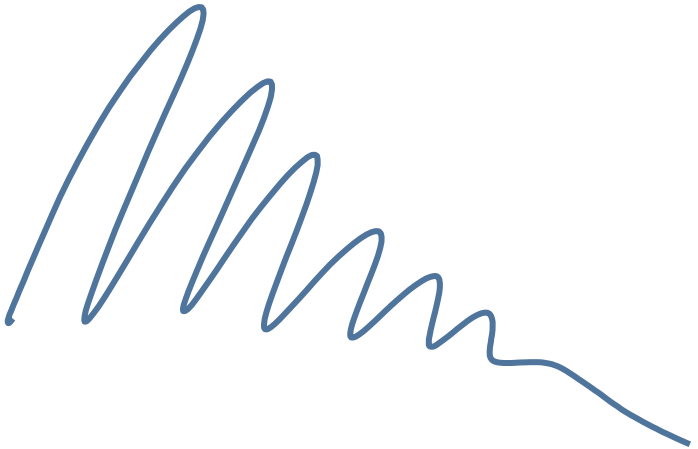
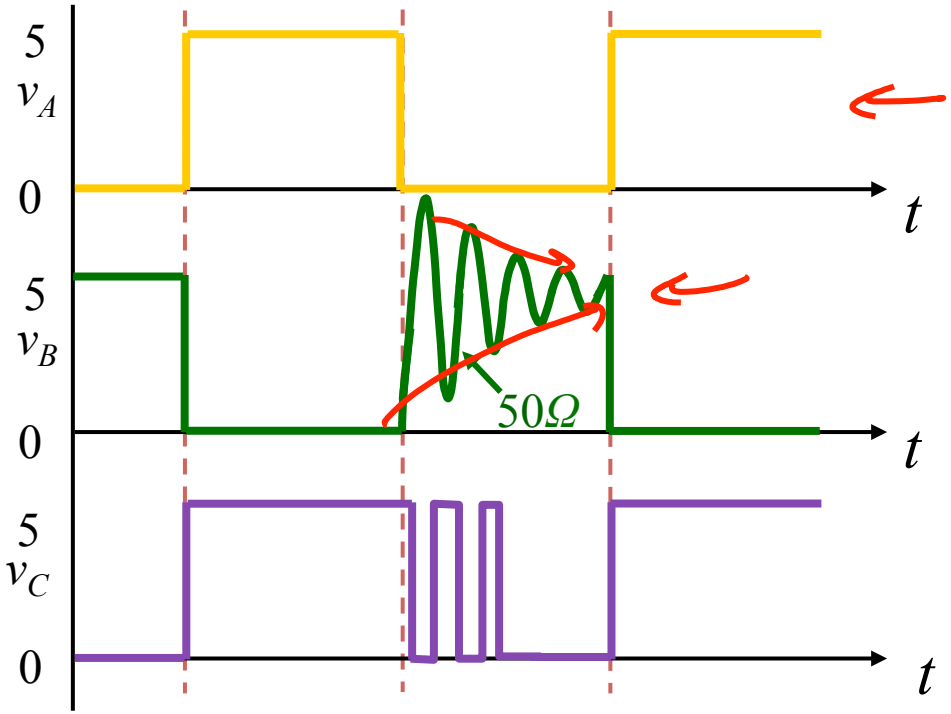
6.002x CIRCUITS AND ELECTRONICS

Damped Second-Order Systems

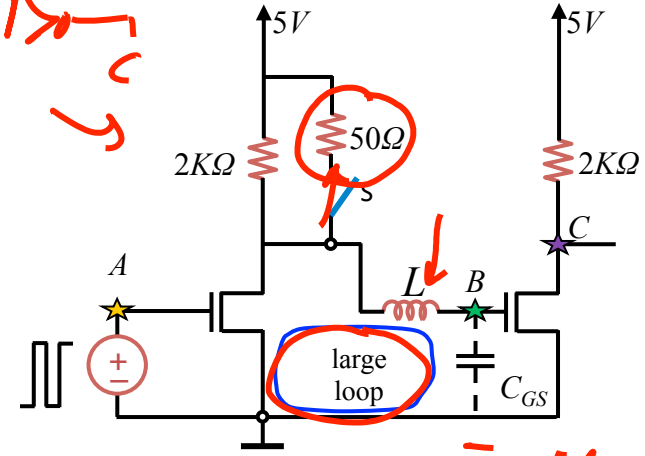


characteristic equation

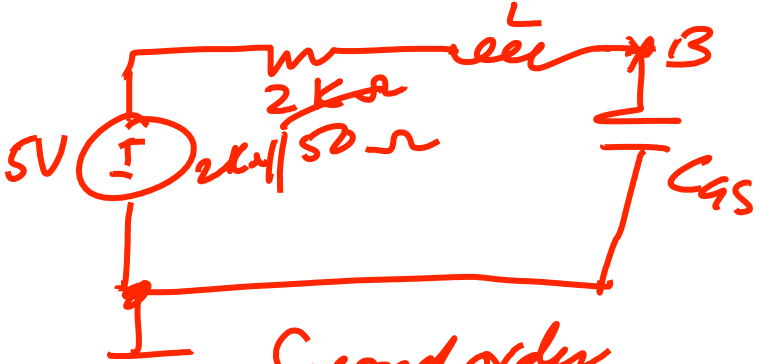
Review





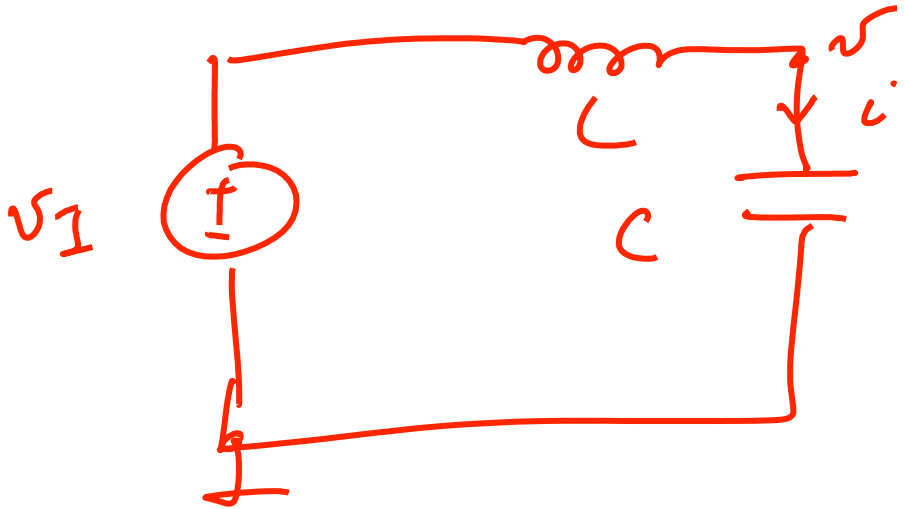


Relevant circuit:



