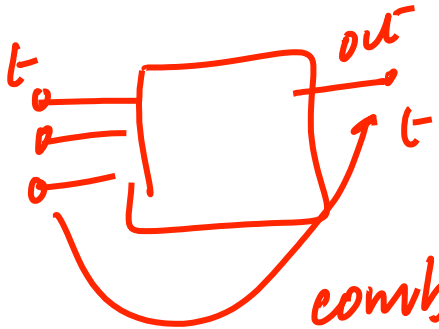
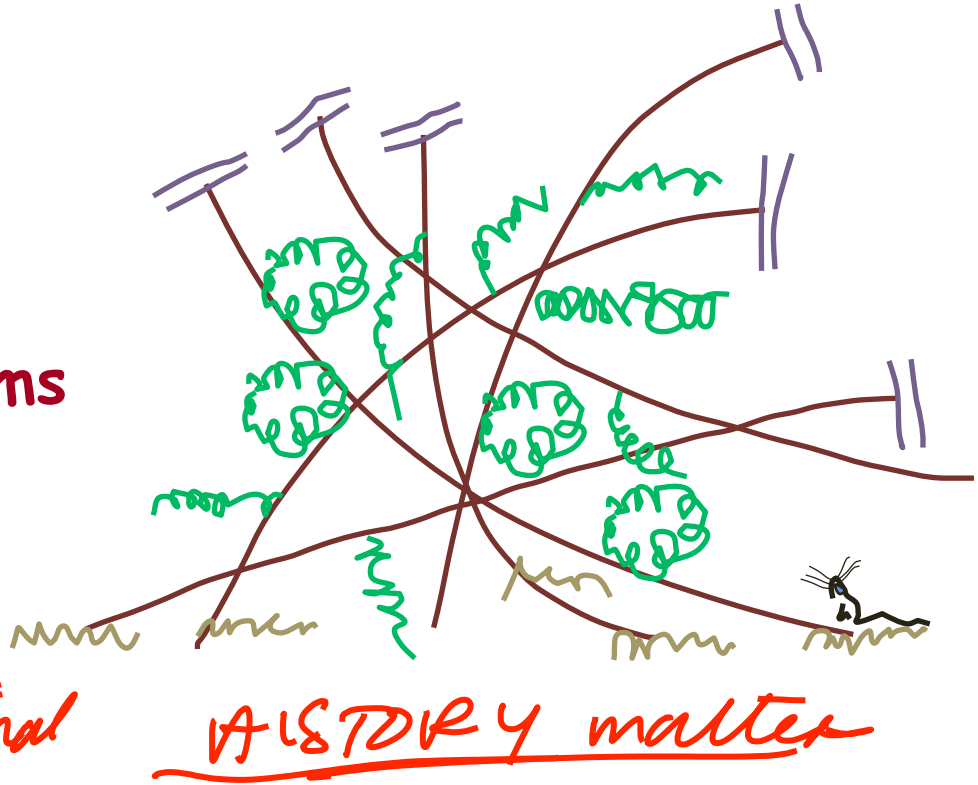
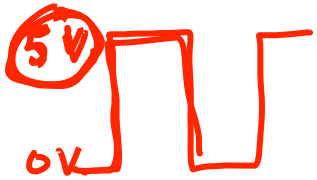
6.002x CIRCUITS AND ELECTRONICS

