6.002x CIRCUITS AND ELECTRONICS

Second-Order Systems

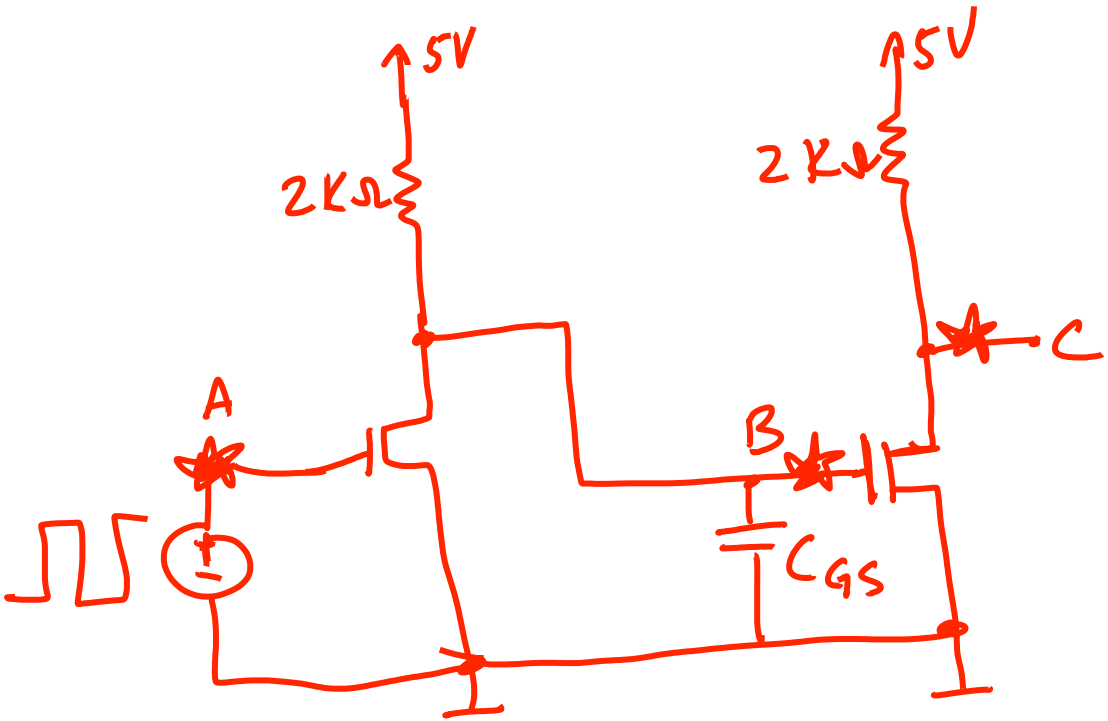


[Review complex algebra Appendix C in textbook]

Motivating Example – Slow Case

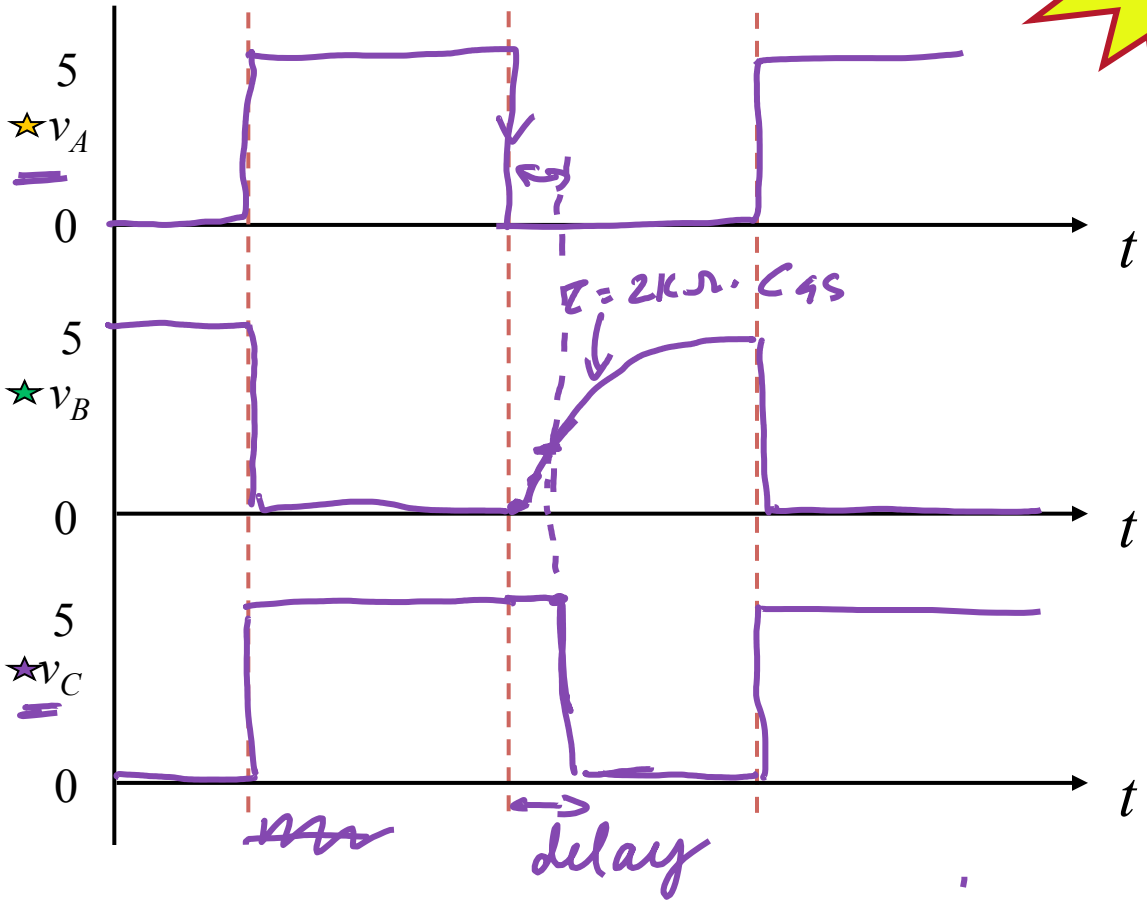


Our old friend, the inverter, driving another.