Second order systems

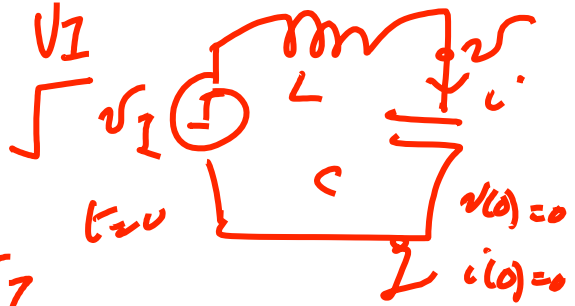
In the last lecture, we started by analyzing the simpler LC circuit to build intuition



In the last lecture… LC circuit

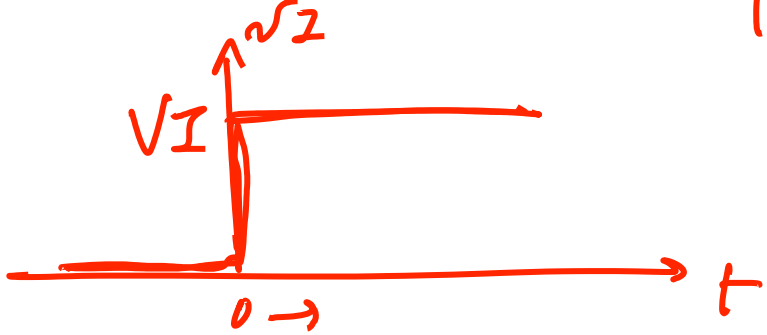
We solved

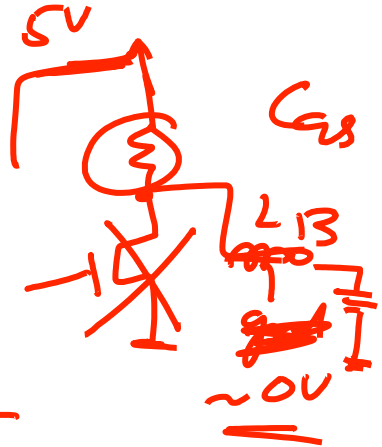
$$\frac{d^2v}{dt^2} + \frac{1}{LC}v = \frac{1}{LC}v_I$$



For input

$V_I\, u(t)$



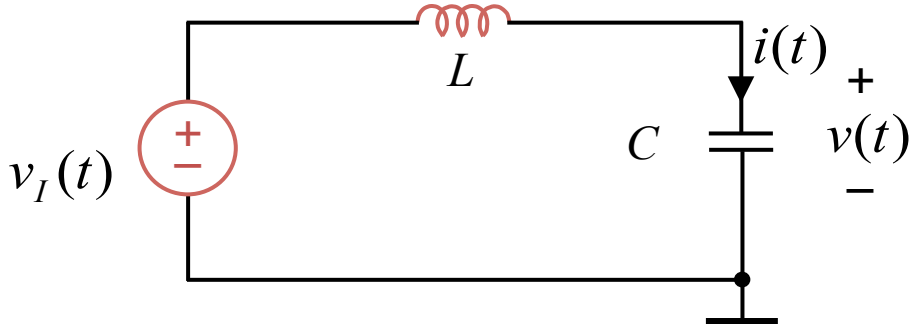


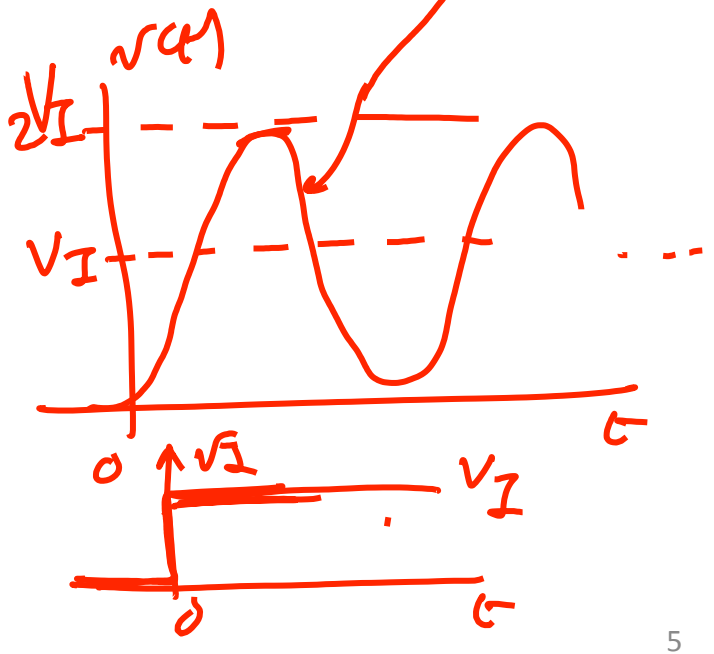
And for initial conditions

$v(0) = 0 \qquad i(0) = 0$ [ZSR]

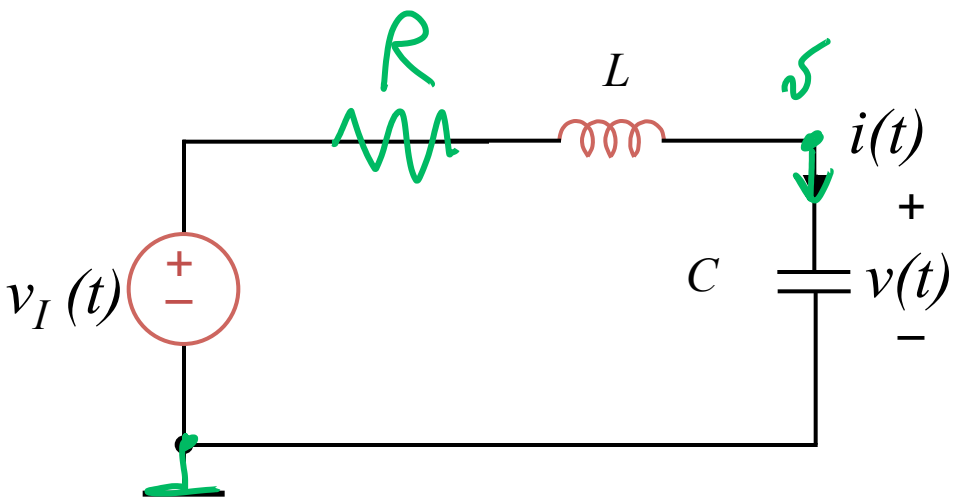
In the last lecture… LC circuit

Total solution $v(t) = V_I - V_I \cos \omega_0 t$ where $\omega_0 = \frac{1}{\sqrt{LC}}$ undamped natural frequency

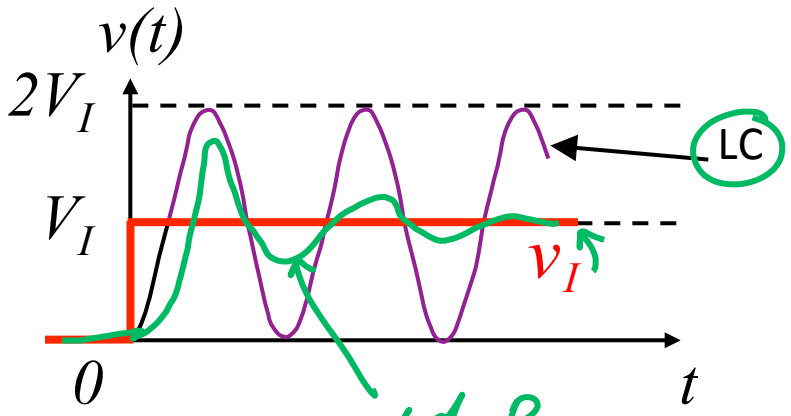




Today, we will close the loop on our observations in the demo by analyzing the RLC circuit



Series RLC ckt.

