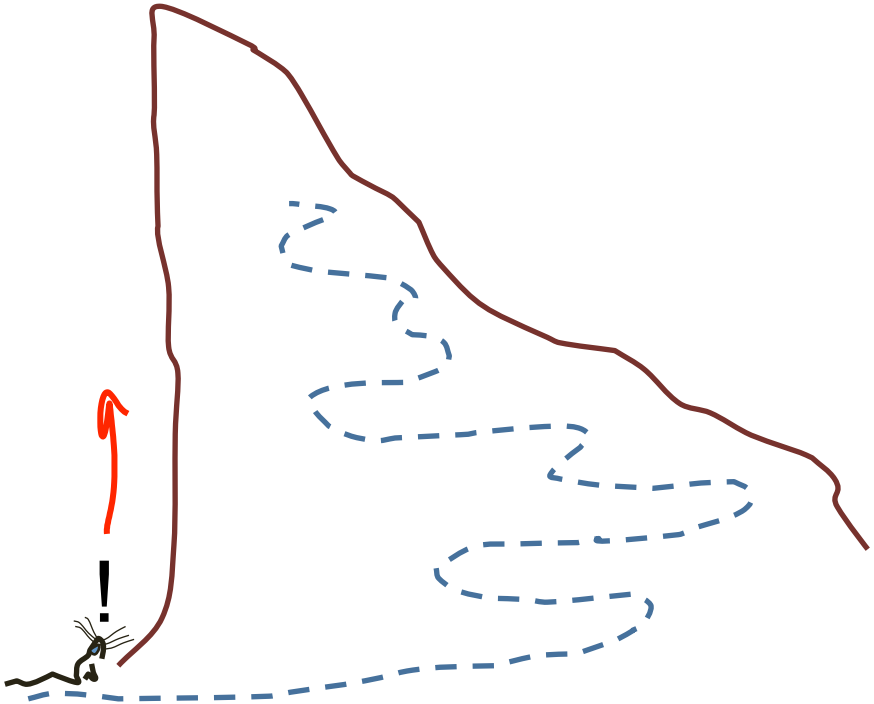
6.002x

CIRCUITS AND ELECTRONICS

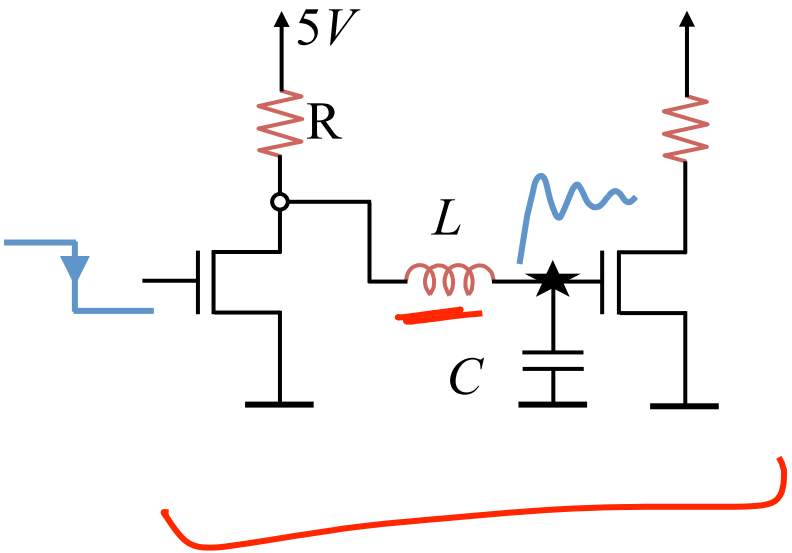
Sinusoidal Steady State

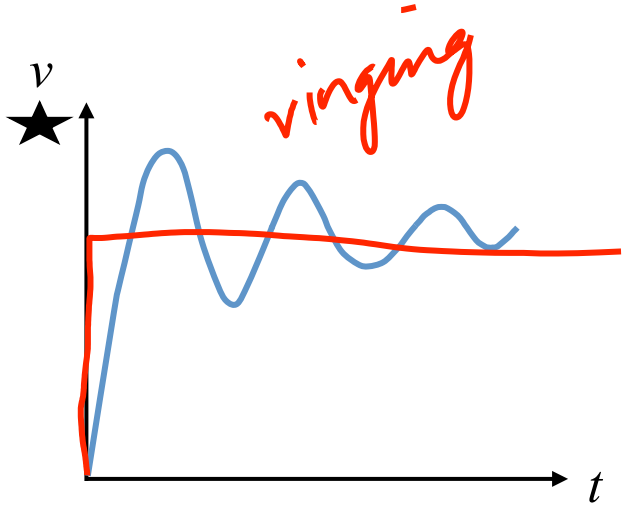


Reading Section 13.1 of the textbook

Review

- We now understand the why of:

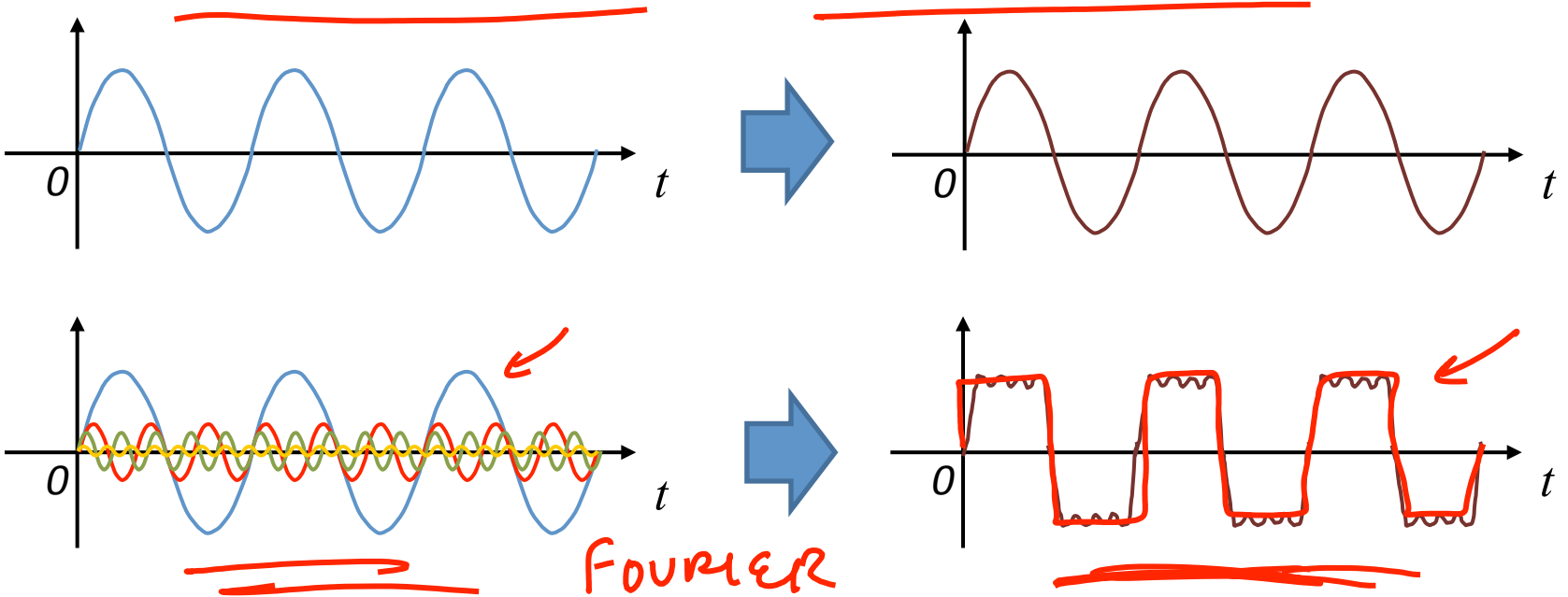




Today

- Today, look at response of networks to sinusoidal drive.

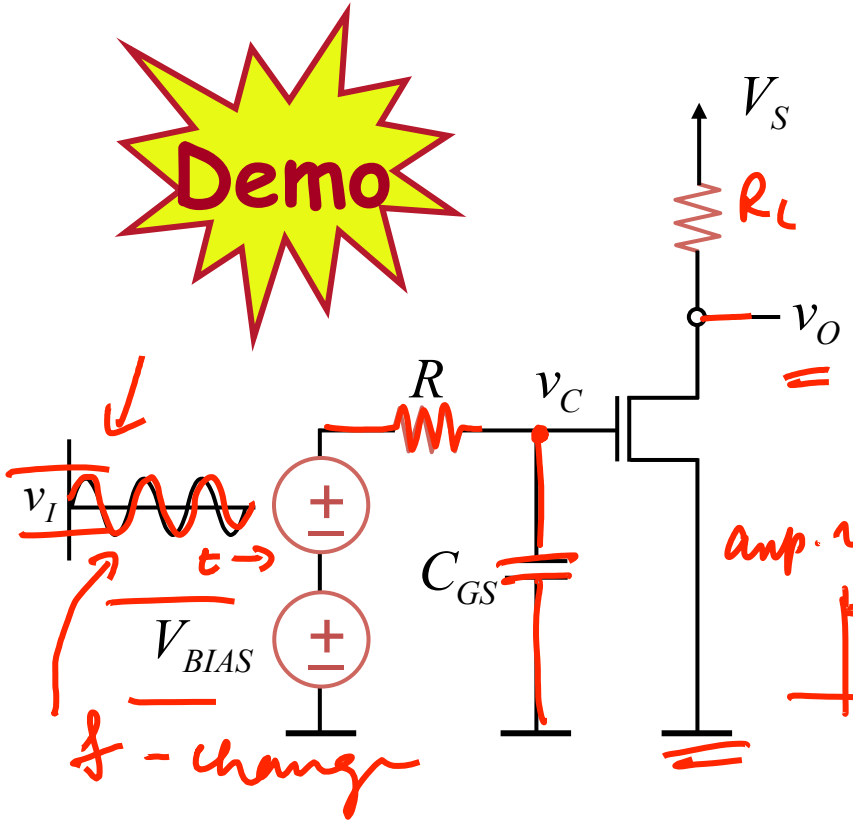
  Sinusoids important because signals can be represented as a sum of sinusoids. Response to sinusoids of various frequencies -- aka frequency response -- tells us a lot about the system