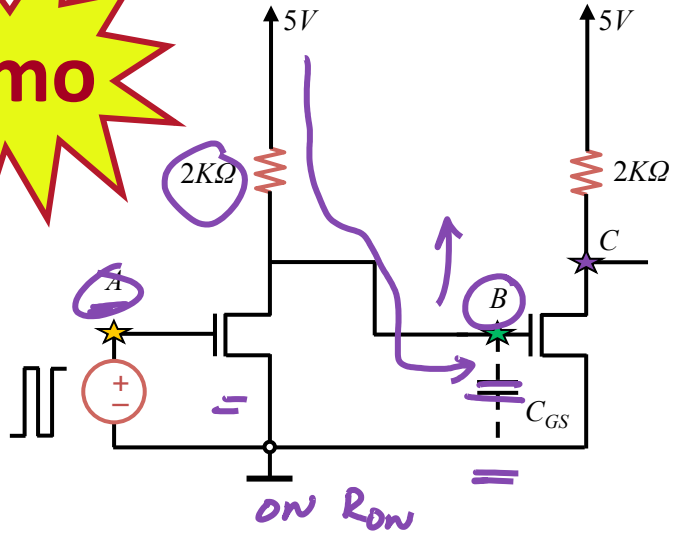


The gate-to-source capacitance of the MOSFET C_{GS} is shown

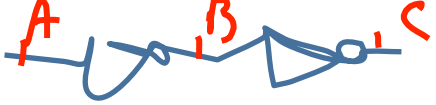
Observed Output - Slow

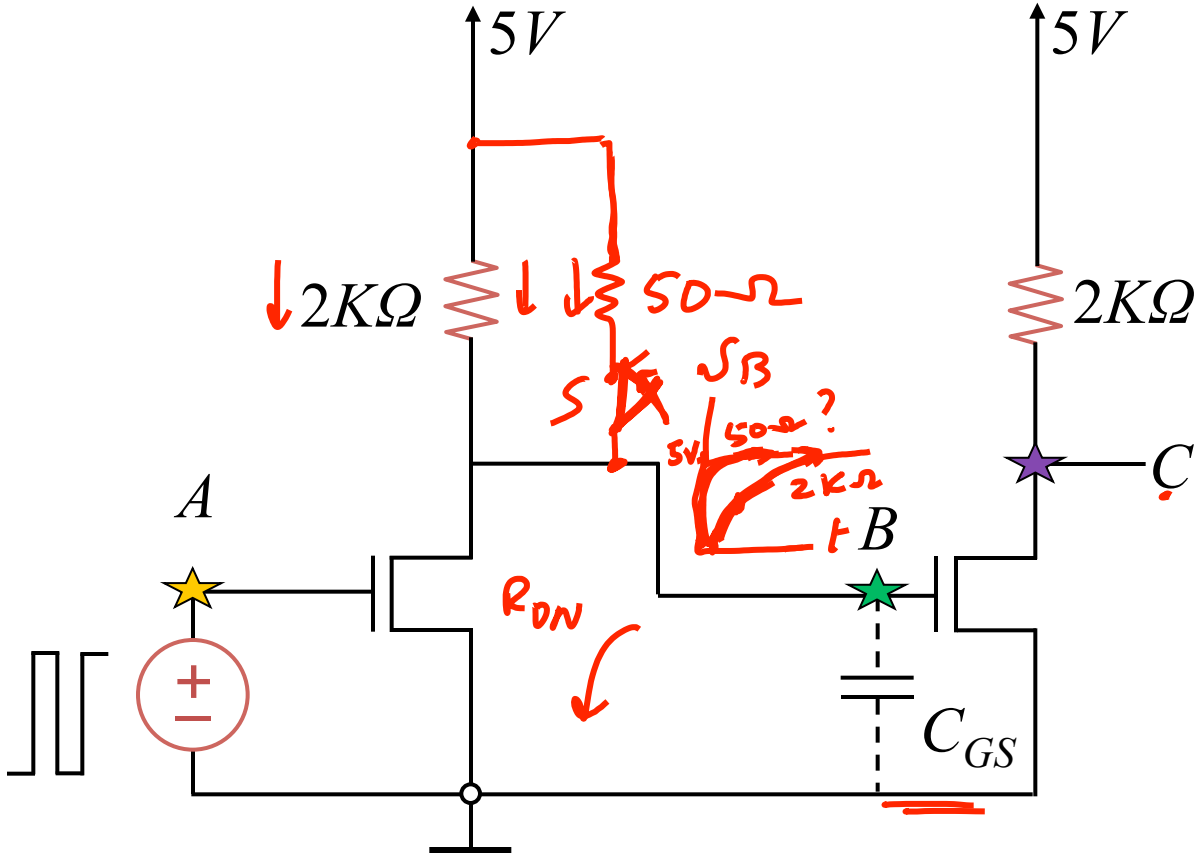


Demo



Motivating Example – Fast Case