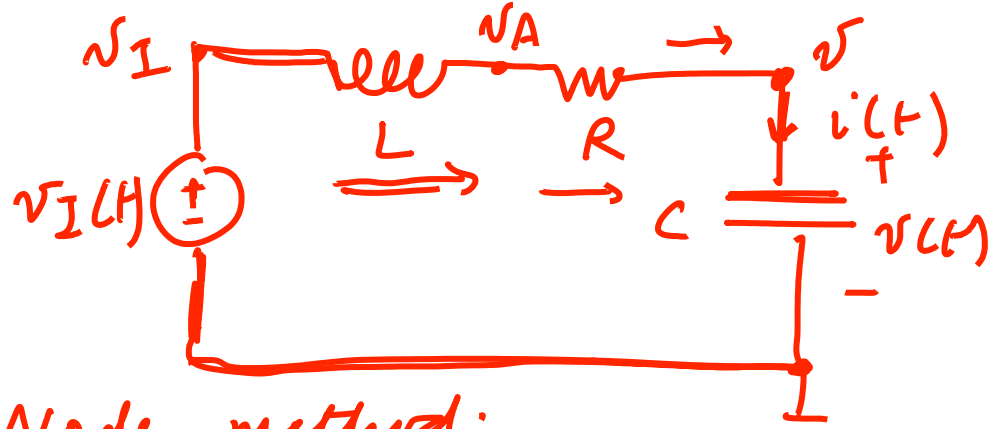


add R

Damped sinusoids with R — remember demo!

See A&L Section 12.5 13.6

Let's analyze the RLC network



Node method:

v: $\frac{v_A - v}{R} = C\frac{dv}{dt}$

v_A: $\frac{1}{L}\int_{-\infty}^{t}(v_I - v_A)dt = \frac{v_A - v}{R}$

Two state vars - i_L, v

focus on v, need to get rid of v_A.

Recall element rules

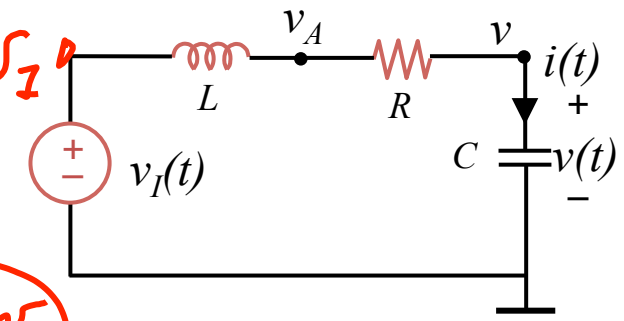
L: $v_L = L\frac{di_L}{dt}$

$\frac{1}{L}\int_{-\infty}^{t} v_L\,dt = i_L$

C: $i_C = C\frac{dv_C}{dt}$

$\frac{1}{C}\int_{-\infty}^{t} i_C\,dt = v_C$

Set up the differential equation



Need to get rid of v_A — must get rid of v_A

v: $\frac{v_A - v}{R} = C\frac{dv}{dt}$ → $v_A = RC\frac{dv}{dt} + v$

v_A: $\frac{1}{L}\int_{-\infty}^{t}(v_I - v_A)\,dt = \frac{v_A - v}{R}$

$\frac{1}{L}\int_{-\infty}^{t}(v_I - v_A)\,dt = C\frac{dv}{dt}$

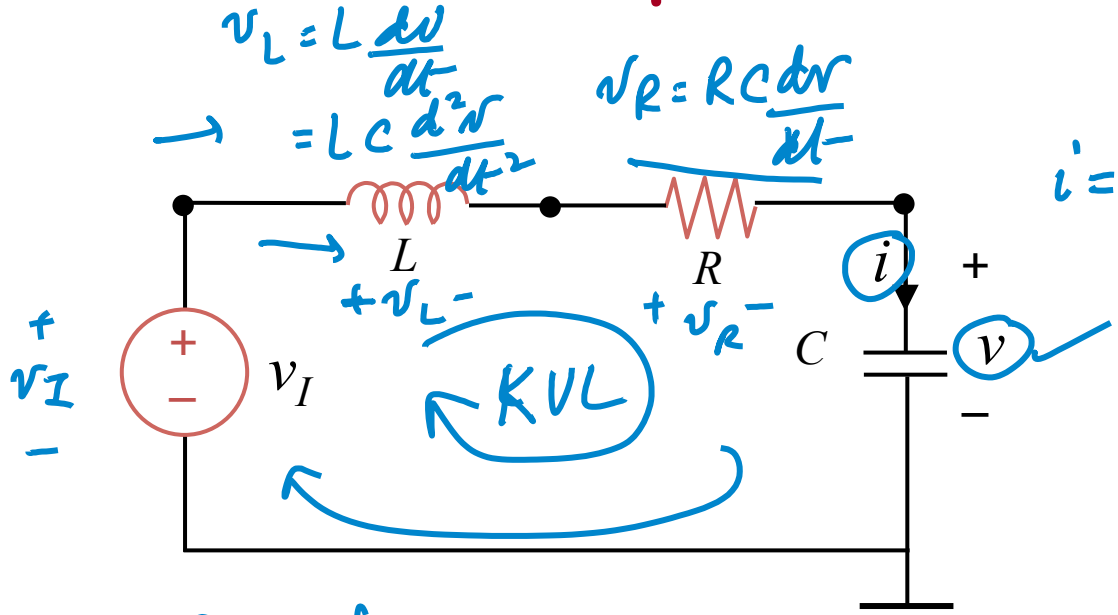
$\frac{d}{dt}$: $\frac{1}{L}(v_I - v_A) = C\frac{d^2v}{dt^2}$ — Node method

$\frac{1}{LC}(v_I - v_A) = \frac{d^2v}{dt^2}$

$\frac{1}{LC}\left(v_I - RC\frac{dv}{dt} - v\right) = \frac{d^2v}{dt^2}$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{1}{LC}v_I$$

Set up the differential equation differently



$$+v_L + v_R + v - v_I = 0$$

$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = v_I$$

Solve

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}v_I$$

Recall the method of homogeneous and particular solutions:

