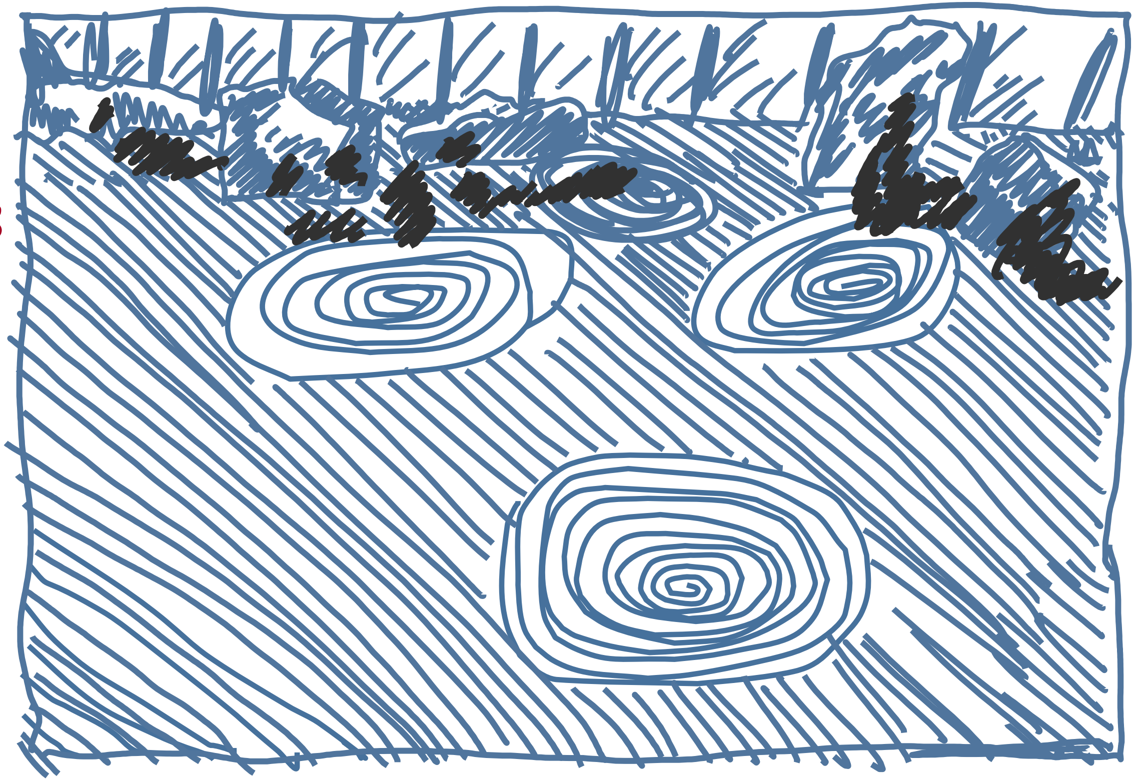
6.002x

CIRCUITS AND ELECTRONICS

Incremental Analysis

Small signal trick



Review

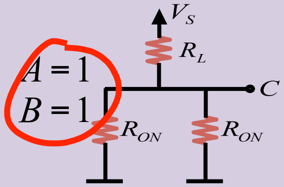
The 6.002x world

Lumped Circuit Abstraction

KVL/KCL
Node
Composition

Linear

Superposition
Thevenin, Norton

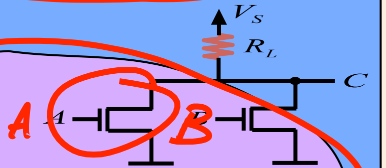


Nonlinear

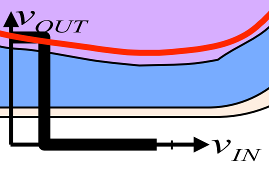
Analytical
Graphical

Small Signal Model

Digital