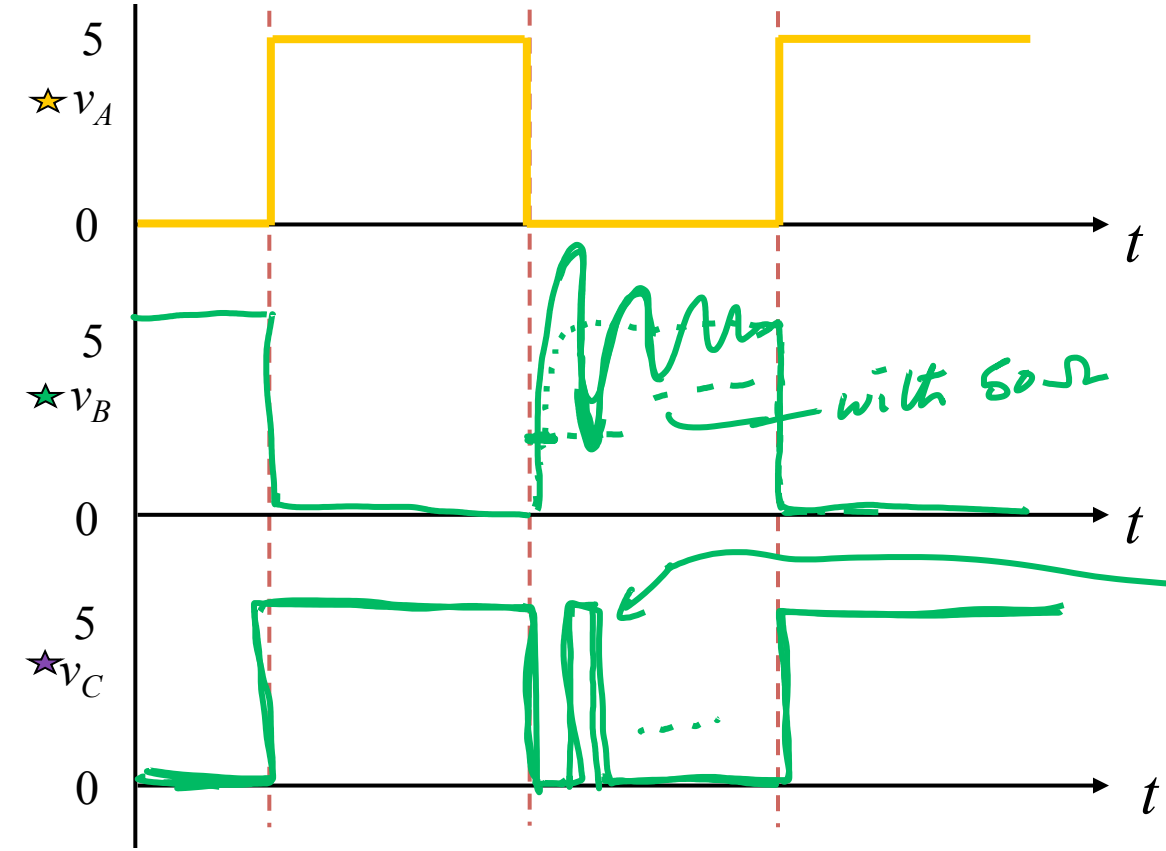


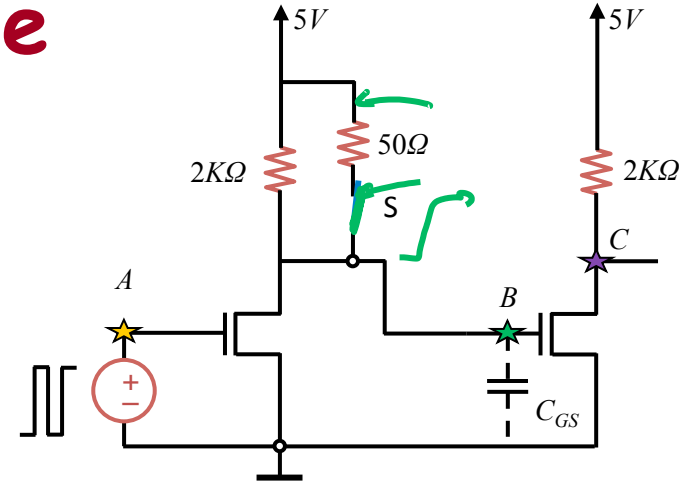


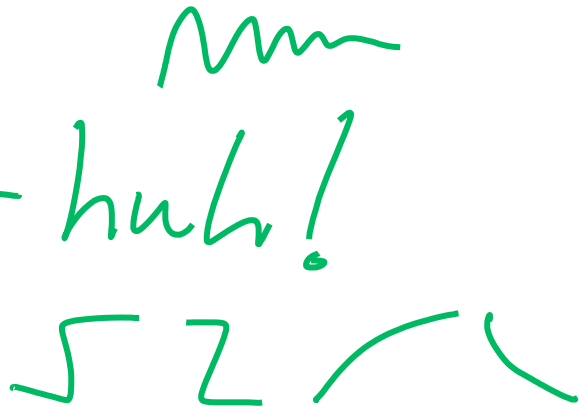
Demo

Now, let's try to speed up our inverter by closing the switch S to lower the effective resistance (pull-up)