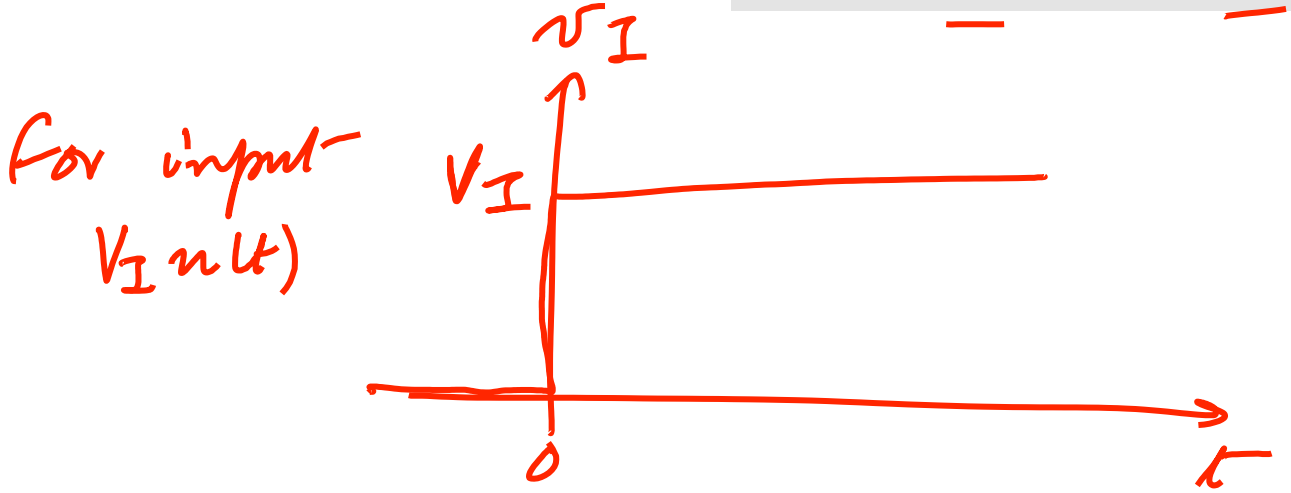
1. Find the particular solution. v_p
2. Find the homogeneous solution $\rightarrow v_H$ 4 step process
3. The total solution is the sum of the particular and homogeneous solutions.

$$v = v_p(t) + v_H(t)$$

Use initial conditions to solve for the remaining constants

Let's solve

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}v_I$$



[ZSR] for initial condition $v(0) = 0$, $i(0) = 0$

1 Particular solution

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}v_I$$

$$\frac{d^2v_p}{dt} + \frac{R}{L}\frac{dv_p}{dt} + \frac{1}{LC}v_p = \frac{1}{LC}V_I \quad \text{for } t \geq 0$$

what is a particular solution ?

$v_p = ?$

1 Particular solution

$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{LC}v=\frac{1}{LC}v_I$$

$$\frac{d^2v_P}{dt^2}+\frac{R}{L}\frac{dv_P}{dt}+\frac{1}{LC}v_P=\frac{1}{LC}V_I$$

guess & check!

$\frac{dv_P}{dt}=0$

$v_P = V_I$

This works!

so $v_P = V_I$ is a particular solution.

(2) Homogeneous solution

$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{LC}v=\frac{1}{LC}v_I$$

Solutions to $\frac{d^2v_H}{dt^2}+\frac{R}{L}\frac{dv_H}{dt}+\frac{1}{LC}v_H=0$

Recall: v_H: Soln. to the homogeneous eq (demo set 6o)

Four-step method:

(2A) Assume a solution of the form $v_H = Ae^{st}$ when A, s?

(2B) Form the characteristic equation: $f(s)$

(2C) Find the roots of the char. eq: s_1, s_2

(2D) General soln. of the homogeneous equation:

$$v_H = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

2 Homogeneous solution

$$\frac{d^2 v_H}{dt^2} + \frac{R}{L}\frac{dv_H}{dt} + \frac{1}{LC} v_H = 0$$

2A anime solution of this from $v_H = A e^{st}$ where $A, s = ?$

find the polynomial in s that must be satisfied.

$A e^{st}$

$\frac{d}{dt}$ ↓ $A s e^{st}$

$\frac{d}{dt}$ ↓ $A s^2 e^{st}$

(2) Homogeneous solution

Solution to $\frac{d^2 v_H}{dt^2} + \frac{R}{L}\frac{dv_H}{dt} + \frac{1}{LC} v_H = 0$

(2A) Assume solution of the form

$v_H = Ae^{ST}$, $A, S = ?$

$\frac{R}{L} \cdot A S e^{St}$

$\frac{1}{LC} e^{St}$

$A S^2 e^{St}$

$A S^2 e^{St} + \frac{R}{L} S A e^{St} + \frac{1}{LC} A e^{St} = 0$

$S^2 + \frac{R}{L} S + \frac{1}{LC} = 0$

↳ char eq.

2 Homogeneous solution

Solution to $\frac{d^2 v_H}{dt^2} + \frac{R}{L}\frac{dv_H}{dt} + \frac{1}{LC} v_H = 0$

2B

$s^2 + \frac{R}{L} s + \frac{1}{LC} = 0$ → Characteristic equation $s^2 + \frac{R}{L} s + \frac{1}{LC} = 0$

$s^2 + 2\alpha s + \omega_0^2 = 0$

'a' 'b' 'c'

$\omega_0 = \frac{1}{\sqrt{LC}}$

$\alpha = \frac{R}{2L}$

2C

Roots

$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$

$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$

2 Homogeneous solution

Solution to $\frac{d^2 v_H}{dt^2} + \frac{R}{L}\frac{dv_H}{dt} + \frac{1}{LC} v_H = 0$

characteristic equation $s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$

Roots $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2}$

$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$

$\omega_o = \sqrt{\frac{1}{LC}}$

$\alpha = \frac{R}{2L}$

2D General solution to the homogenous equations.

$v_H = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$v_H = A_1 e^{-\alpha + \sqrt{\alpha^2 - \omega_o^2}} + A_2 e^{-\alpha - \sqrt{\alpha^2 - \omega_o^2}}$

find from init conds

3 Total solution

$e^{-\alpha t} \cdot \cos(\omega t)$

$v(t) = v_p(t) + v_{hf}(t)$

$$v(t) = V_I + A_1 e^{\left(-\alpha + \sqrt{\alpha^2 - \omega_0^2}\right)t} + A_2 e^{\left(-\alpha - \sqrt{\alpha^2 - \omega_0^2}\right)t} \quad \text{for } t \geq 0$$

$\omega_0 = \frac{1}{\sqrt{LC}}$

$\alpha = \frac{R}{2L}$

find unknowns from initial condition

$v(0) = 0$: $\quad 0 = V_I + A_1 + A_2$ in A_1, A_2

$i(0) = 0$: $\quad i(t) = C\frac{dv}{dt} = CA_1\left(-\alpha + \sqrt{\alpha^2 - \omega_0^2}\right)e^{\left(-\alpha + \sqrt{\alpha^2 - \omega_0^2}\right)t} + CA_2\left(-\alpha - \sqrt{\alpha^2 - \omega_0^2}\right)e^{\left(-\alpha - \sqrt{\alpha^2 - \omega_0^2}\right)t}$

So, at $t = 0$, $i(0) = 0$ @ $t = 0$

in A_1, A_2 $\quad 0 = CA_1\left(-\alpha + \sqrt{\alpha^2 - \omega_0^2}\right) + CA_2\left(-\alpha - \sqrt{\alpha^2 - \omega_0^2}\right)$

mathematically, solve for unknowns A_1, A_2 ... DONE

3 Let's stare at this a while longer...

$$v(t) = V_I + A_1 e^{\left(-\alpha+\sqrt{\alpha^2-\omega_o^2}\right)t} + A_2 e^{\left(-\alpha-\sqrt{\alpha^2-\omega_o^2}\right)t}$$

$$v(t) = V_I + A_1 e^{-\alpha t} e^{\left(\sqrt{\alpha^2-\omega_o^2}\right)t} + A_2 e^{-\alpha t} e^{\left(-\sqrt{\alpha^2-\omega_o^2}\right)t}$$

