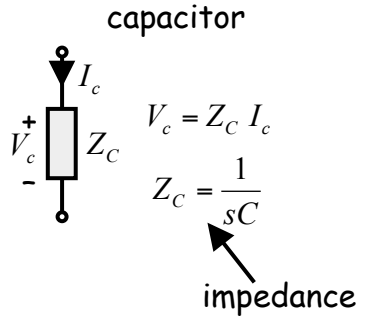


Find V_c

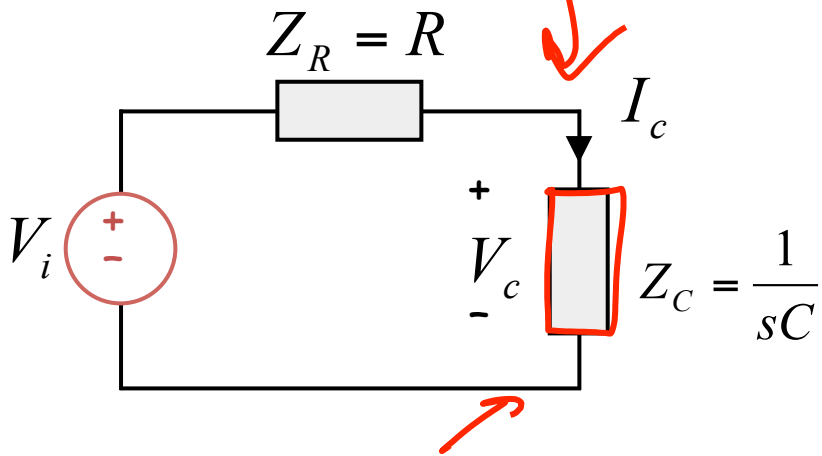
Back to RC example…



Is there an even simpler way to get V_p ?



Impedance model:



$$V_c = \frac{Z_c}{Z_c + Z_R} \cdot V_i = \frac{\frac{1}{sC}}{\frac{1}{sC} + R} \cdot V_i$$

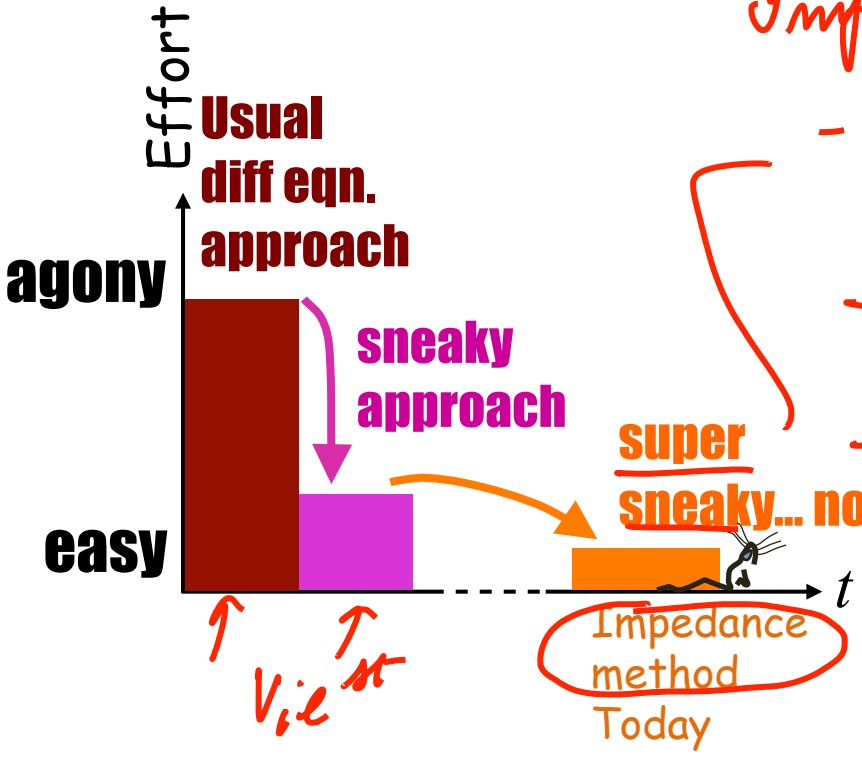
$$\times \frac{sC}{sC} \qquad V_c = \frac{1}{1 + sCR} V_i$$

done!