Observed Output – Fast Case

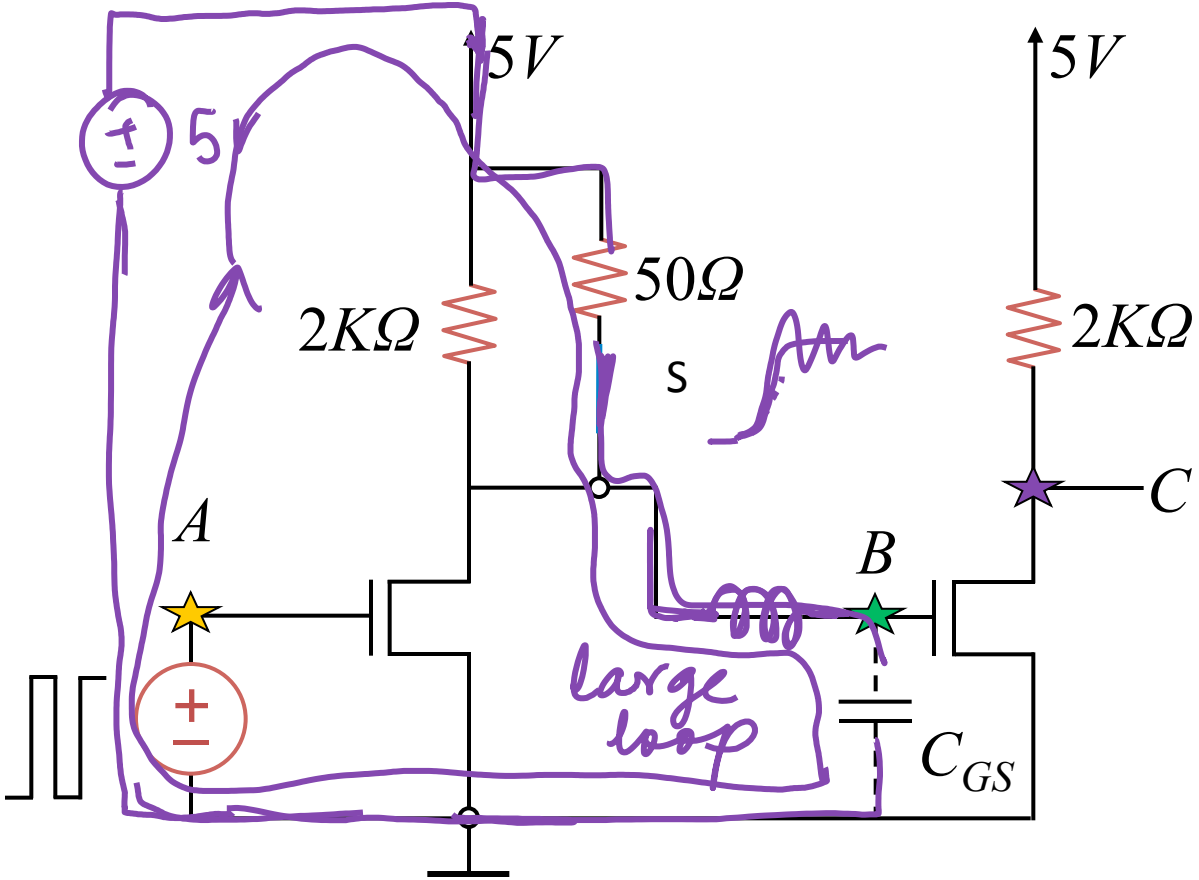






Fast Case – What's Really Going On

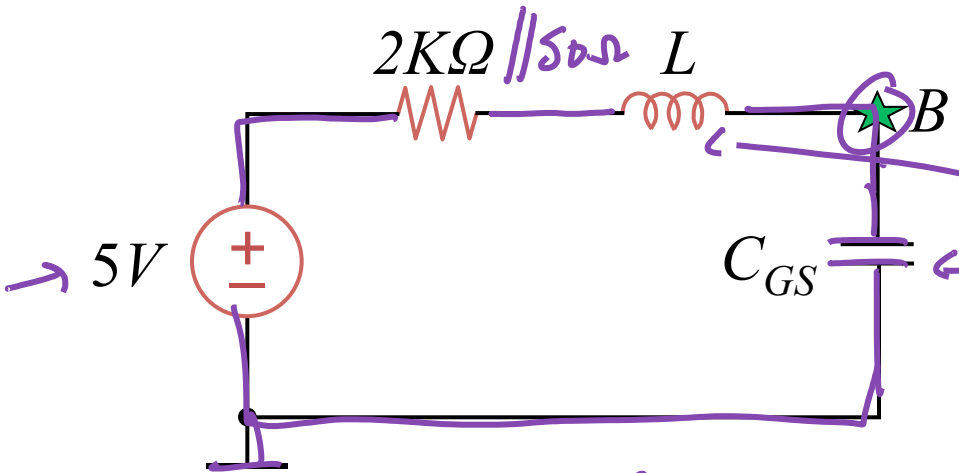


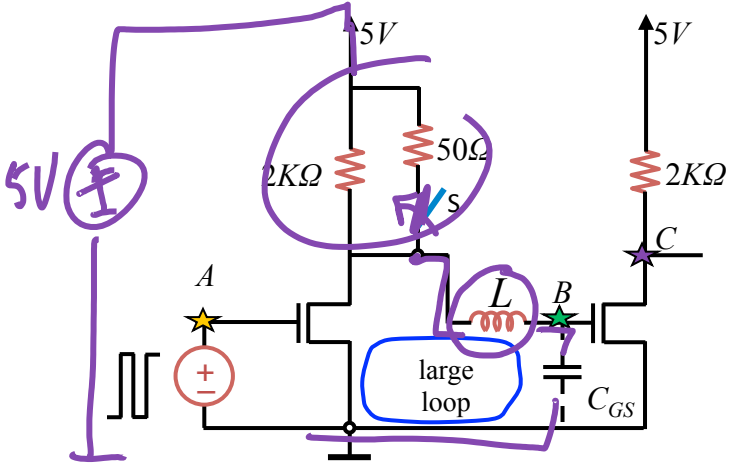


The parasitic inductance of the wire is included.

Second-Order Systems

Relevant circuit:



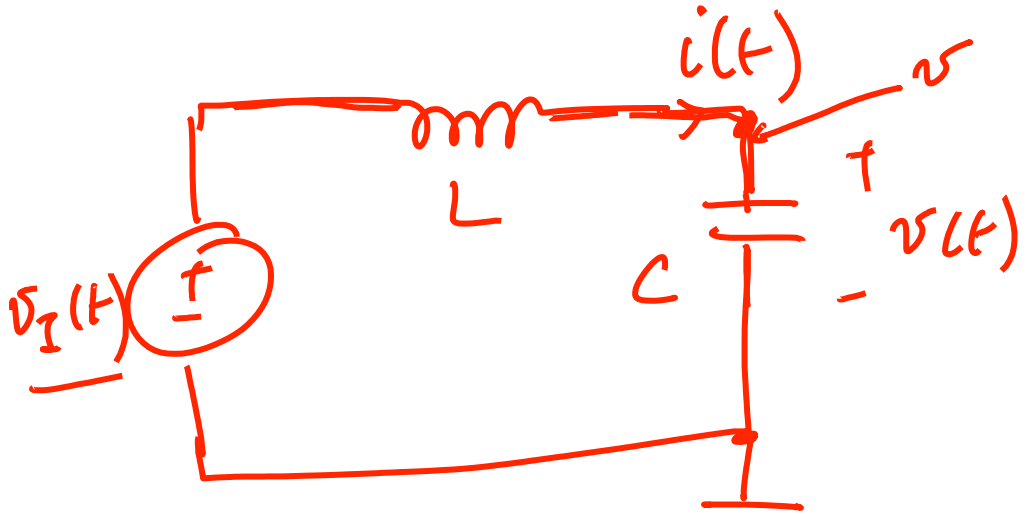


2 energy storage elements

2nd order circuit

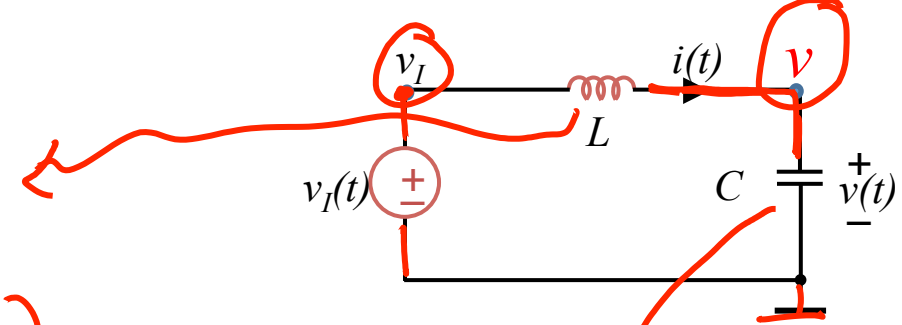
First, let's analyze the LC network

previous: RLC ckt



We will introduce R into the circuit in the next sequence

Analyzing the LC network



"v" across inductor $\left.\right\} \frac{1}{L}(v_I - v) = K\frac{di}{dt}$

$$\frac{1}{L}\int_{-\infty}^{t}(v_I - v)\,dt = i(t) \longleftrightarrow i(t) = C\frac{dv}{dt}$$

$$\frac{1}{L}\int_{-\infty}^{t}(v_I - v)\,dt = C\frac{dv}{dt}$$

2 linear

$$\frac{1}{L}(v_I - v) = C\frac{d^2v}{dt^2}$$

v_I, i state variables

$$LC\frac{d^2v}{dt^2} + v = v_I$$

Solving

Recall the method of homogeneous and particular solutions: $LC\frac{d^2v}{dt^2} + v = v_I$