Motivation

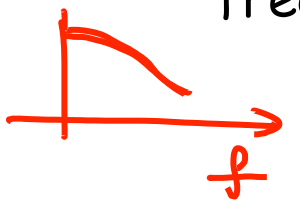
For motivation, consider our old friend, the amplifier:

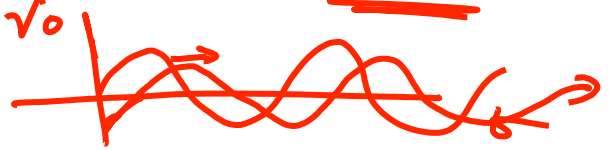
Need to study behavior of networks for sinusoidal drive.



Observe v_O amplitude as the frequency of the input v_I changes. Notice it decreases with frequency.

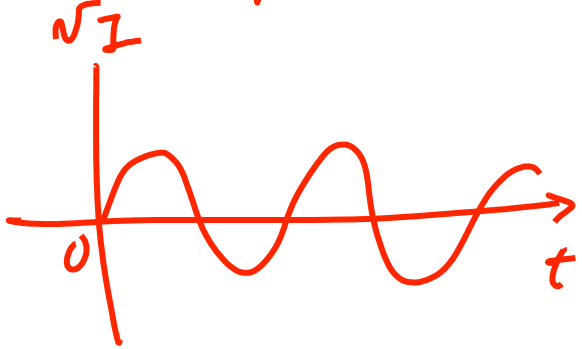
Also observe v_O shift as frequency changes (phase).

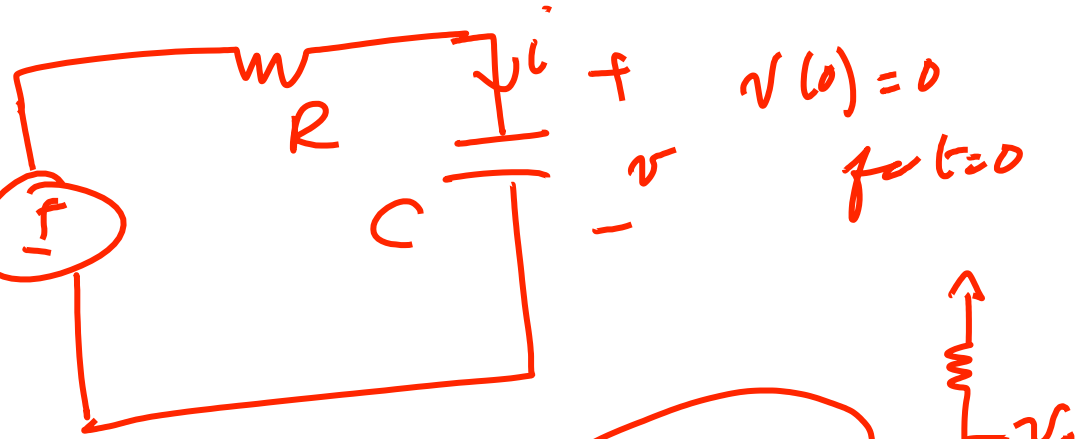




Sinusoidal Response of RC Network

Example.