$\sqrt{\alpha^2 - \omega_o^2}$

$\alpha = \frac{R}{2L}$

$\omega_o = \frac{1}{\sqrt{LC}}$

3 cases:

$\alpha > \omega_o$ — Overdamped case

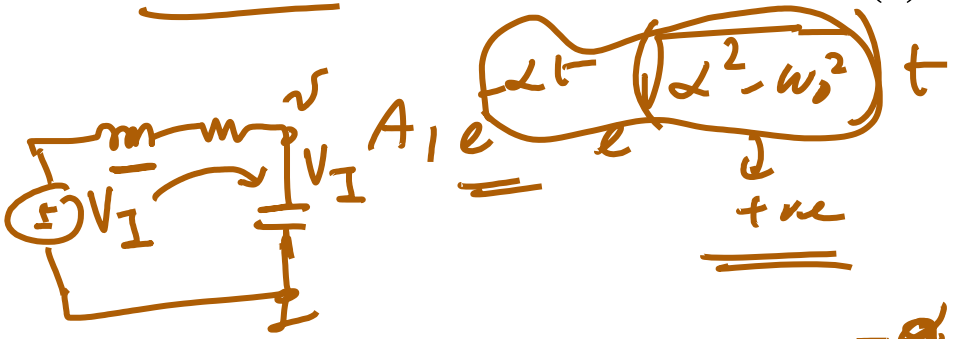
$\alpha < \omega_o$ — Underdamped case

$\alpha = \omega_o$ — critically damped case

3

$\alpha > \omega_o$ Overdamped

$$v(t) = V_I + A_1 e^{-\alpha t} e^{\left(\sqrt{\alpha^2-\omega_o^2}\right)t} + A_2 e^{-\alpha t} e^{\left(-\sqrt{\alpha^2-\omega_o^2}\right)t}$$

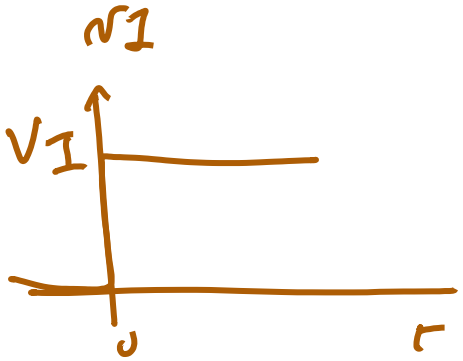


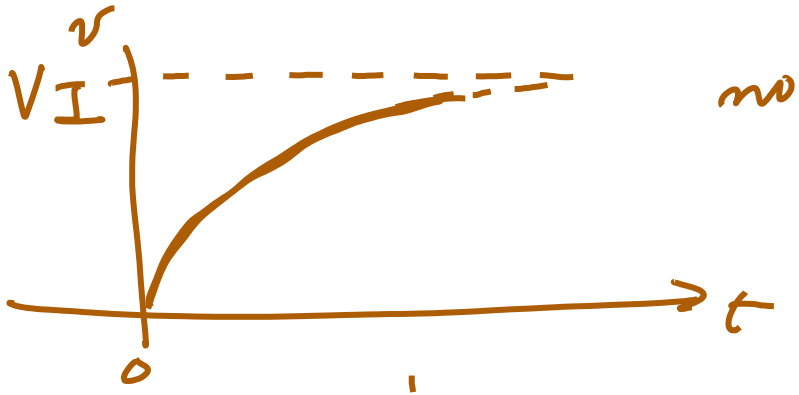
$\alpha > \omega_o$ Overdamped

$\alpha < \omega_o$ Underdamped

$\alpha = \omega_o$ Critically damped

$$v(t) = V_I + A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t}$$





no sinusoids here!

3 $\alpha < \omega_o$ Underdamped $v(t) = V_I + A_1 e^{-\alpha t} e^{\left(\sqrt{\alpha^2-\omega_o^2}\right)t} + A_2 e^{-\alpha t} e^{\left(-\sqrt{\alpha^2-\omega_o^2}\right)t}$

$e^{\sqrt{\alpha^2-\omega_o^2}}$ $\alpha < \omega_o$ $\sim -1 \cdot (\omega_o^2 - \alpha^2)$

$e^{j\sqrt{\omega_o^2-\alpha^2}}$ $j = \sqrt{-1}$

$\alpha > \omega_o$ Overdamped

$\alpha < \omega_o$ Underdamped

$\alpha = \omega_o$ Critically damped

$v(t) = V_I + A_1 e^{-\alpha t} e^{(j\sqrt{\omega_o^2-\alpha^2})t} + A_2 e^{-\alpha t} e^{(-j\sqrt{\omega_o^2-\alpha^2})t}$

$v(t) = V_I + A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$

$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$

ω_d : damped natural frequency

ω_o = undamped natural freq

real → RHS must be real

$v(t) = V_I + K_1 e^{-\alpha t} \cos\omega_d t + K_2 e^{-\alpha t} \sin\omega_d t$

$e^{j\omega_d t} = \cos\omega_d t + j\sin\omega_d t$

$e^{-j\omega_d t} = \cos - j\sin$

3 $\alpha < \omega_o$ Underdamped contd...

$$v(t) = V_I + K_1 e^{-\alpha t}\cos\omega_d t + K_2 e^{-\alpha t}\sin\omega_d t$$

$v(0) = 0$: $\quad 0 = V_I + K_1 \Rightarrow K_1 = -V_I$ (*)

$i(0) = 0$: $\quad i(t) = C\frac{dv}{dt} = -CK_1\alpha e^{-\alpha t}\cos\omega_d t - CK_1 e^{-\alpha t}\omega_d \sin\omega_d t$

$\quad - CK_2\alpha e^{-\alpha t}\sin\omega_d t + CK_2 e^{-\alpha t}\omega_d\cos\omega_d t$

$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$, $\omega_o = \frac{1}{\sqrt{LC}}$, $\alpha = \frac{R}{2L}$

$0 = -K_1\alpha + K_2\omega_d \Rightarrow K_2 = \frac{K_1\alpha}{\omega_d} = \frac{-V_I\alpha}{\omega_d}$ (*)

$\rightarrow v(t) = V_I - V_I e^{-\alpha t}\cos\omega_d t - \frac{V_I\alpha}{\omega_d} e^{-\alpha t}\sin\omega_d t$

Check: For $R = 0$, $\alpha = 0$, $\omega_d = \omega_o$

$v(t) = V_I - V_I\cos\omega_o t$

Correct for LC ckt!