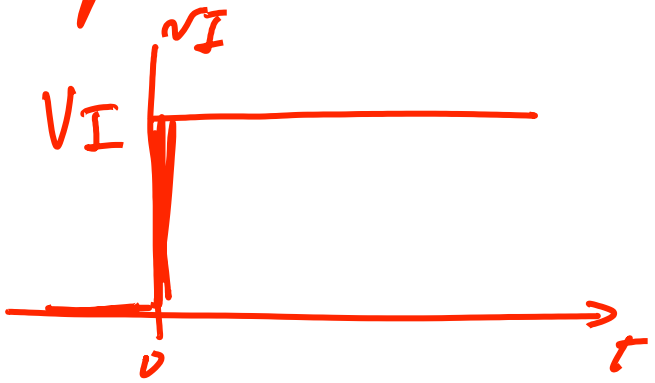
1. Find the particular solution $v_p(t)$
2. Find the homogeneous solution - $v_H(t)$
   - 4 step process
3. The total solution is the sum of the particular and homogeneous solns. Then use initial condition to solve for the remaining constants.

$$v(t) = v_p(t) + v_H(t)$$

Let's solve

$$LC\frac{d^2v}{dt^2} + v = v_I$$

For step input



and for initial condition

$v(0) = 0$, $i(0) = 0$ ZSR

1 Particular solution

$$LC\frac{d^2v}{dt^2}+v=v_I$$

$$LC\frac{d^2v_p}{dt^2}+v_p=V_I$$

$t>0$

1 Particular solution

$$LC\frac{d^2v_P}{dt^2} + v_P = V_I$$

$v_P = ?$

$v_P = V_I$ is a solution

(2) Homogeneous solution

Solution to $LC\frac{d^2 v_H}{dt^2} + v_H = 0$

Recall v_H is solution to homogenous equations (drive set to zero)

4-step method. (2A) ... (2D)

(2A) assume a solution of the form* → $v_H = Ae^{st}$, $A, s = ?$

* Diff eq are commonly solved by guessing solution

equations in s ? → characteristic eq.

(2) Homogeneous solution

Solution to $$LC\frac{d^2 v_H}{dt^2} + v_H = 0$$

Recall, v_H : solution to homogeneous equation (drive set to zero)

Four-step method:

2A - 2D

(2A) Assume solution of the form*

$$\rightarrow v_H = Ae^{st} \quad , \; A, \; s \; = \; ?$$

*Differential equations are commonly solved by guessing solutions

So, $LCAs^2e^{st} + Ae^{st} = 0$

$$LCs^2 + 1 = 0$$ ← characteristic equation

(2) Homogeneous solution

$$LC\frac{d^2 v_H}{dt^2} + v_H = 0$$

(2A) assume soln $A e^{st}$

(2B) write char. eq

$$LCs^2 + 1 = 0 \rightarrow s^2 = -\frac{1}{LC}$$ { char. eq.

(2C) Roots of char. eq.

$$s = \pm j\sqrt{\frac{1}{LC}}$$

use $\omega_0 = \sqrt{\frac{1}{LC}}$, so $s = \pm j\omega_0$

app. C of text for complex alg.

$j = \sqrt{-1}$

(2D) General solution to homogeneous equation

$$v_H = A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t} \quad (*)$$

③ Total solution

$$v(t) = v_p(t) + v_H(t)$$

$$v(t) = V_I + A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t}$$

Find unknowns from initial conditions

$v(0) = 0$ $\quad t=0$

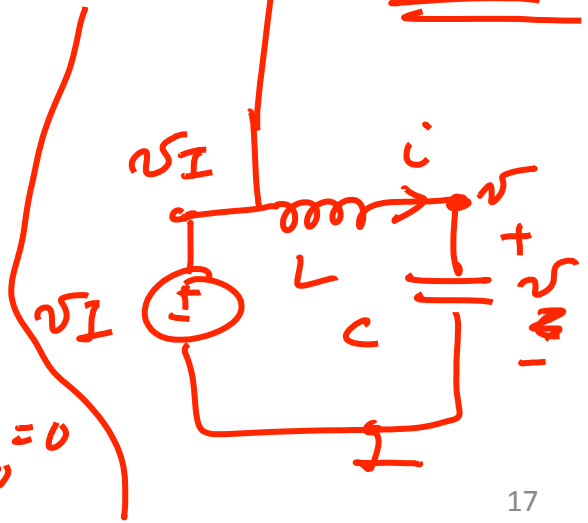
$i(0) = 0$

$$i(t) = C\frac{dv}{dt}$$

$i(0) = 0 \Rightarrow C\left.\frac{dv}{dt}\right|_{t=0} = 0$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Hints:



3 Total solution

$$v(t) = v_P(t) = v_H(t)$$

Find unknowns from initial conditions.

$$v(t) = V_I + A_1 e^{j\omega_o t} + A_2 e^{-j\omega_o t}$$

$v(0) = 0 \rightarrow 0 = V_I + A_1 + A_2$

$i(0) = 0$

$$i(t) = C\frac{dv}{dt} = CA_1 j\omega_o e^{j\omega_o t} - CA_2 j\omega_o e^{-j\omega_o t}$$

at $t = 0$,

$$0 = CA_1 j\omega_o - CA_2 j\omega_o$$

or

$$A_1 = A_2, \quad A_1 = -\frac{V_I}{2} = A_2$$

so,

$$\boxed{v(t) = V_I - \frac{V_I}{2}\left(e^{j\omega_o t} + e^{-j\omega_o t}\right)}$$

appendix C of text

③ Total solution

Remember Euler relation

$$v(t) = V_I - \frac{V_I}{2}\left(e^{j\omega_o t} + e^{-j\omega_o t}\right)$$

$e^{jx} = \cos x + j\sin x$ (verify using taylor's expansion)