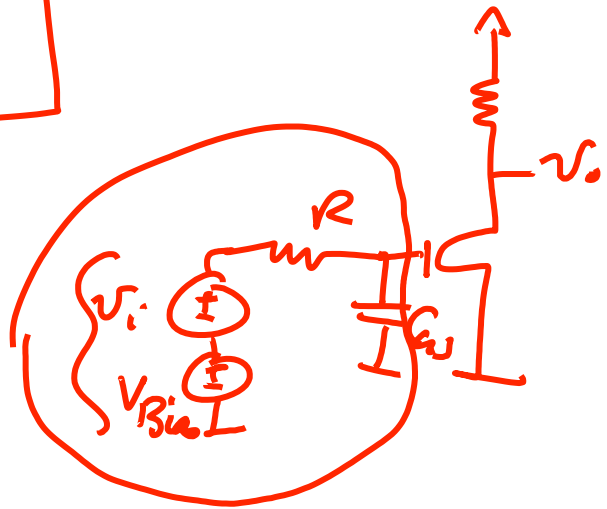


$v_I(t) = V_i \cos \omega t$ for $t \geq 0$

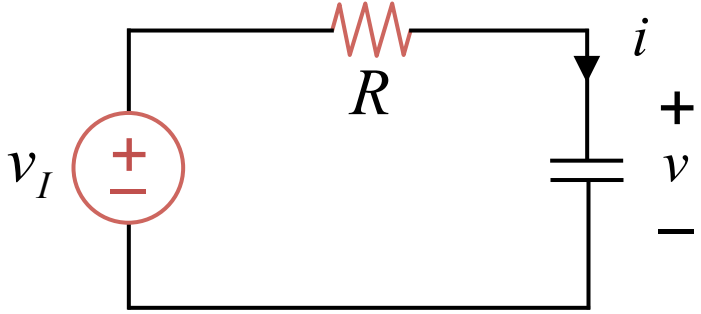
V_i is real

$V_i \cos \omega t \cdot u(t)$

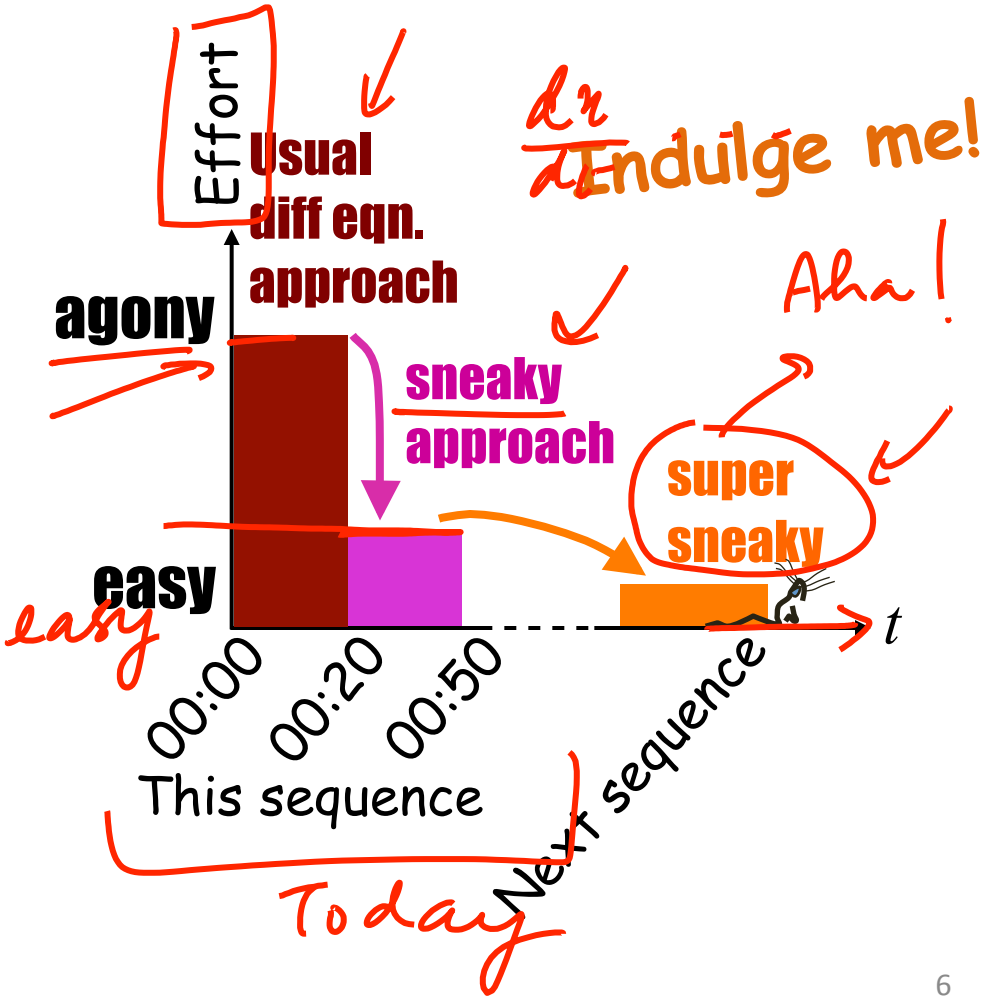
$= 0$ for $t < 0$



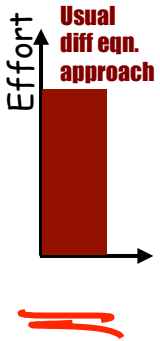
Our Approach



Determine $v(t)$



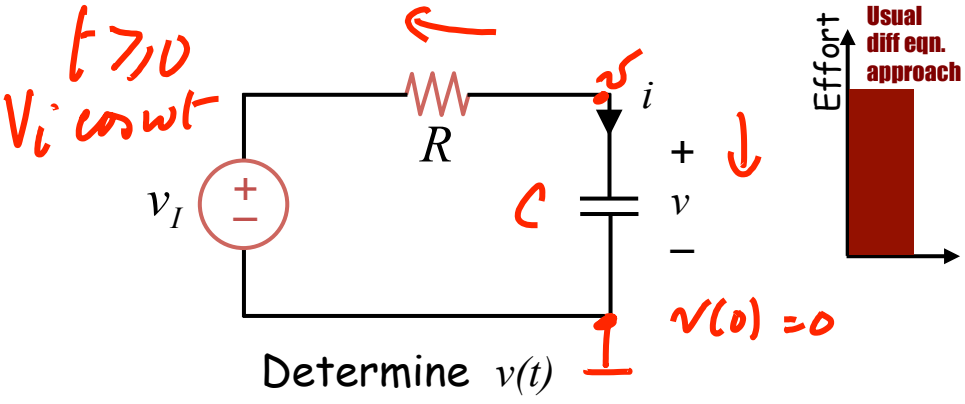
Let's try the usual approach…



1. Set up DE
2. v_p
3. find v_H
4. $v = v_p + v_H$

solve for unknown using initial condition

Usual approach...