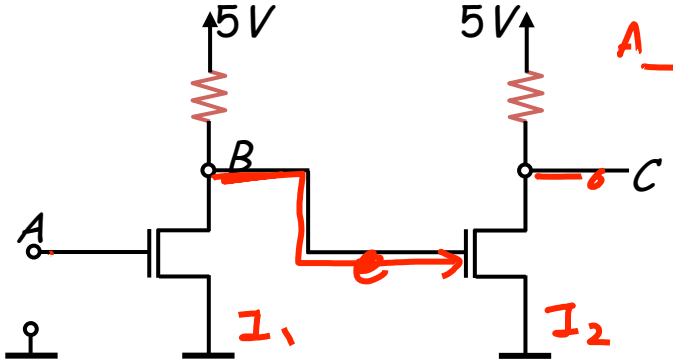
Capacitors and First-Order Systems



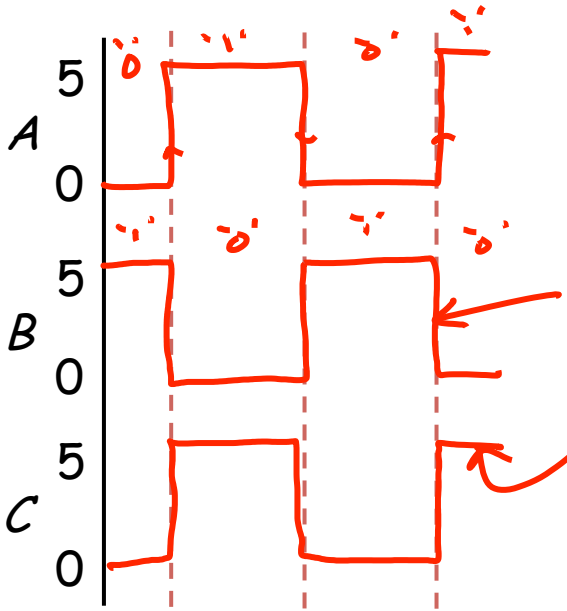


Motivation





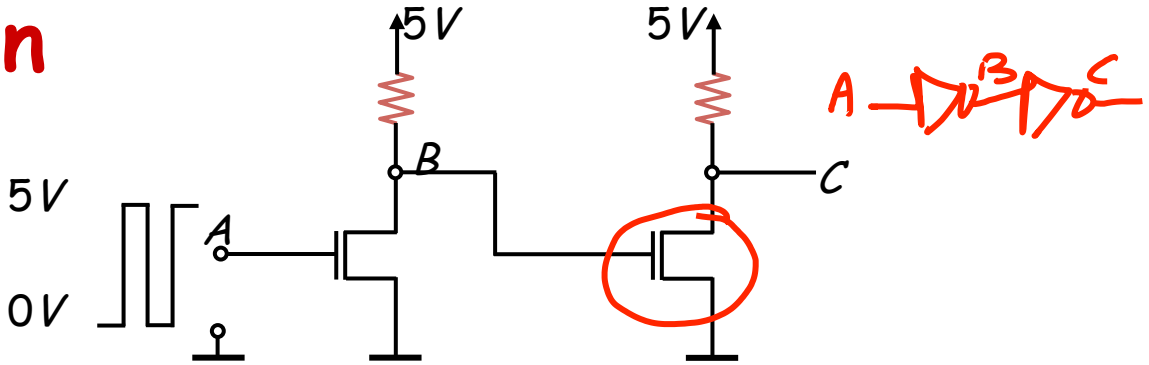


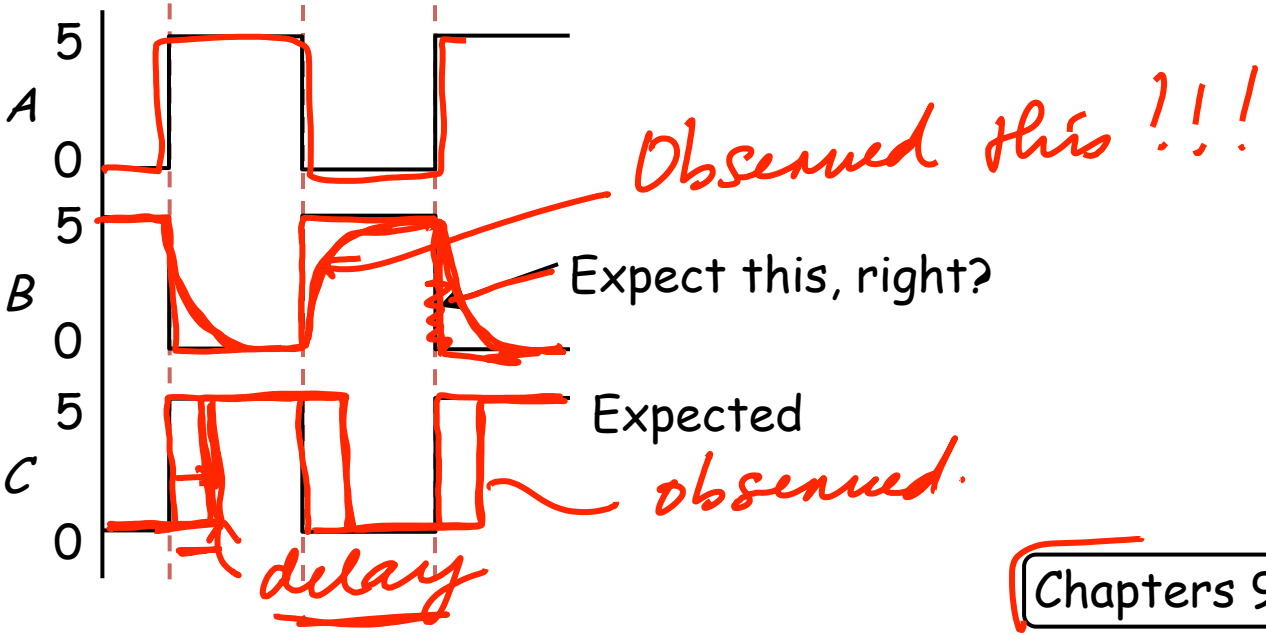


Expect to see this, right?!