$\frac{e^{jx} + e^{-jx}}{2} = \cos x$

$\frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2} \xi \cos\omega_o t$

for $t \geq 0$ so,

$$v(t) = V_I - V_I \cos\omega_o t$$

where $\omega_o = \frac{1}{\sqrt{LC}}$

$$i(t) = C V_I \omega_o \sin\omega_o t$$

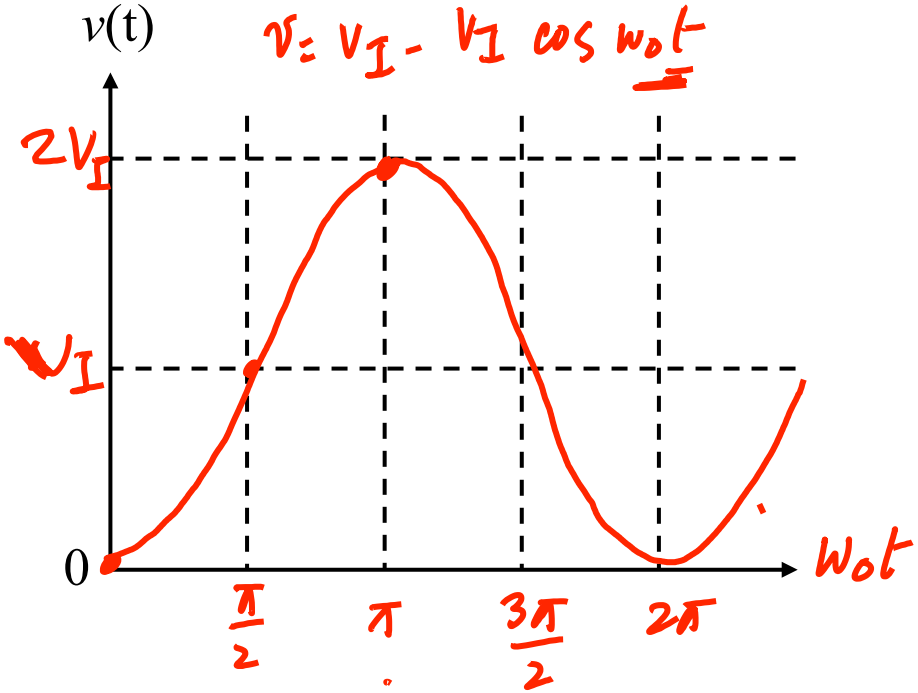
$i(t) = C\frac{dv}{dt}$

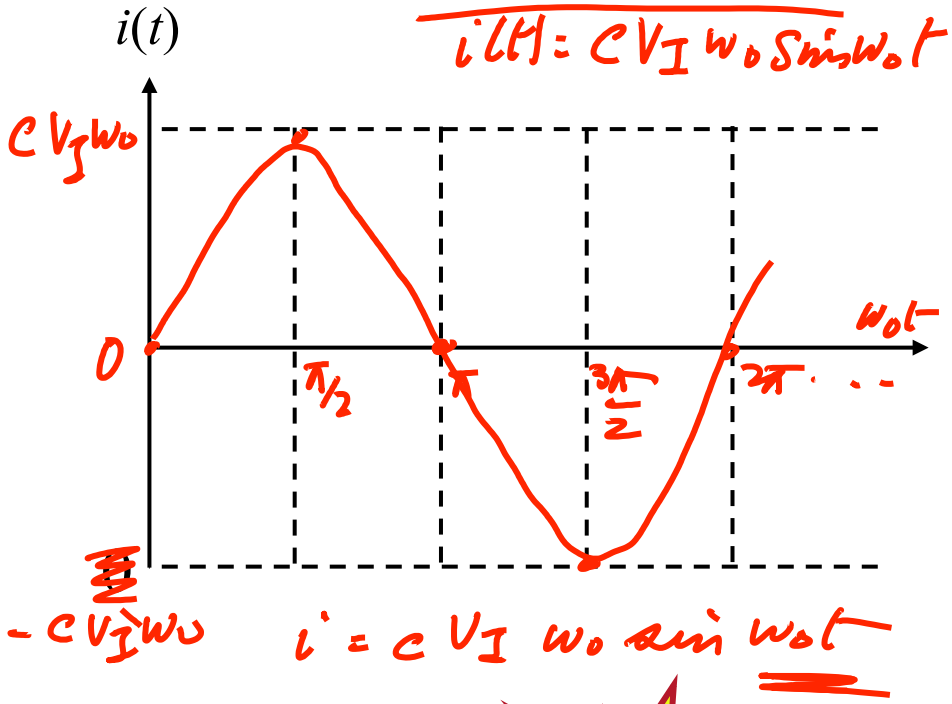
The output looks sinusoidal !!!

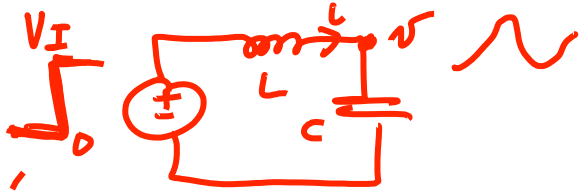


Plotting the Total Solution

$$v(t) = V_I - V_I \cos \omega_o t$$







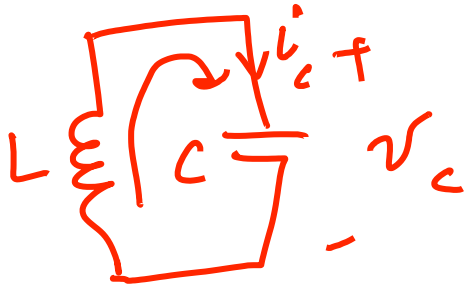
Demo

Summary of Method

1. Write DE for circuit by applying node method.
2. Find particular solution v_P by guessing and trial & error.
3. Find homogeneous solution v_H
   - A. Assume solution of the form Ae^{st}.
   - B. Obtain characteristic equation.
   - C. Solve characteristic equation for roots s_i.
   - D. Form v_H by summing $A_i e^{s_i t}$ terms.
4. Total solution is $v_P + v_H$, then solve for remaining constants using initial conditions.