① Set up DE

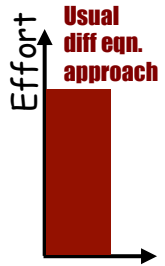
$$\frac{v - v_I}{R} + C\frac{dv}{dt} = 0$$

$$RC\frac{dv}{dt} + v = v_I$$

$$= V_i \cos \omega t$$

② Find v_P

$$RC\frac{dv}{dt} + v = V_i \cos \omega t$$



$RC\frac{dv_p}{dt} + v_p = V_i \cos \omega t$

First try: $v_p = A$ → NOPE

Second try: $v_p = A\cos \omega t$ → NOPE

Third try: $v_p = A\cos(\omega t + \phi)$ ← freq

amplitude ↑ ← phase

$-RC\omega A\sin(\omega t+\phi) + A\cos(\omega t+\phi) = V_i \cos \omega t$

$-RC\omega A \sin\omega t\cos\phi - RC\omega A\cos\omega t\sin\phi + A\cos\omega t\cos\phi - A\sin\omega t\sin\phi = V_i\cos\omega t$

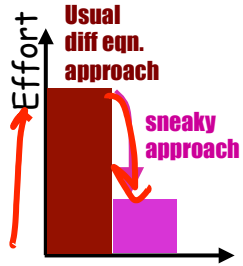
work!

but trig nightmare!

Detour: Let's get sneaky!

② $RC\frac{dv_p}{dt} + v_p = v_I$

Find particular solution to another input: $V_i \cos \omega t$

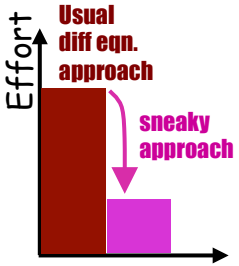


$RC\frac{dv_{ps}}{dt} + v_{ps} = v_{Is}$ (s: sneaky :-))

$= V_i e^{st}$ ←

Trial solution $v_{ps} = V_p e^{st}$

Detour: Let's get sneaky!



Find particular solution to another input…

$$RC\frac{dv_{PS}}{dt} + v_{PS} = v_{IS} = V_i e^{st}$$

(s: sneaky :-))

Try solution $v_{PS} = V_p e^{st}$ **Find** V_p

$$RC\frac{d}{dt}V_p e^{st} + V_p e^{st} = V_i e^{st}$$

$$RC\,V_p s e^{st} + V_p e^{st} = V_i e^{st}$$

nice property of exponential

$$(RCs + 1)V_p = V_i \quad \text{so} \quad V_p = \frac{V_i}{1+sRC}$$

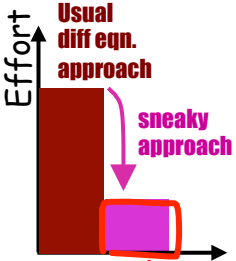
That was easy!

Sneaky detour: find particular solution to another input

$$V_p = \frac{V_i}{1 + sRC}$$

Thus, particular solution to my sneaky input $v_{ps} = V_p e^{st}$

$$= \frac{V_i}{1 + sRC} e^{st}$$



EASY!

particular solution to input $V_i e^{st}$

suppose $s = j\omega$

$V_i e^{j\omega t}$

Why V_p $\frac{V_i}{1 + j\omega RC}$ $e^{j\omega t}$ → part. sol. for $V_i e^{j\omega t}$

complex amplitude $\frac{V_i}{R e^{j\theta}}$

? $V_i \cos \omega t$? ? ?

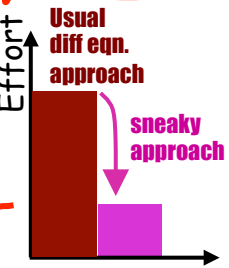
② Fourth try to find v_P... using the sneaky way $V_p = \frac{V_i}{1+sRC}$

Fact 1: Finding the response to $V_i e^{jwt}$ was easy

Fact 2: $v_I = V_i \cos wt = \text{Real}\left[V_i e^{jwt}\right] = \text{real}[v_{Is}]$ (sneaky)

from Euler Relation $e^{jwt} = \cos wt + j \sin wt$

$s = j\omega$





real part of $v_I \rightarrow$ Response $\rightarrow v_P$???

sneaky v_{Is} (= $V_i e^{jwt}$) $\rightarrow$ Response $\rightarrow v_{Ps}$ $\rightarrow$ real part of sneaky

an inverse superposition argument, assuming system is linear, real

② Fourth try to find v_P...

v_P particular response to $V_i \cos \omega t$

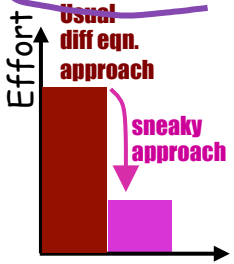
v_{PS} particular response to $V_i e^{j\omega t}$