For fixed input values …



Review

Nonlinear Analysis

- Analytical method
- Graphical method
- Piecewise Linear method

Today

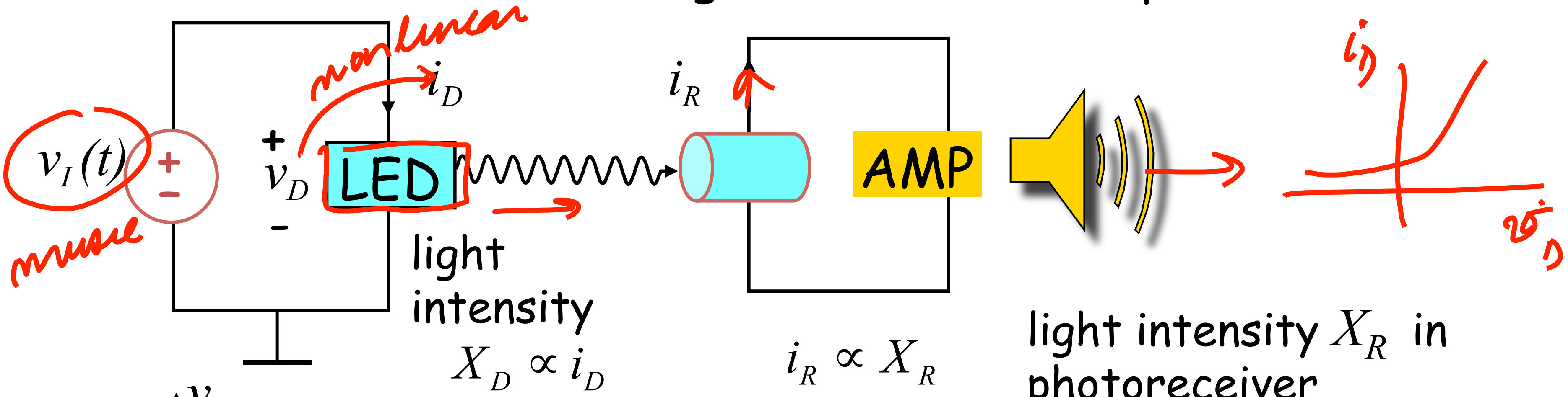
- Incremental analysis or small signal method

Reading: Section 4.5

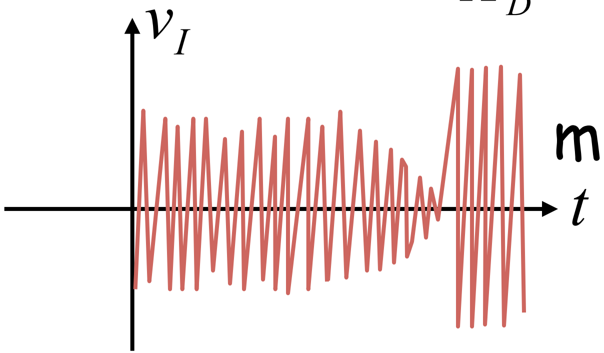
Method 3: Incremental Analysis

(Actually a particular disciplined use of a circuit)

Motivation: music over a light beam. Can we pull this off?