Example

What if we have:



$v_c(0) = V$

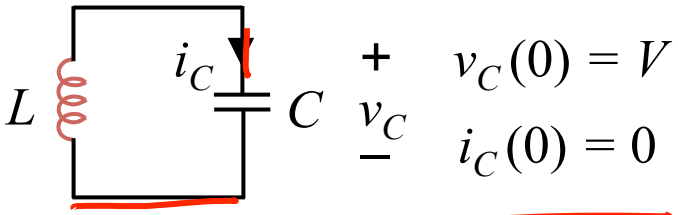
$i_c(0) = 0$

ZIR

We can obtain the answer directly from the homogeneous solution ($V_I = 0$)

Example

We can obtain the answer directly from the homogeneous solution ($V_I = 0$).



$\omega_0 = \frac{1}{\sqrt{LC}}$

$$v_C(t) = A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t}$$ from (*) with $V_I = 0$

$v_C(0) = V$; $V = A_1 + A_2$

$i_C(0) = 0$ $\quad 0 = jA_1\omega_0 - jA_2\omega_0 \Rightarrow A_1 = A_2$

$A_1 = A_2 = V/2$

Remember $i(t) = C\frac{dv}{dt}$

or $v_c = \frac{V}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$

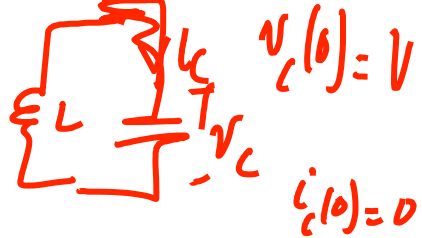
$v_c = V\cos\omega_0 t$

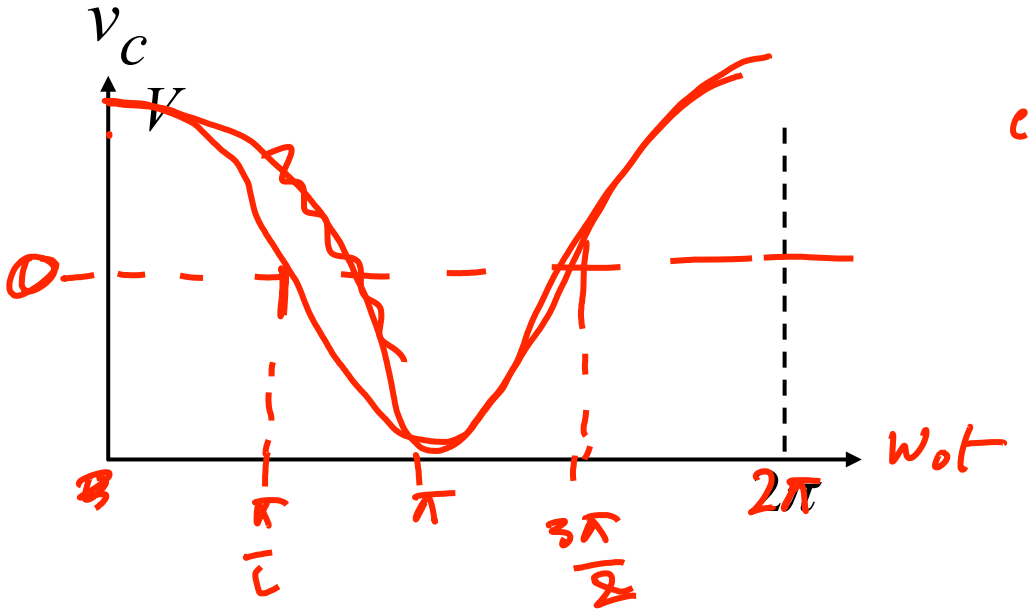
$i_c = -CV\omega_0 \sin\omega_0 t$

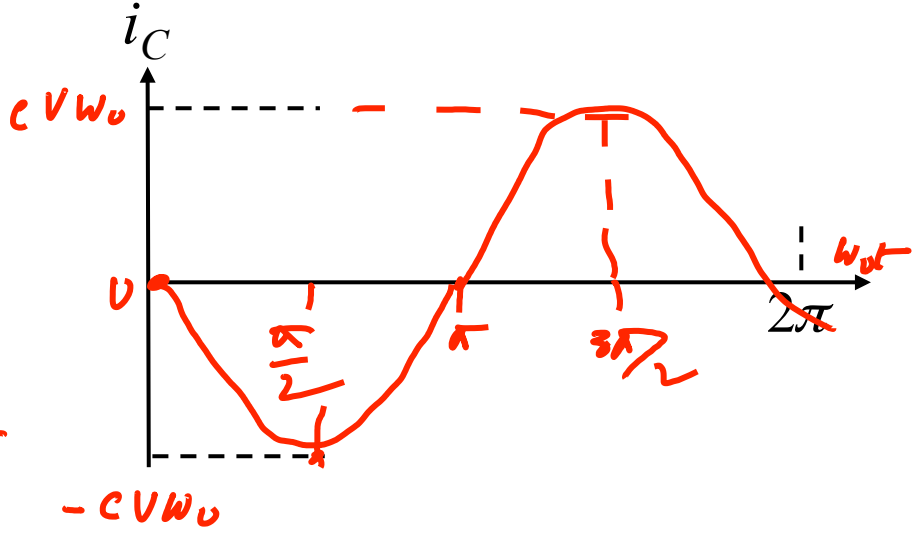