Chapters 9 & 10

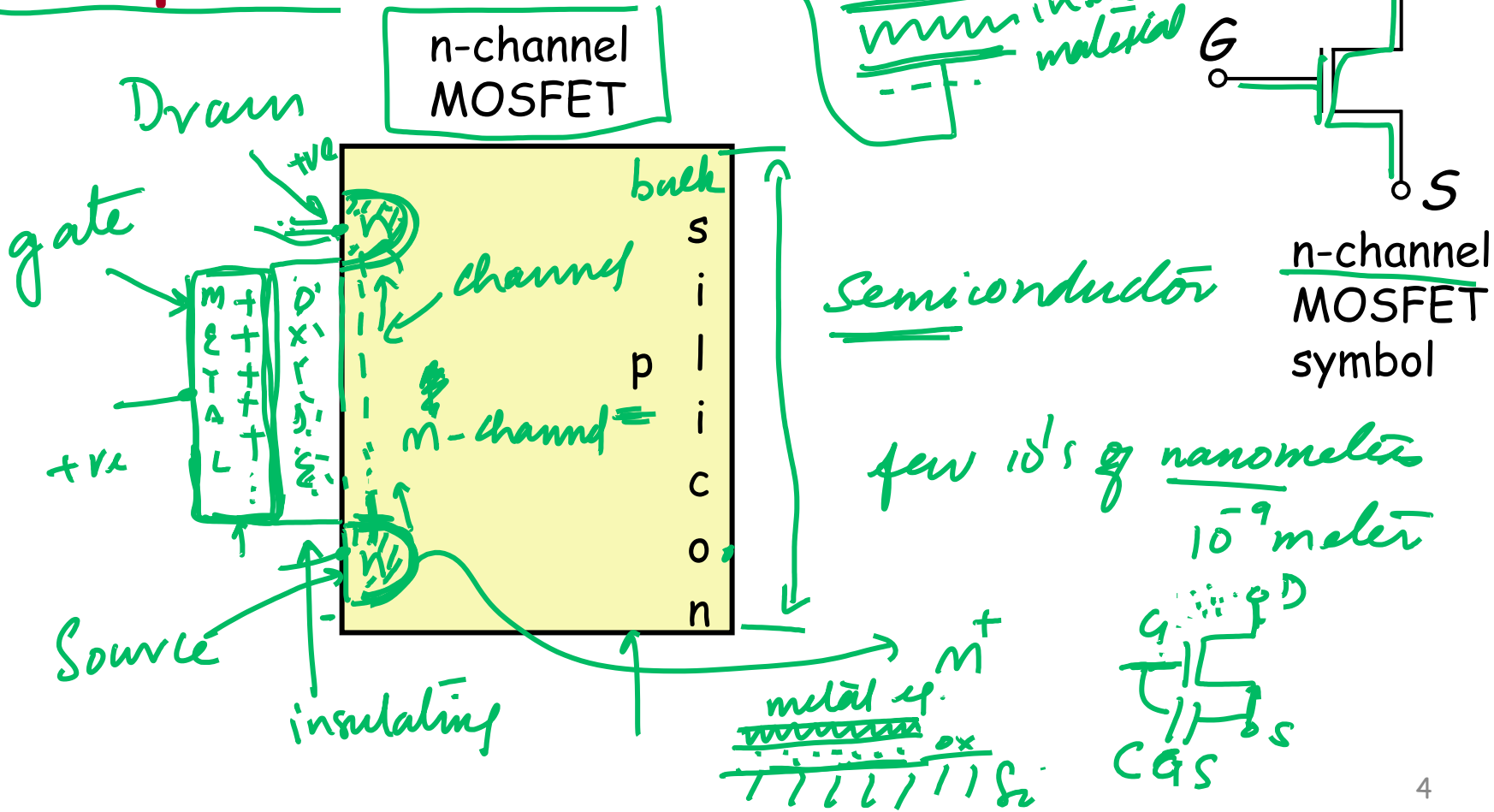
Motivation



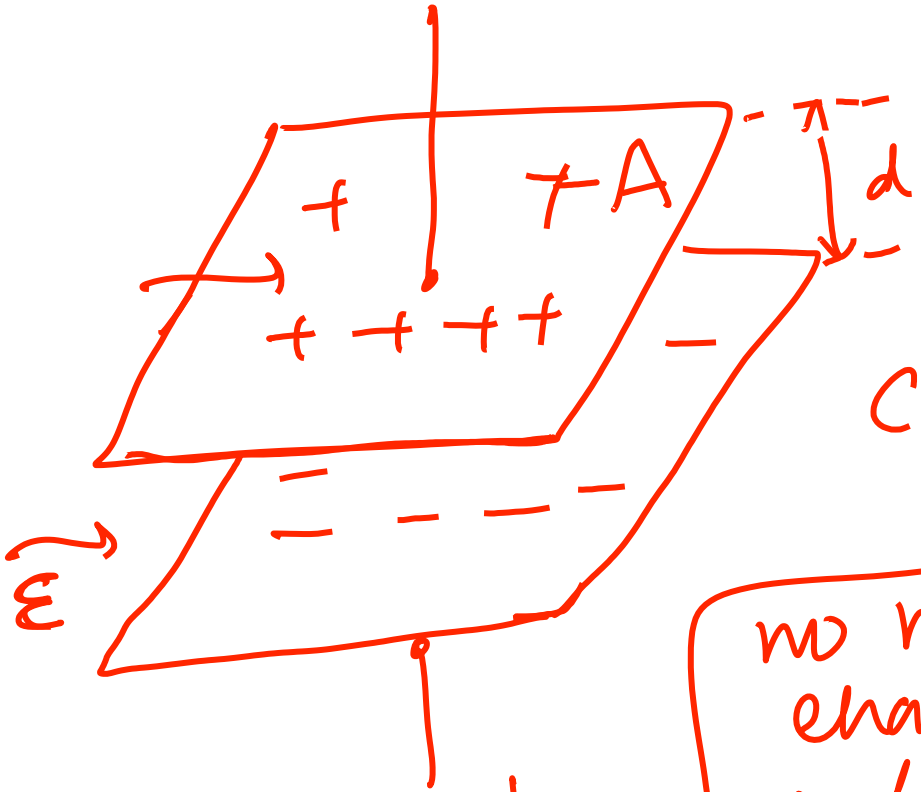


Chapters 9 & 10

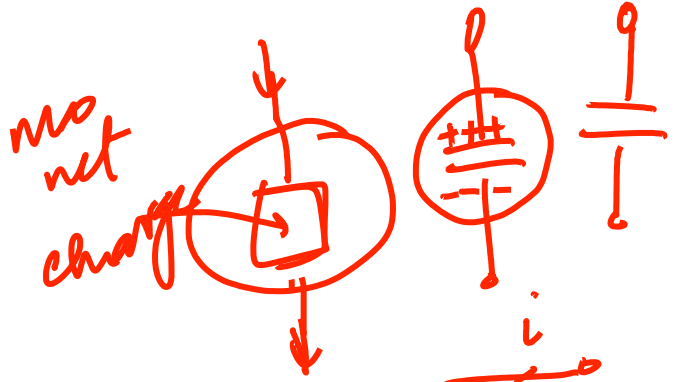
The Capacitor



Ideal Linear Capacitor



obeys DMD!



$$C = \frac{\varepsilon A_{//}}{d_{//}}$$

no net charge in element

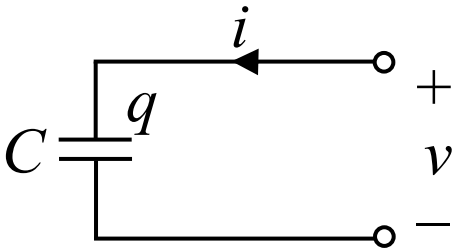
$$q = C\,v$$

coulombs = farads · volts

tot at charge on cap = $+q - q = 0$ //

elst-element!