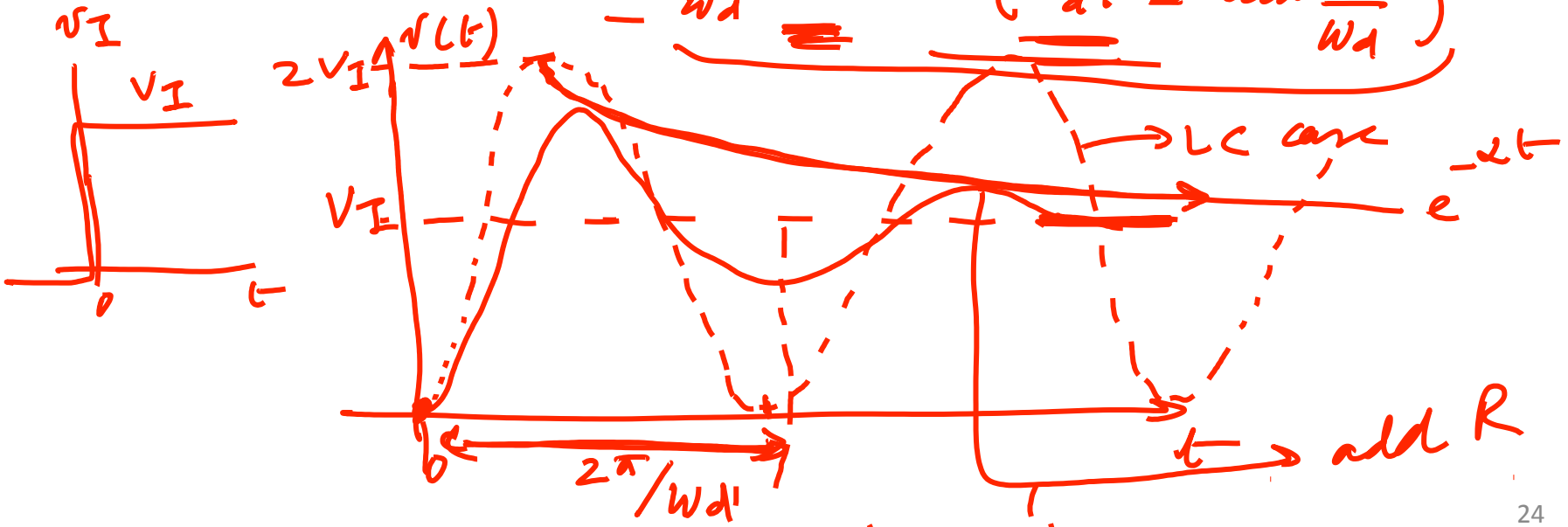
3 $\alpha < \omega_o$ Underdamped contd...

$$v(t) = V_I - V_I e^{-\alpha t}\cos\omega_d t - V_I \frac{\alpha}{\omega_d} e^{-\alpha t}\sin\omega_d t$$

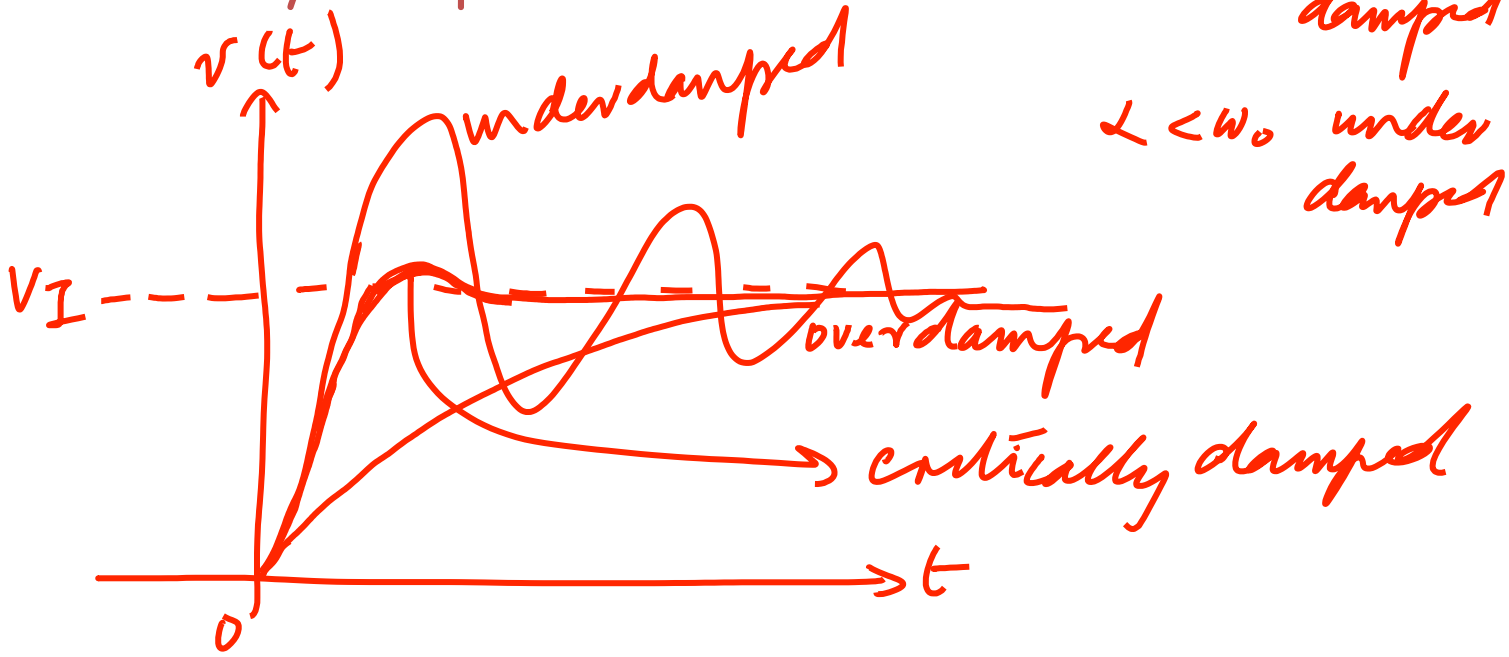
Remember, scaled sum of sines (of the same frequency) are also sines! -- Appendix B.7

$$A_1\cos\theta + A_2\sin\theta = \sqrt{A_1^2 + A_2^2}\cos\left(\theta - \tan^{-1}\frac{A_2}{A_1}\right)$$

$$v(t) = V_I - V_I \frac{\omega_0}{\omega_d} e^{-\alpha t}\cos\left(\omega_d t - \tan^{-1}\frac{\alpha}{\omega_d}\right)$$