Re

$$V_p = \frac{V_i}{1 + sRC}$$

$$s = j\omega$$

so $v_p = \mathrm{Re}\left[\underbrace{v_{ps}}_{\text{complex}}\right] = \mathrm{Re}\left[V_p e^{j\omega t}\right] = \mathrm{Re}\left[\underbrace{\frac{V_i}{1 + j\omega RC}}_{V_p} \cdot e^{j\omega t}\right]$

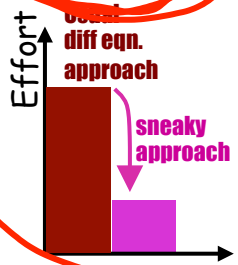


② Fourth try to find v_P...

v_P particular response to $V_i \cos\omega t$

v_{PS} particular response to $V_i e^{j\omega t}$

$$V_p = \frac{V_i}{1 + sRC}, \quad s = j\omega$$



so, (complex)

$$v_P = \mathrm{Re}[v_{PS}] = \mathrm{Re}[V_p e^{j\omega t}] = \mathrm{Re}\left[\frac{V_i}{1 + j\omega RC} \cdot e^{j\omega t}\right]$$

V_p

this

$$= \mathrm{Re}\left[\frac{V_i(1 - j\omega RC)}{1 + \omega^2 R^2 C^2} \cdot e^{j\omega t}\right] = \mathrm{Re}\left[\frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cdot e^{j\phi} \cdot e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[\frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cdot e^{j(\omega t + \phi)}\right]$$

$\tan\phi = -\omega RC$

$|V_p|$

$\angle V_p$

$$v_P = \frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cos(\omega t + \phi)$$

$|V_p|$ $\angle V_p$

Notice I got this directly from

look at complex analysis appendix in text

③ Find v_H

$$RC\frac{dv_H}{dt} + v_H = 0$$

$$v_H = A\,e^{\frac{-t}{RC}}$$

① DE

② $v_p \rightarrow v_{ps}$

③ v_H

④ $v = v_p + v_H$

④ Find total solution

$$v = v_p + v_H$$

$$v = \frac{V_i}{\sqrt{1+\omega^2 R^2 C^2}} \cdot \cos(\omega t + \phi) + A e^{-t/RC}$$

where $\phi = \tan^{-1} -\omega RC$

Given $v(0) = 0$ for $t = 0$,

$$A = -\frac{V_i}{\sqrt{1+\omega^2 R^2 C^2}} \cdot \cos\phi$$

Done! Phew!

Sinusoidal Steady State

$$V_p = \frac{V_i}{1 + sRC}$$

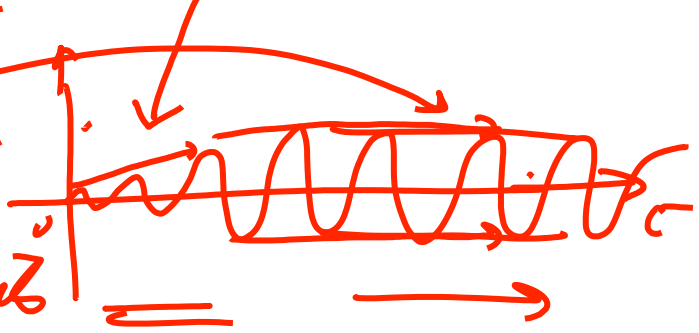
$$s = j\omega$$

$$v = \frac{V_i}{\sqrt{1+\omega^2R^2C^2}}\cos(\omega t + \phi) - \frac{V_i}{\sqrt{1+\omega^2R^2C^2}}\cos\phi \cdot e^{-t/RC}$$

transient

v_p

where $\phi = \tan^{-1}(-\omega RC)$



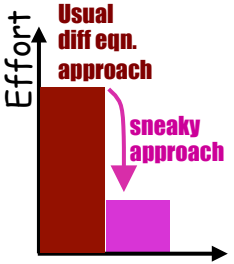
We are usually interested only in the particular solution for sinusoids, i.e., after the transient have died.