Ideal Linear Capacitor



$$q = C\,v$$

Capacitors are linear devices*

$$i = \frac{dq}{dt} = \frac{d(Cv)}{dt}$$

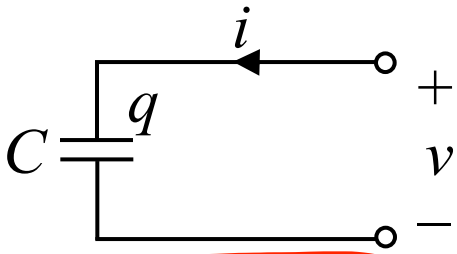
6.002x assumptions

$$i = C\frac{dv}{dt} \quad \leftarrow L \quad \leftarrow 1$$

$$v = \frac{1}{C}\int_{-\infty}^{t} i\,dt$$

*Verify for yourselves that capacitors satisfy both the properties of homogeneity and superposition

Ideal Linear Capacitor



$$q = C\,v$$

$$i = C\frac{dv}{dt}$$

$$P = vi = vC\frac{dv}{dt} = \frac{d}{dt}\left(\frac{1}{2}Cv^2\right)$$

Demo

$$E = \frac{1}{2}Cv^2$$

ultra caps

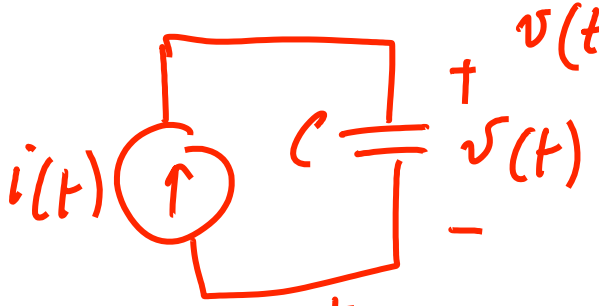
→ memory devices

DRAM → 1GB

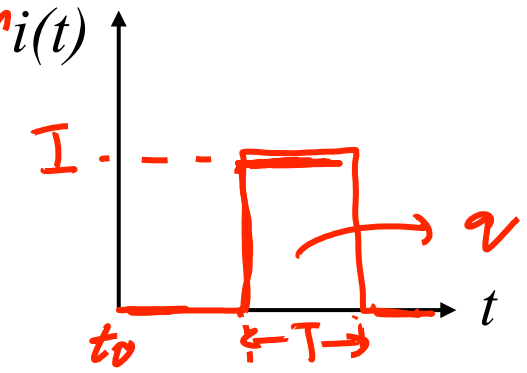


A capacitor is an energy storage device → memory device → history matters!

Capacitor and a Current Source



$v(t_0) \rightarrow$ given



$$v = \frac{1}{C}\int_{-\infty}^{t} i\,dt$$

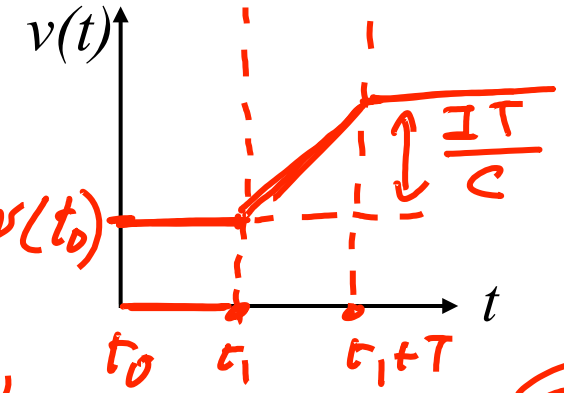
$$= \underbrace{\frac{1}{C}\int_{-\infty}^{t_0} i\,dt}_{-q} + \frac{1}{C}\int_{t_0}^{t} i\,dt$$

$$= \underbrace{v(t_0)}_{\text{given}} + \frac{1}{C}\int_{t_0}^{t} i\,dt$$

$v(t_0) = C\,v(t_0)$

$\hookrightarrow t_0 - t_0$

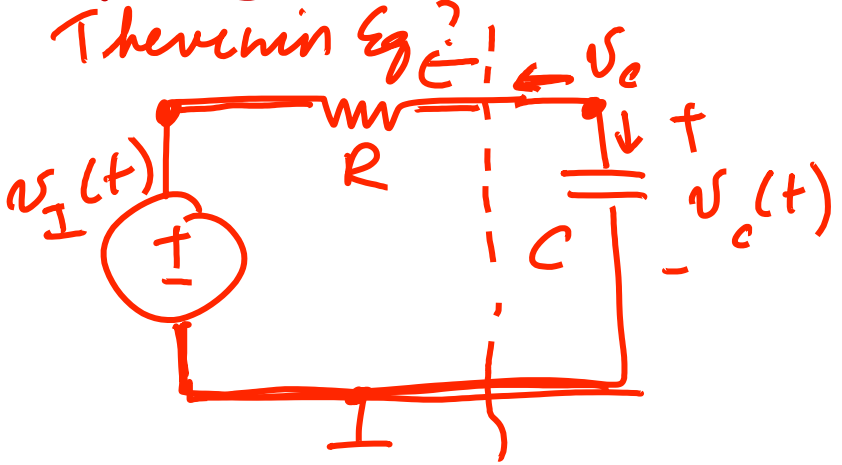
$\hookrightarrow t_1 \rightarrow t_1 + T,\ t_1 + T \rightarrow$