3 $\alpha = \omega_o$ Critically damped



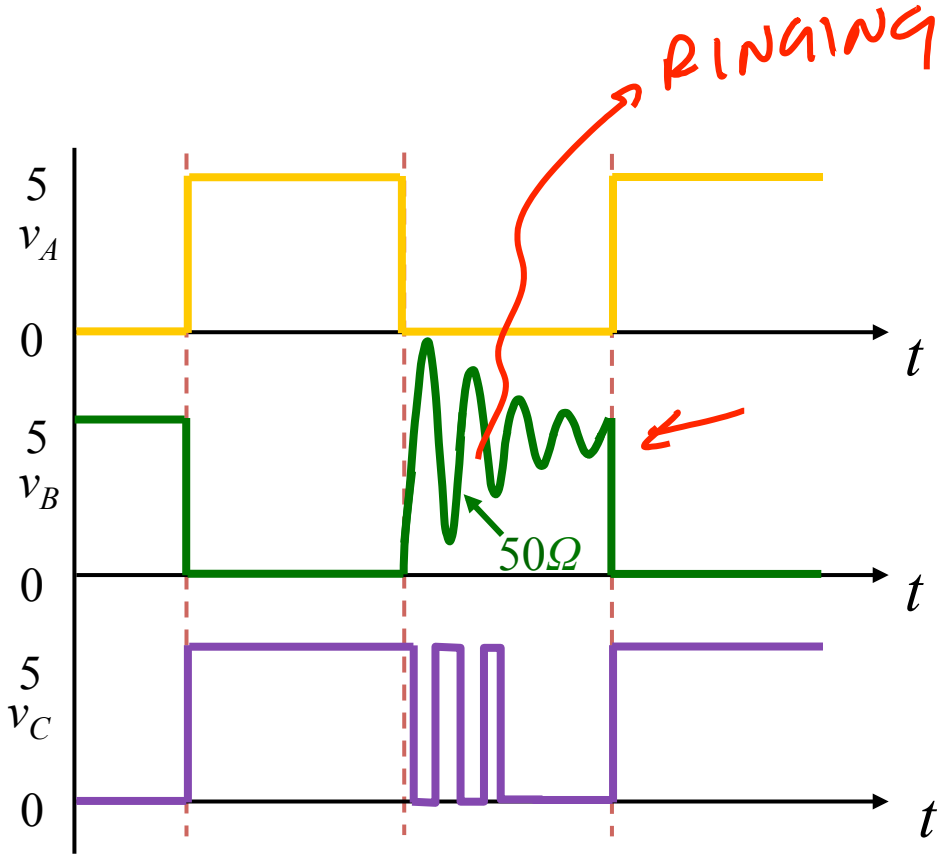
$\alpha > \omega_o$ over damped

$\alpha < \omega_o$ under damped

12.2.3

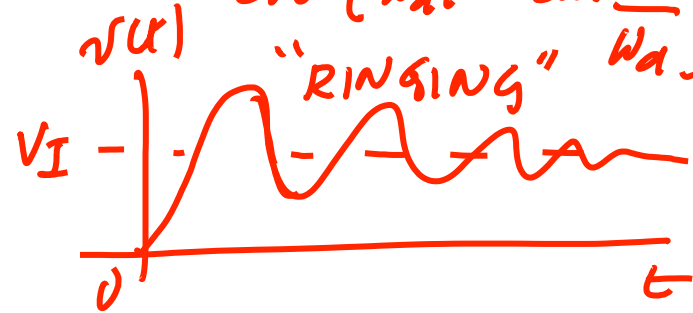
Section 12.2.3

Remember this? We have now closed the loop…



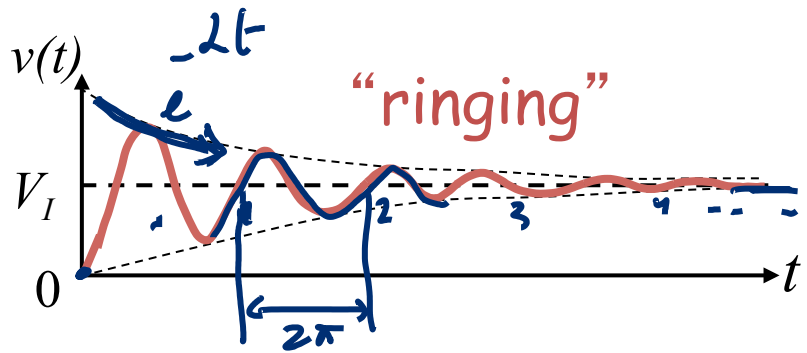
Underdamped

$$v(t) = V_I - V_I \frac{\omega_0}{\omega_d} e^{-\alpha t} \cos\left(\omega_d t - \tan^{-1}\frac{\alpha}{\omega_d}\right)$$



See Example 12.9 on page 664 of this textbook for inverter pair analysis

Easy way: Characteristic equation tells the whole story!



Characteristic equation tells the story (for series RLC ckt)

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

canonic form

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

NO DIFF EQs!

from char. eq

$\alpha = \frac{R}{2L}$, $\omega_0 = \frac{1}{\sqrt{LC}}$

$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

↳ oscillation freq

α = governs the rate of decay

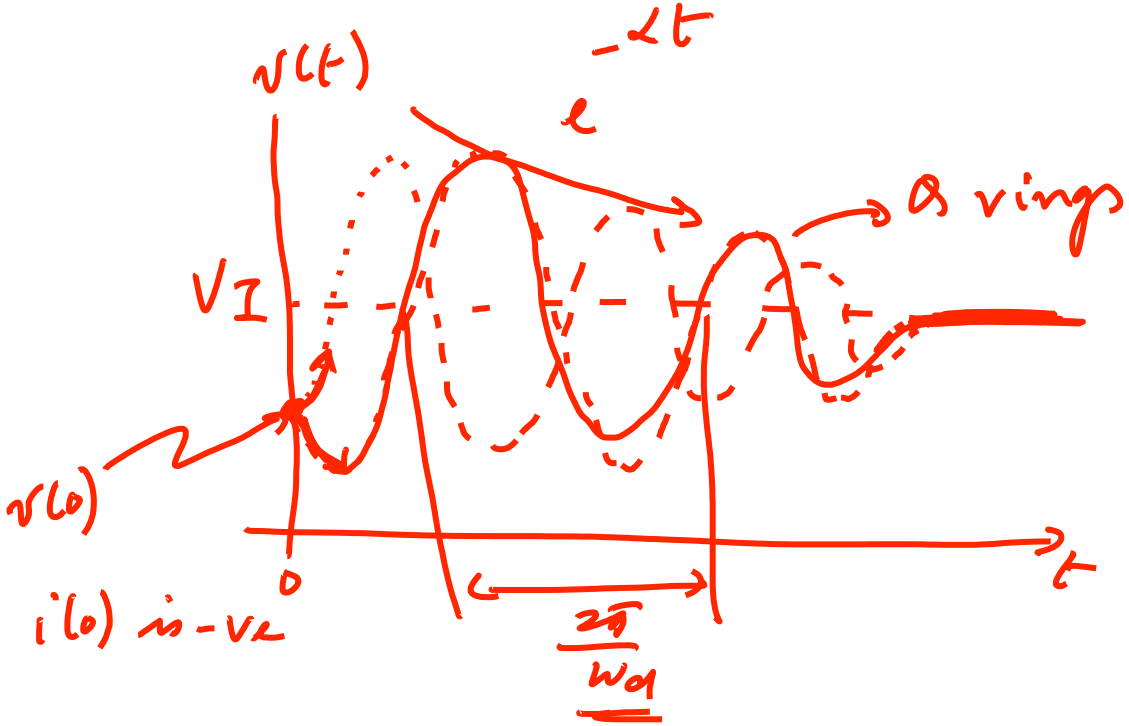
V_I = final val

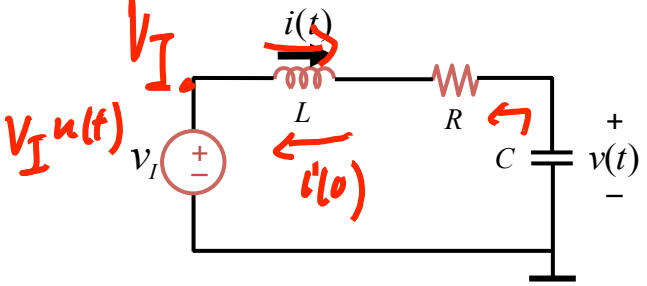
$v(0)$ = initial val

$Q = \frac{\omega_0}{2\alpha}$: Quality factor (approx. the number of cycles of ringing