This is the famed complex amplitude "V_p"

- all of our old friends apply, KVL, KCL, superpo.
- $1 + sCR$? char eq !!!!

There you have it



Impedance method:

- Elements → impedance boxes
- circuit topology same
- analyze algebraically

KVL, KCL, node, superposition, Thevenin...

$R, C - v_c$

$\tau_v \dot{v}_c - V_c$

Signal Notation

V_a $\underline{V_p}$

- V_A upper case symbols, upper case subscripts - DC or op. pt. variables
- v_A Total instantaneous variables
- v_a Incremental or small signal instantaneous variables
- V_a Complex amplitudes (total var or incremental vars). Recall $\underline{V_i}$ comp- they also represent real amplitude of sinusoidal source

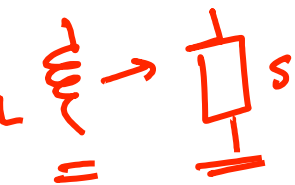
Impedance Method Summary

looking for SSS

page 716 textbook

1. Replace sinusoidal sources by their complex (or real) amplitudes. $V_i \cos\omega t \rightarrow \underline{V}_i$

2. Circuit elements replaced with boxes (impedances), R → R boxes, L's with Ls boxes, C's with $\frac{1}{sC}$ boxes. Here $s = j\omega$ Circuit topology same. Impedance model of network.