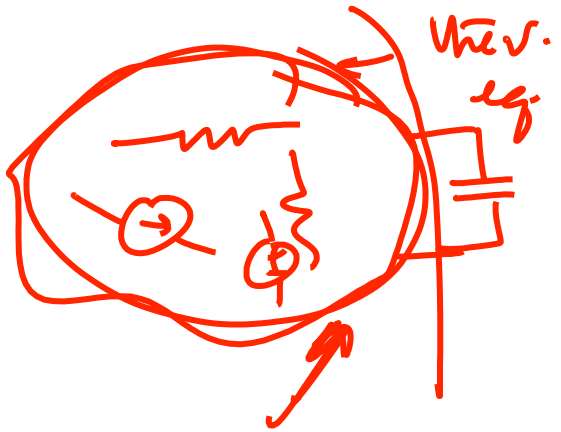


$T: \Delta q = I\,T$

See 9.4 in the text

Analyzing an RC circuit





apply node method:

$$\frac{v_c - v_I}{R} + C\frac{dv_c}{dt} = 0$$

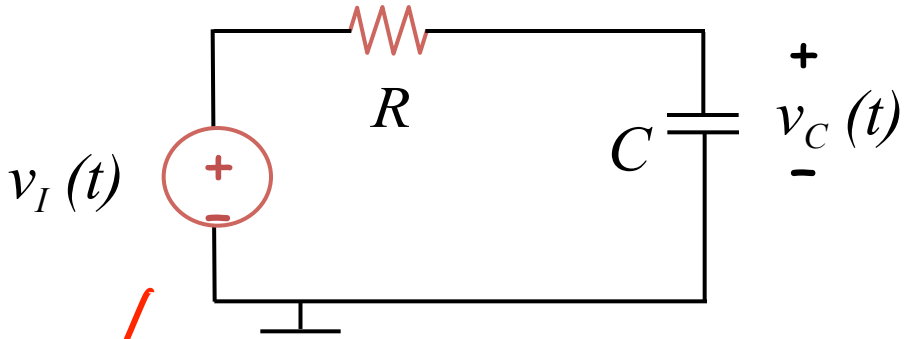
$$RC\frac{dv_c}{dt} + v_c = v_I$$

units of time

Circuits with one energy storage element are called first order circuits

$$\text{If } \begin{cases} t \geq t_0 \\ v_c(t_0) \text{ given} \end{cases}$$

Let's do an example RC circuit



$\Longrightarrow v_c(0) = V_0$ given

$\Longrightarrow v_I(t) = V_I$

$RC\frac{dv_c}{dt} + v_c = V_I$ ——— (X)

next steps – math!

How to solve the differential equation

Method of homogeneous and particular solutions:

1. Find the particular solution.
2. Find the homogeneous solution.
3. The total solution is the sum of the particular and homogeneous solutions. Use the initial conditions to solve for the remaining constants.

$$v_I(t) = V_I$$

$v_C(0) = V_0$ given

$$RC\frac{dv_C}{dt} + v_C = V_I$$

$v_C(t) = v_{CH}(t) + v_{CP}(t)$

total = homogeneous + particular

soln. = guesswork

Solving differential equations is actually quite easy – commonly guesswork and applying patterns

1 Particular solution

v_{CP} : any solution that satisfies the original equation (X)

$$v_I(t) = V_I$$

$$v_C(0) = V_0 \text{ given}$$

$$RC\frac{dv_C}{dt} + v_C = V_I$$

(X)

In general, use trial and error (guesswork!)