See Sec. 12.7 of A&L textbook

Intuitive Analysis





given

$i(0)$ -ve

$v(0)$ +ve

$\alpha = \frac{R}{2L}$

$\omega_0 = \frac{1}{\sqrt{LC}}$

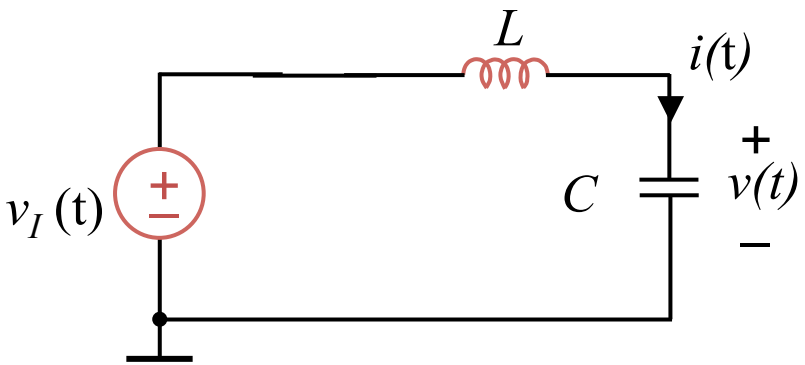
$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

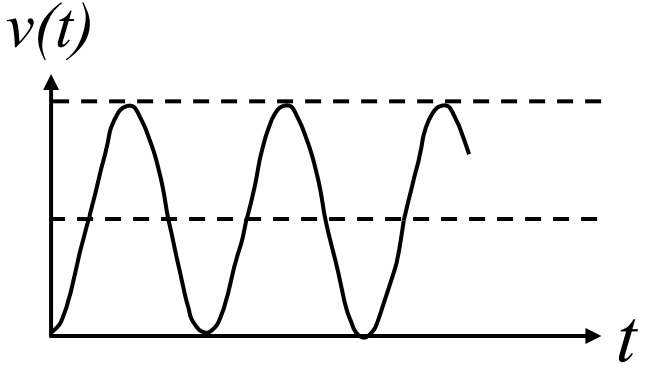
period $= \frac{2\pi}{\omega_d}$, $\alpha < \omega_0$ underdamped case

$Q = \frac{\omega_0}{2\alpha}$

Next, introduce R: RLC Circuits

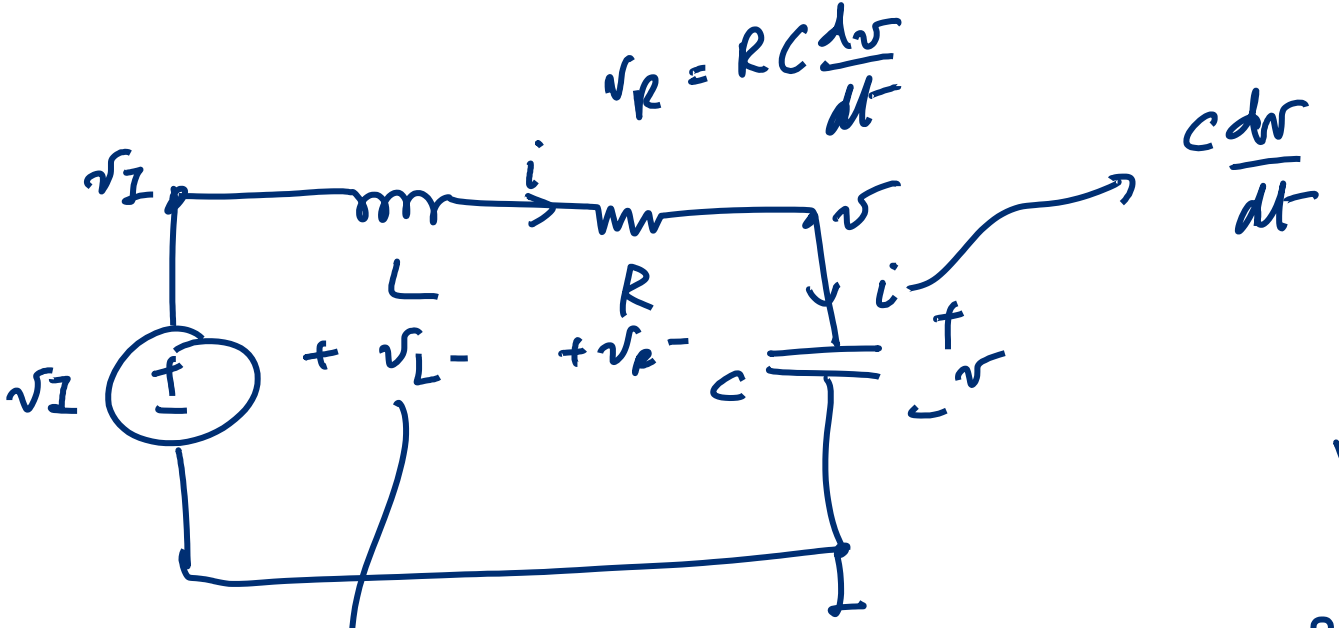
Ignore this slide!



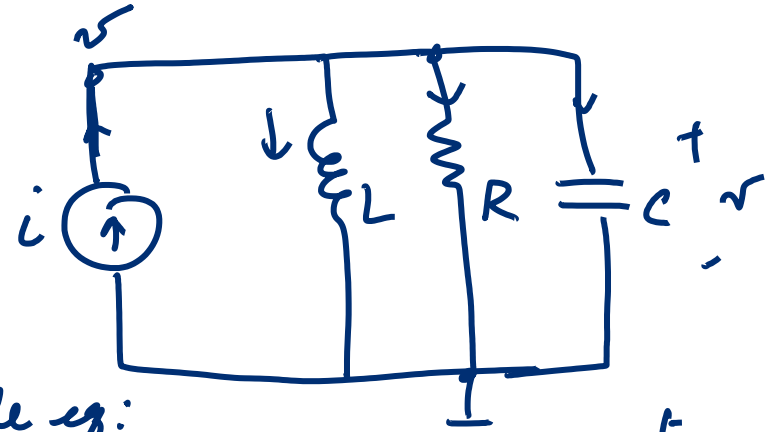


More in the next sequence! If you are impatient, see A&L Section 12.5

What about other variables?



Parallel RLC – Characteristic equation says it all



node eq:

$$C\frac{dv}{dt} + \frac{v}{R} - i + \frac{1}{L}\int_{-\infty}^{t} v\,dt = 0$$

$\frac{1}{C}\frac{d}{dt}$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

substitute $Ae^{st} \rightarrow v$

$$\longrightarrow s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

What is the char eq.

from $s^2 + 2\alpha s + \omega_0^2 = 0$

char eq $s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$

for parallel RLC

$L\frac{di}{dt} = v$

$\rightarrow 0$ for homogeneous eq.

$$\alpha = \frac{1}{2RC},\quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$Q = \frac{\omega_0}{2\alpha}$$