Example

$v_L = V \cos \omega_0 t$

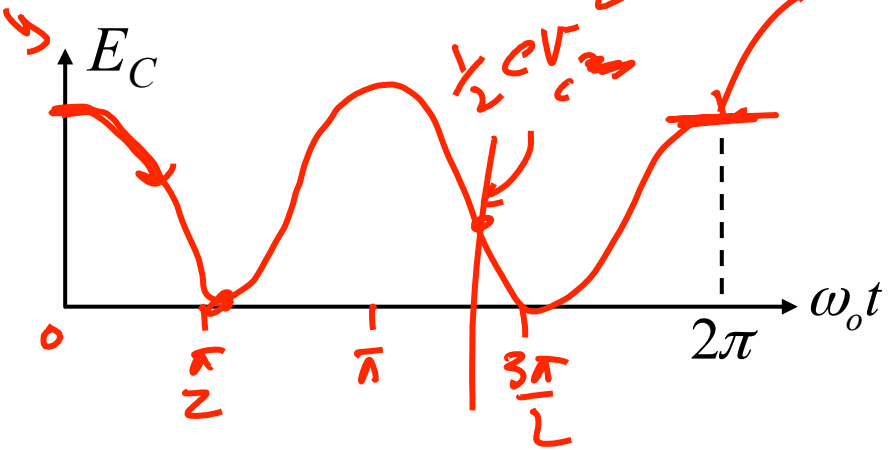
$i_C = -CV\omega_0 \sin \omega_0 t$

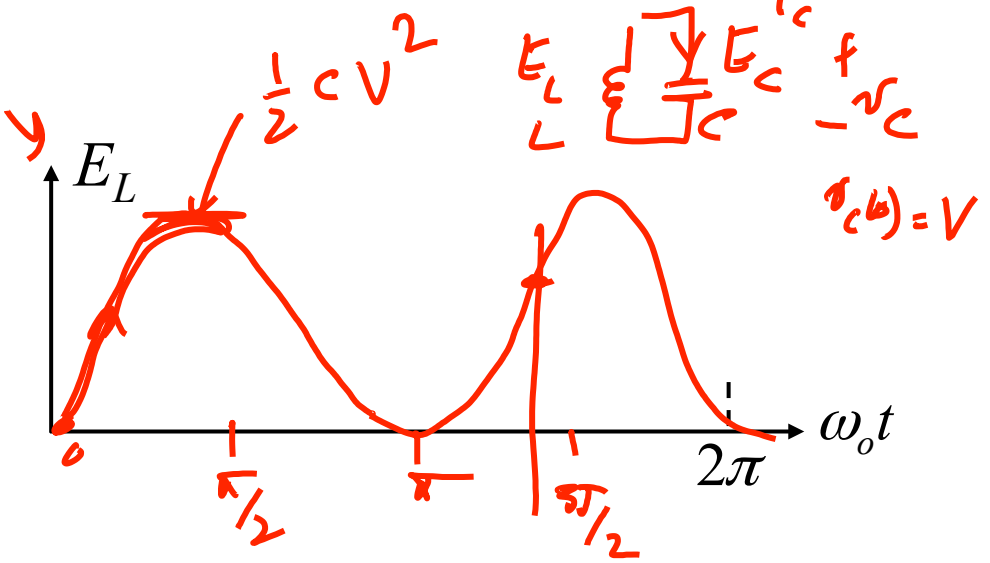






Energy



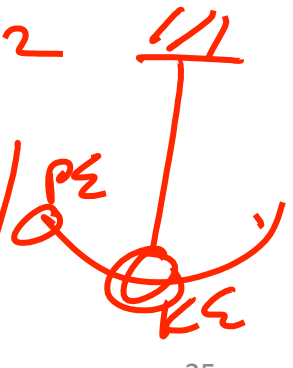


C: $\frac{1}{2}Cv_c^2$

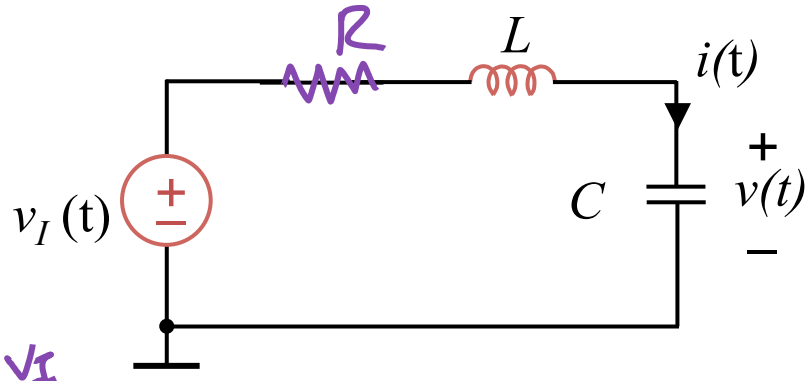
L: $\frac{1}{2}Li_c^2$

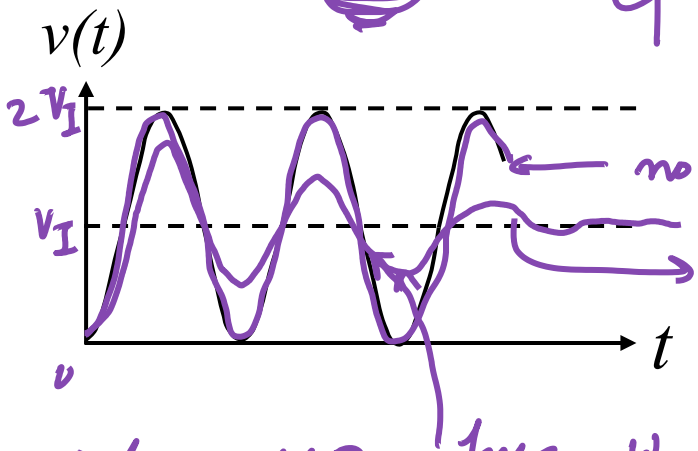
notice $\frac{1}{2}Cv_c^2 + \frac{1}{2}Li_c^2 = \frac{1}{2}CV^2$

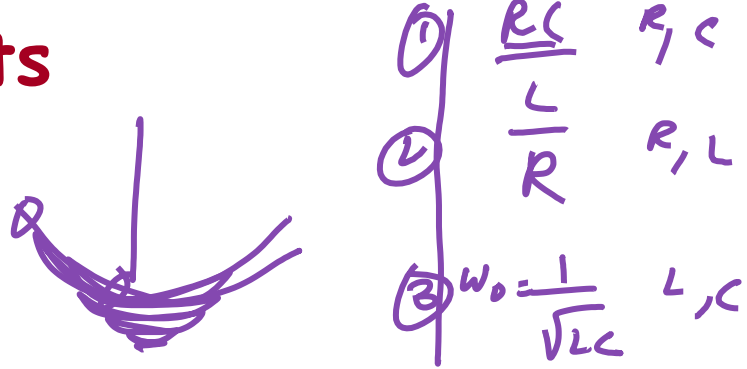
Total energy in the system is a constant but it sloshes back and forth between the capacitor and the inductor.



Next, introduce R: RLC Circuits









Damped sinusoids with R freq $\omega_0 = \frac{1}{\sqrt{LC}}$

Recall demo

Next sequence! If you are impatient, see section 13.2 in the textbook

More in the next sequence! If you are impatient, see A&L Section 13.2