$$RC\frac{dv_{CP}}{dt} + v_{CP} = V_I$$

$$\boxed{v_{CP} = V_I}$$ works

$$RC\frac{dV_I}{dt} + V_I = V_I$$ check ✓ twice

$\frac{dV_I}{dt} = 0$

② Homogeneous solution

v_{CH}: solution to the homogeneous equation (set drive to zero) (Y)

$$v_I(t) = V_I$$

$$v_C(0) = V_0 \text{ given}$$

$$RC\frac{dv_C}{dt} + v_C = V_I \quad \text{(Y)}$$

$$RC\frac{dv_{CH}}{dt} + v_{CH} = 0$$

$v_{CH} = Ae^{st}$ assume a solution of this form. A? s?

$$RC\frac{d}{dt}Ae^{st} + Ae^{st} = 0$$

A, s? → $RCAse^{st} + Ae^{st} = 0$ Discard trivial A=0 solution

$RCs + 1 = 0$ Characteristic Equation

$s = -\frac{1}{RC}$ → or $v_{CH} = Ae^{-\frac{1}{RC}t}$ $= Ae^{-t/\tau}$

RC is called time constant τ

③ Total solution

$v_I(t) = V_I$

$v_C(0) = V_0$ given

$$RC\frac{dv_C}{dt} + v_C = V_I$$

$\otimes$

$v_C = v_{CP} + v_{CH}$

$\Rightarrow v_C = V_I + A e^{-\frac{1}{RC}t}$

Find remaining unknowns from initial conditions

Given, $v_C = V_0$ at $t = 0$ //

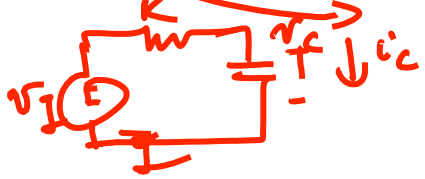
So, $V_0 = V_I + A$ → substituting $v_C = V_0$ at $t = 0$ in