Described as SSS: Sinusoidal Steady State

Sinusoidal Steady State – V_P says it all

$$V_p = \frac{V_i}{1 + sRC}$$

$s = j\omega$



SSS

all information about SSS is contained in V_p, the complex amplitude

Recall $V_p = \frac{V_i}{1 + jwRC}$ → $|V_p| \cos wt + \angle V_p$

steps ③ & ④ mere a waste of time

V_H total

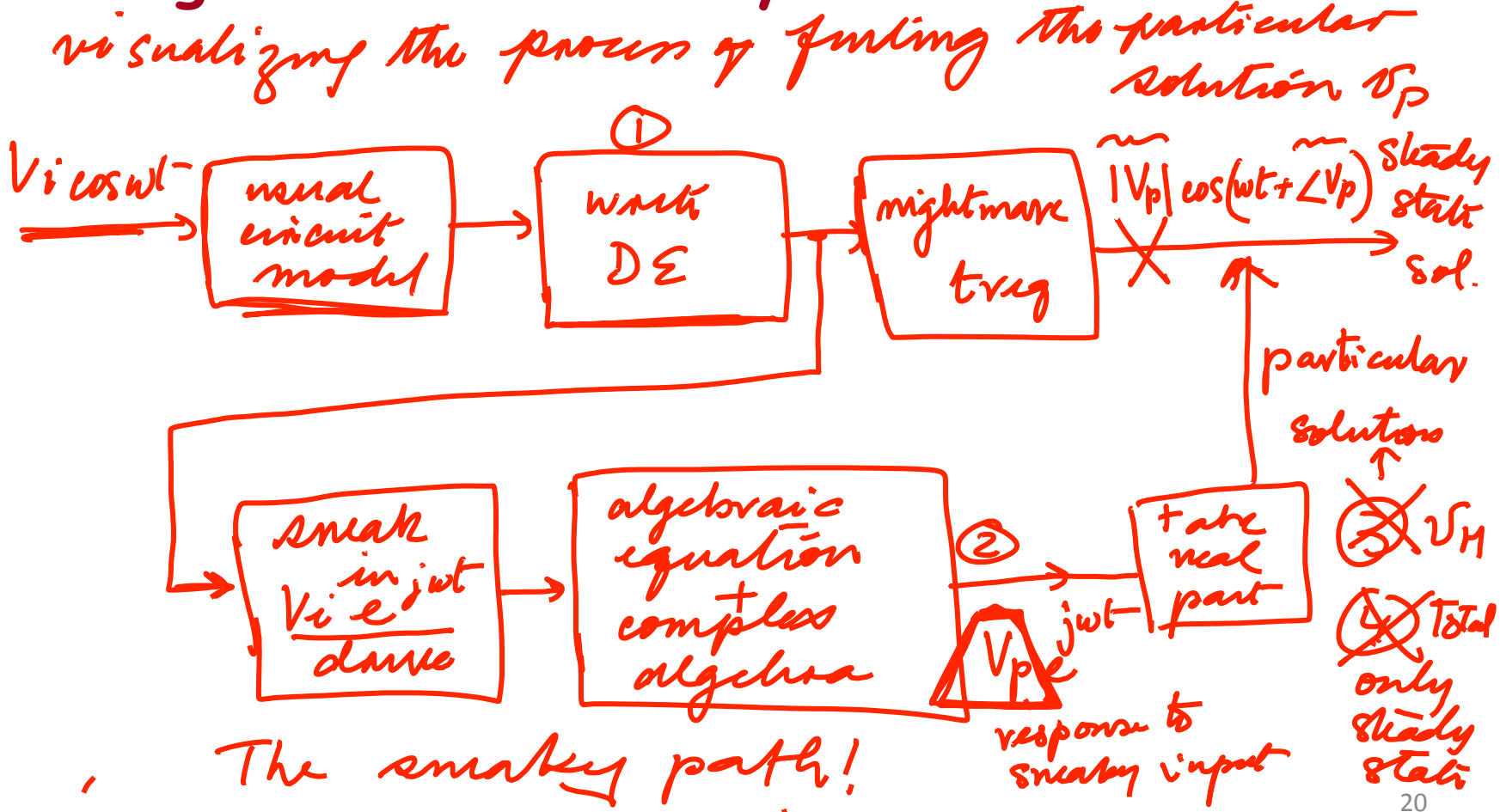
$$\frac{V_p}{V_i} = \frac{1}{1 + jwRC} = \frac{1}{\sqrt{1 + w^2R^2C^2}} \cdot e^{j\phi}$$

where $\phi = \tan^{-1} -wRC$

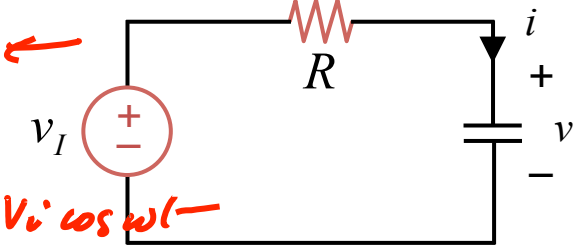
$$\text{mag } \left|\frac{V_p}{V_i}\right| = \frac{1}{\sqrt{1 + w^2R^2C^2}}$$

$$\text{phase } \angle\frac{V_p}{V_i} = -\tan^{-1} wRC$$

Visualizing Sinusoidal Steady State



Summary of SSS approach – fits on one slide



1. Set up DE $RC\frac{dv}{dt} + v = V_i \cos \omega t$ ←
2. Apply sneaky input $V_i e^{st}$ $(s = j\omega)$, then find particular sol $V_p e^{st}$
3. Find $V_p e^{st}$

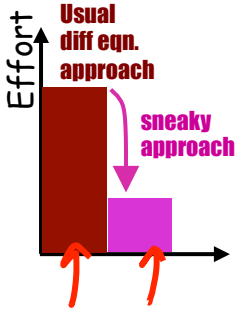
$$RC\frac{dv}{dt} + v = V_i e^{st}$$

$$RC\frac{dV_p e^{st}}{dt} + V_p e^{st} = V_i e^{st}$$

$$RC V_p \cdot s e^{st} + V_p e^{st} = V_i e^{st}$$

$$V_p = \frac{V_i}{1 + sRC}$$

That was easy!