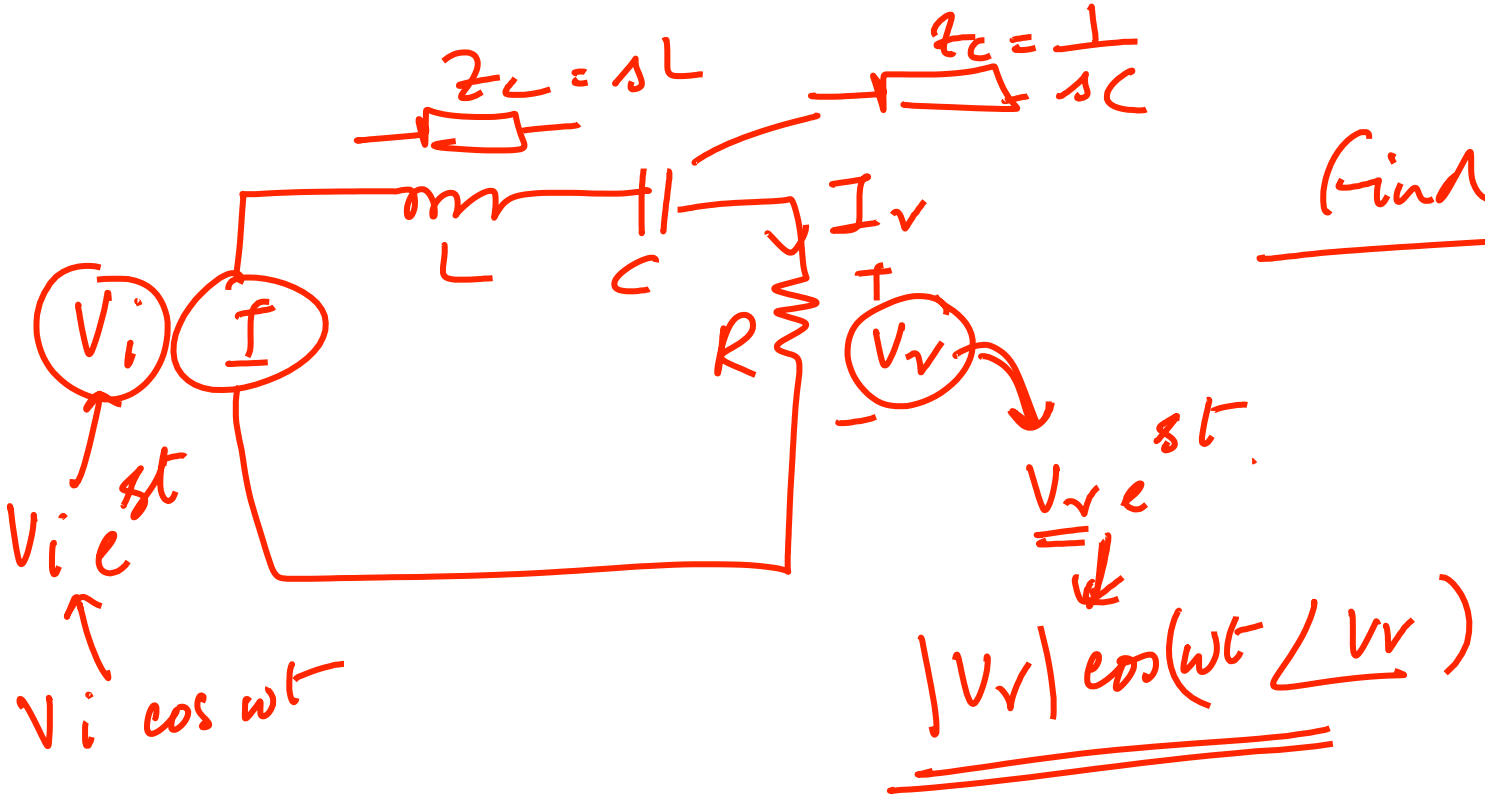


3. Determine complex amplitudes of the node voltages and branch currents using any method such as node

4. Not super, ele- usually needed. Obtain time vars. from complex amplitudes. $V_a \rightarrow v_A = |V_a| \cos(\omega t + \angle V_a)$

Another example, recall series RLC

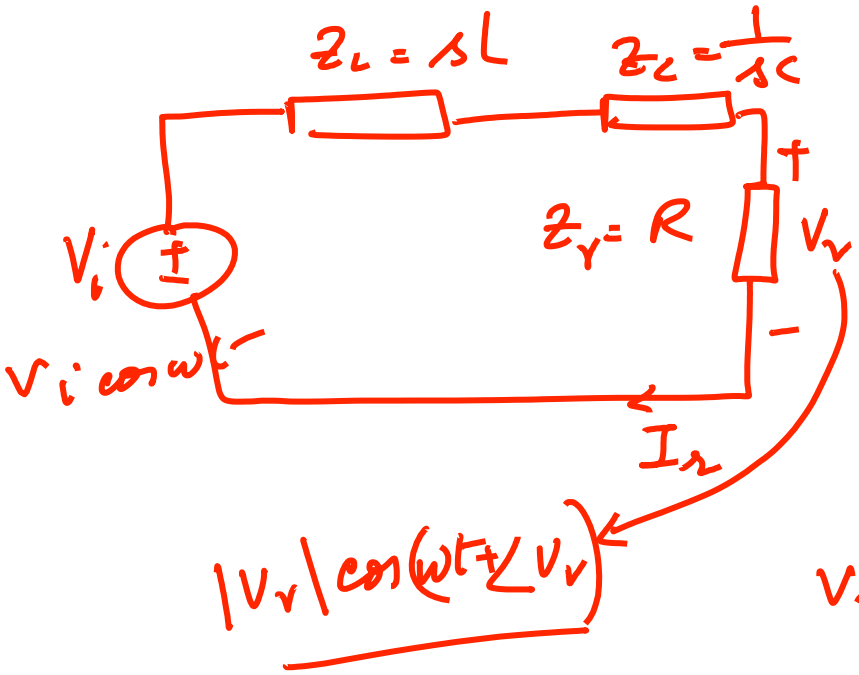
Remember, we want only the steady-state response to sinusoid (SSS)



Find V_V

Another example, recall series RLC

Remember, we want only the steady-state response to sinusoid (SSS)



$s = j\omega$

$$V_r = \frac{V_i \cdot Z_R}{Z_L + Z_C + Z_R}$$

$$V_r = \frac{V_i R}{sL + \frac{1}{sC} + R} = \frac{V_i R \cdot \frac{s}{L}}{s^2 + \frac{1}{LC} + \frac{R}{L}s}$$