or $A = V_0 - V_I$

Thus $\boxed{v_C = V_I + (V_0 - V_I)\, e^{-\frac{1}{RC}t}}$

$v_{CP} = V_I$

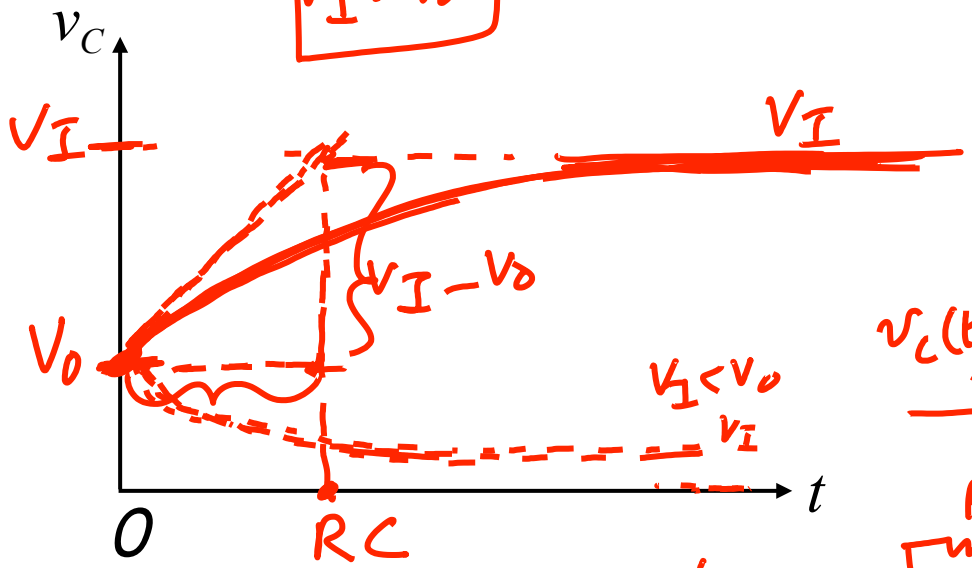
$v_{CH} = Ae^{\frac{-t}{RC}}$



$$i_C = C\cdot\frac{dv_C}{dt} = -\frac{1}{R}(V_0 - V_I)\, e^{-\frac{1}{RC}t}$$

Plot Solution

$V_I > V_0$



$$v_C = V_I + (V_0 - V_I)\, e^{\frac{-t}{RC}}$$

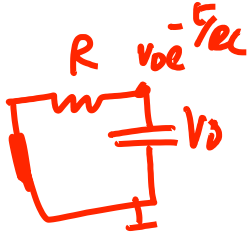
Rearrange

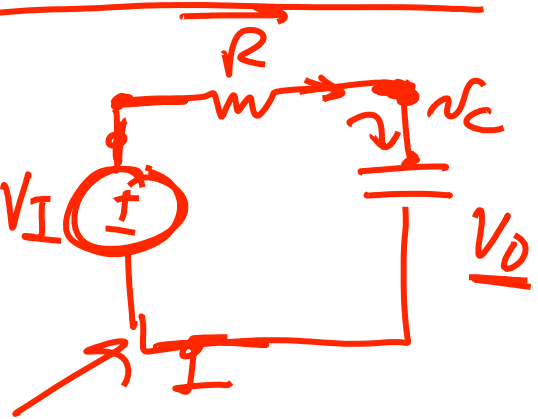
$$v_C = V_0\, e^{-t/RC} + V_I\left(1 - e^{\frac{-t}{RC}}\right)$$

$$= v_C(t) = v_C(0)\, e^{-t/RC} + v_C(\infty)\left(1 - e^{-t/RC}\right)$$

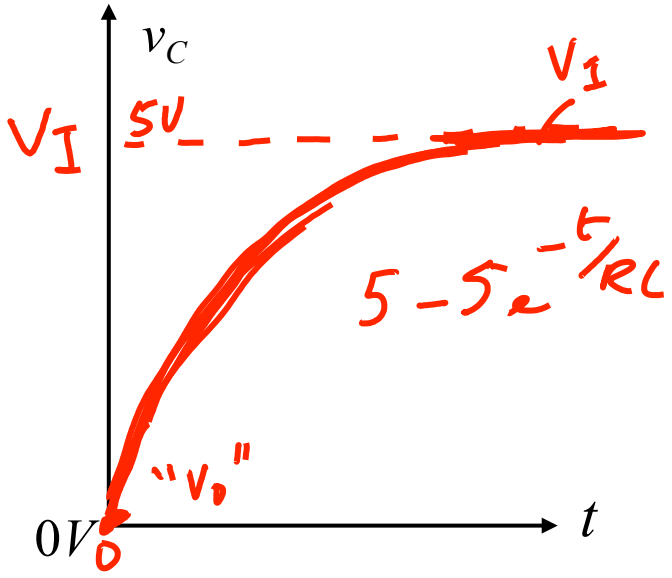
$$\left.\frac{dv_C}{dt}\right|_{t=0} = -\frac{(V_0 - V_I)}{RC}\, e^{\frac{-t}{RC}}\Big|_{t=0}$$

$$= \frac{V_I - V_0}{RC}$$ slope @ $t=0$





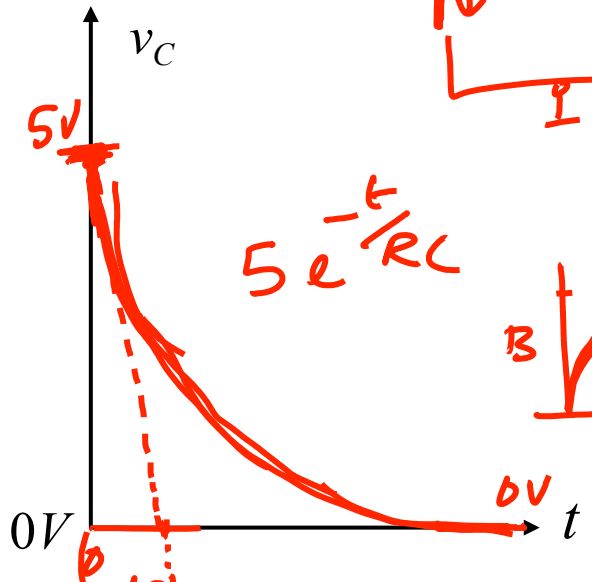
Examples



$v_C(0) \rightarrow V_0 = 0V$

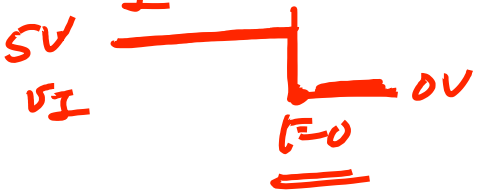
$v_I(H) \rightarrow V_I = 5V$

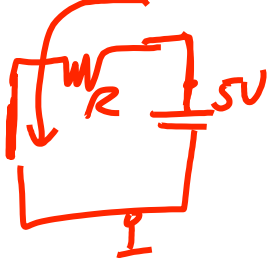
$t \geq 0$



$V_0 = 5V$

$V_I = 0V$







$v_C = V_I + (V_0 - V_I)e^{-\frac{t}{RC}}$