4. Find v_p, use SS to input v_I

$v_p = |V_p| \cos \omega t + \angle V_p$

Magnitude Plot

Freq Response: → magnitude, phase

Transfer function $H(j\omega) = \frac{V_p}{V_i}$

$V_p = \frac{V_i}{1 + sRC}$, $s = j\omega$

$V_i \cos \omega t$

$$|H(j\omega)| = \left|\frac{V_p}{V_i}\right| = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}}$$

$\frac{1}{\sqrt{2}}$

From demo $\sim \frac{1}{\omega RC}$

→ Explain V_o fall off for high frequencies

log scale

log scale → $\omega = \frac{1}{RC}$ ω break frequency

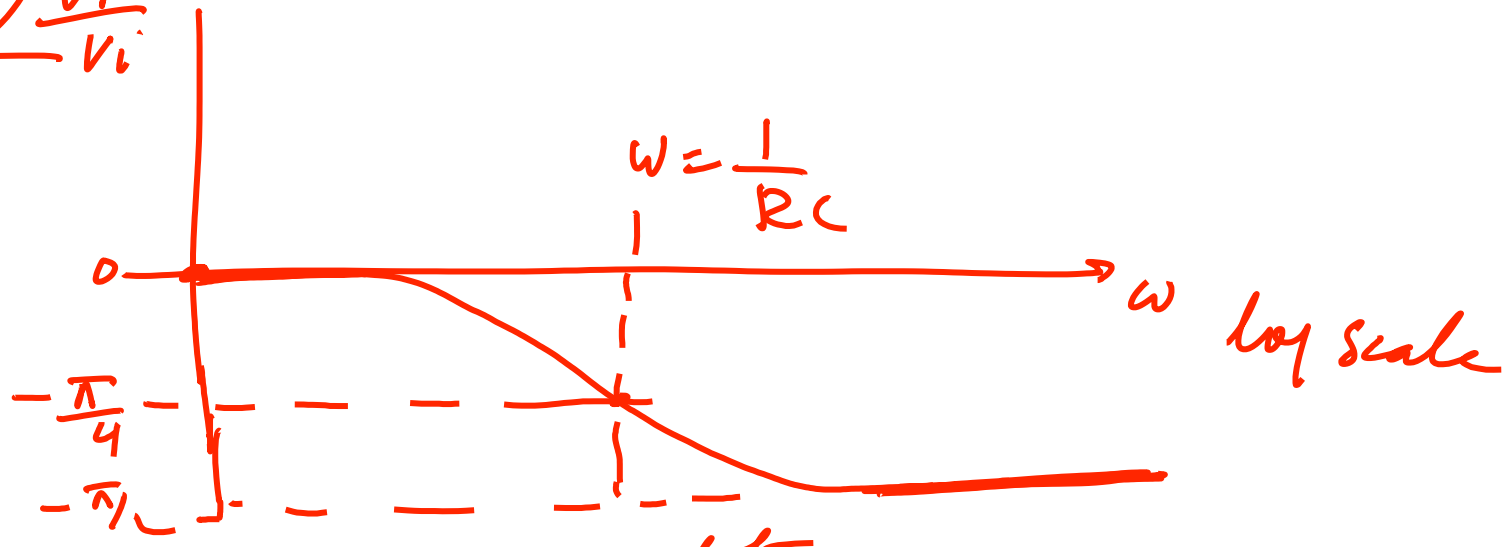
Phase Plot

$$V_p = \frac{V_i}{1 + sRC}$$

$$s = j\omega$$

$\phi = \tan^{-1} - \omega RC$

$\phi = \angle \frac{V_p}{V_i}$



Freq Response { mag plot
phase plot

Preview of upcoming attractions: The next Aha moment!

1. Set up DE $RC\frac{dv}{dt} + v = V_i \cos \omega t$

2. Apply sneaky input $V_i e^{st}$, then find particular solution $V_P e^{st}$

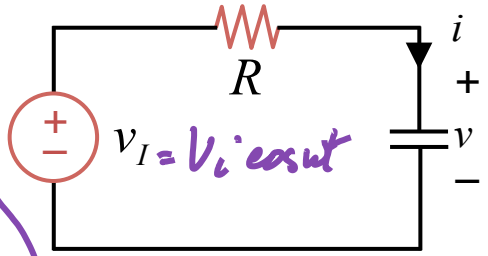
$$RC\frac{dv}{dt} + v = V_i e^{st}$$

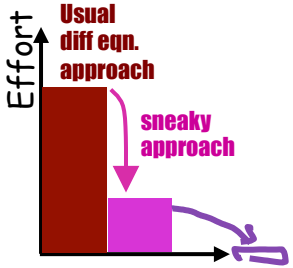
3. Find $V_P e^{st}$

$$RC\frac{dV_p e^{st}}{dt} + V_p e^{st} = V_i e^{st}$$

$$V_p = \frac{V_i}{1 + sRC}$$

$s = j\omega$





4. Find v_P, which is steady state solution for v

$$v_P = |V_P| \cos(\omega t + \angle V_P)$$

Imagine this!
The next big Aha!