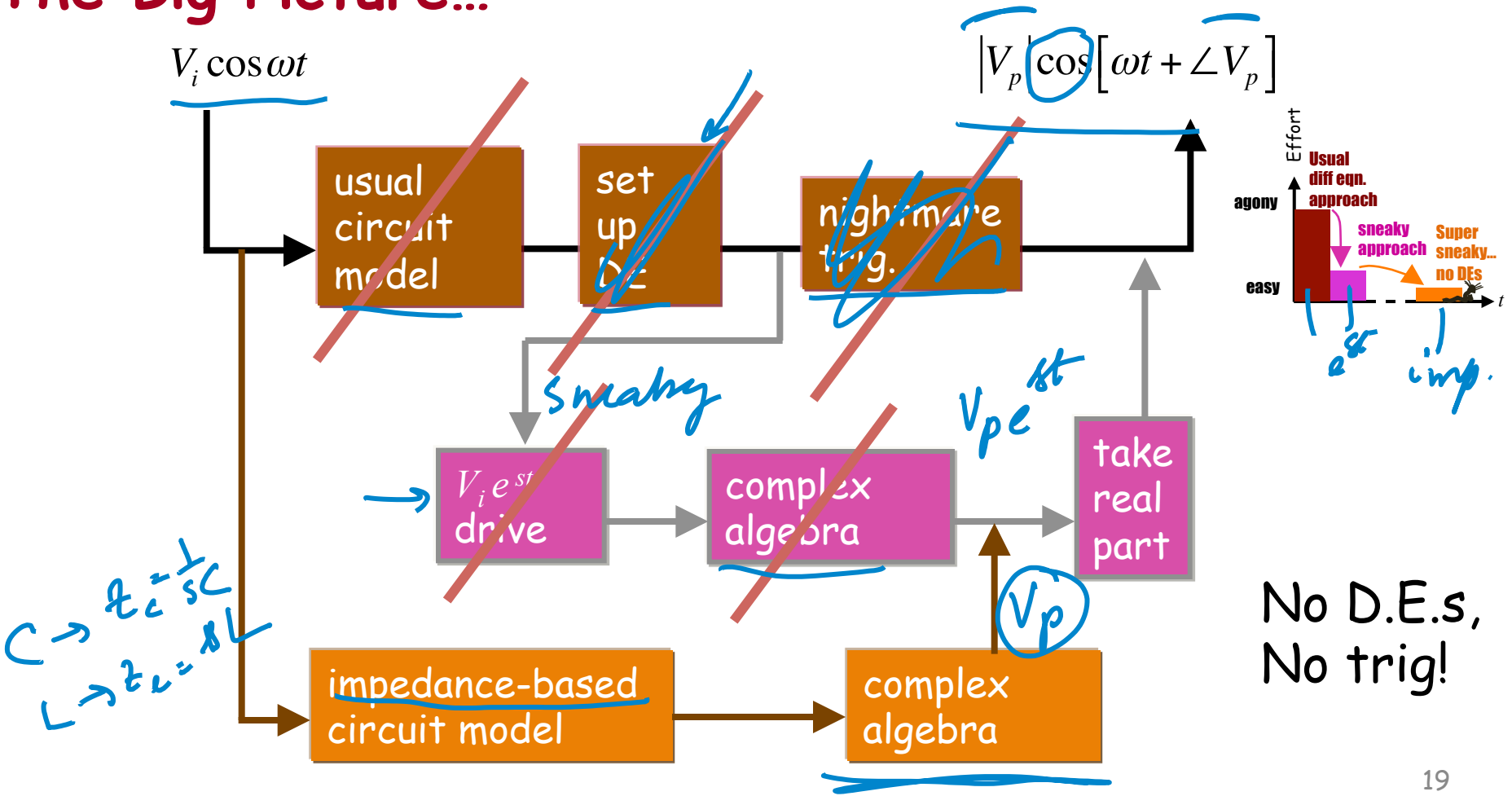
$s^2 + \frac{R}{L}s + \frac{1}{LC}$? char. eq ? !! can get TD behavior !

$s = j\omega$

$$V_r = \frac{V_i \frac{R}{L} \cdot j\omega}{-\omega^2 + \frac{R}{L} j\omega + \frac{1}{LC}}$$

* we will study this and other functions in more detail in the next sequence

The Big Picture…

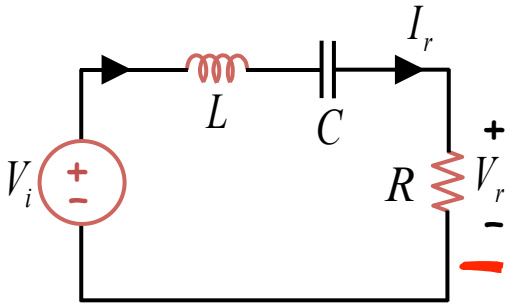


No D.E.s,
No trig!

Back to

Let's study this transfer function

$$\frac{V_r}{V_i} = \frac{\frac{j\omega R}{L}}{\frac{1}{LC} + \frac{j\omega R}{L} - \omega^2}$$



$$\frac{V_r}{V_i} = \frac{jwRC}{1 + jwRC - w^2LC} = \frac{jwRC}{(1 - w^2LC) + jwRC}$$

$$= \frac{jwRC}{[(1-w^2LC)+jwRC]} \cdot \frac{[(1-w^2LC)-jwRC]}{[(1-w^2LC)-jwRC]}$$

$$= \frac{jwRC\,[(1-w^2LC)-jwRC]}{(1-w^2LC)^2 + (wRC)^2}$$

$$\left|\frac{V_r}{V_i}\right| = \frac{wRC}{\sqrt{(1-w^2LC)^2 + (wRC)^2}}$$

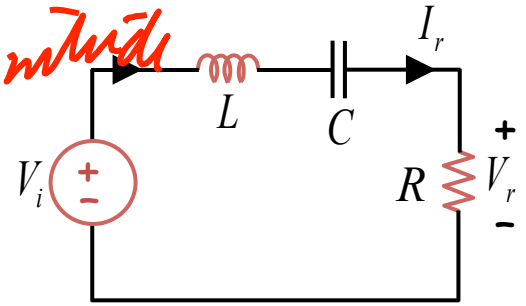
$$\frac{1}{a+jb} \rightarrow \frac{1}{(a+jb)} \cdot \frac{(a-jb)}{(a-jb)} \rightarrow \frac{a-jb}{a^2+b^2} \rightarrow \frac{1}{\sqrt{a^2+b^2}}$$

Review complex algebra in Appendix C of textbook

Graphically

Freq Response — magnitude plot

$$\left|\frac{V_r}{V_i}\right| = \frac{\omega RC}{\sqrt{\left(1-\omega^2 LC\right)^2 + (\omega RC)^2}}$$

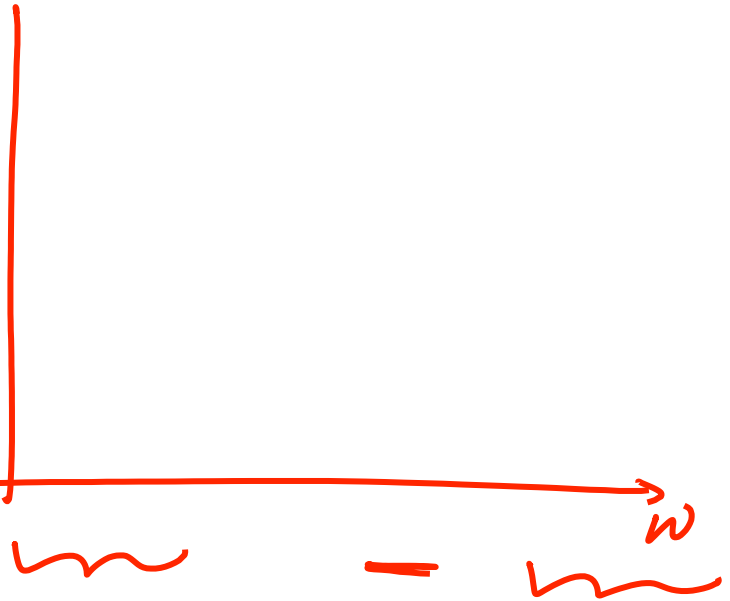


Find limit

Low ω:

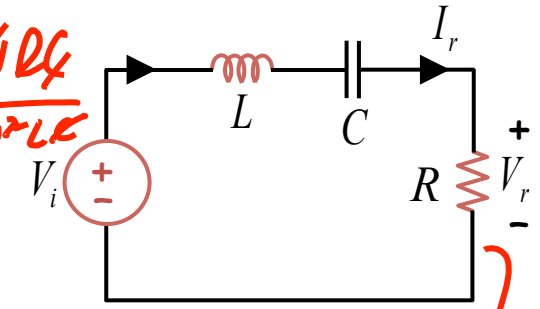
High ω:

$\omega = \frac{1}{\sqrt{LC}}$:



Graphically

$$\left|\frac{V_r}{V_i}\right| = \frac{\omega RC}{\sqrt{\left(1-\omega^2 LC\right)^2 + (\omega RC)^2}}$$



Observe

Low ω: $\approx \omega RC$

High ω: $\approx \frac{R}{\omega L}$

$\omega = \frac{1}{\sqrt{LC}}$: ≈ 1