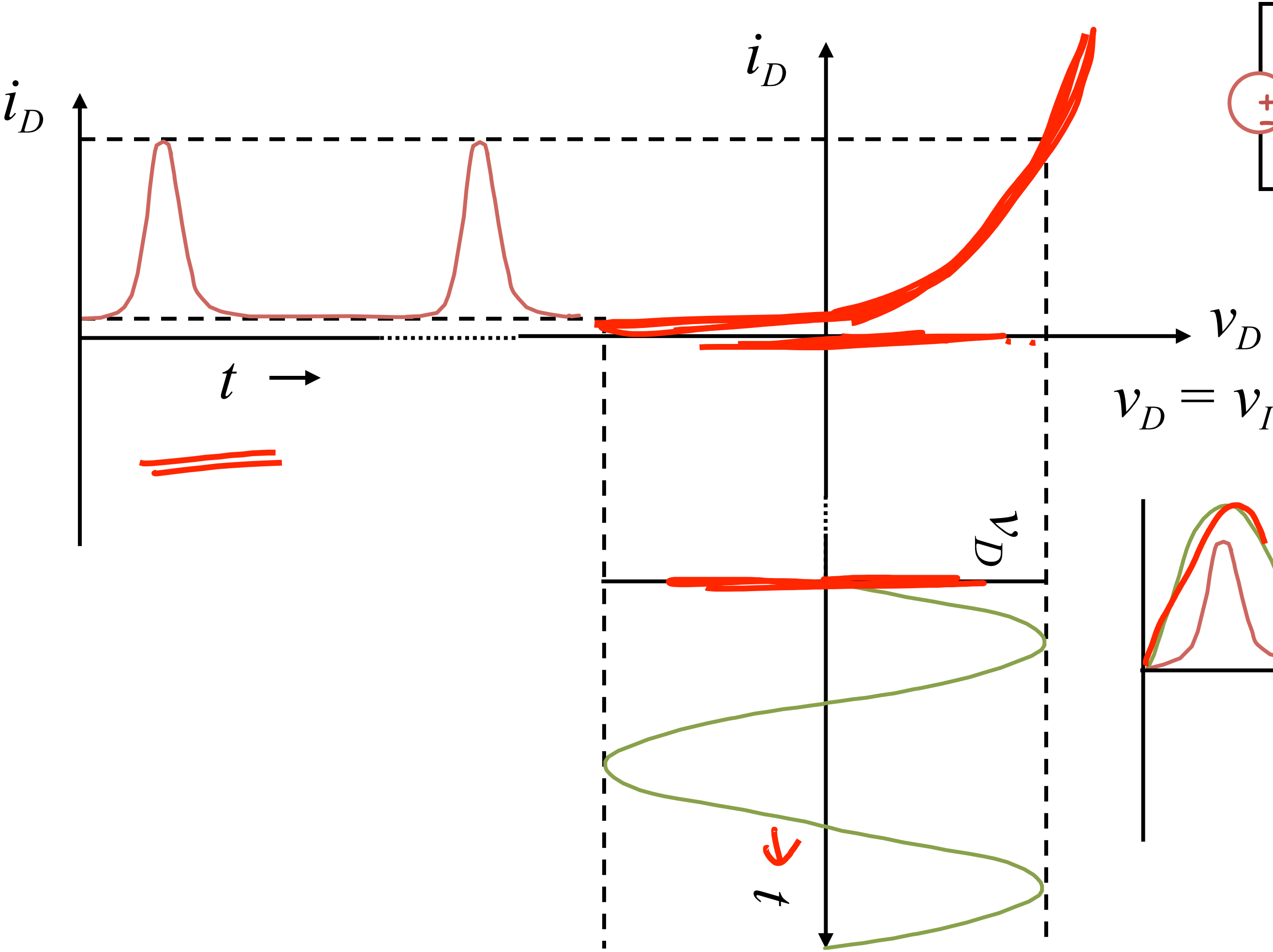
light intensity X_R in photoreceiver

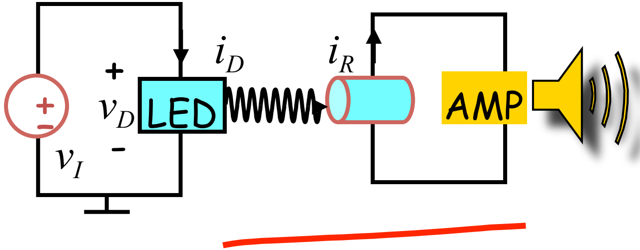


music signal

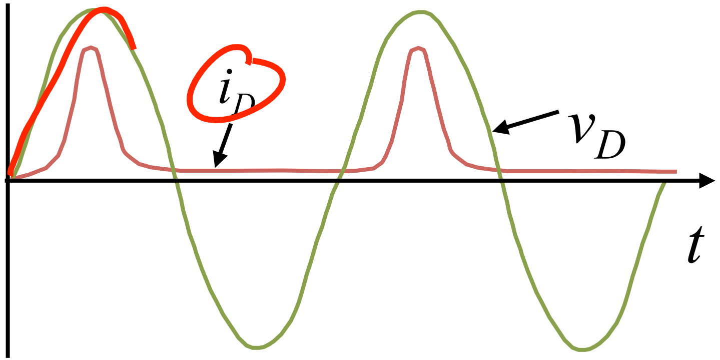
Remember LED: Light Emitting expoDweep ☺

Problem: The LED is nonlinear → distortion

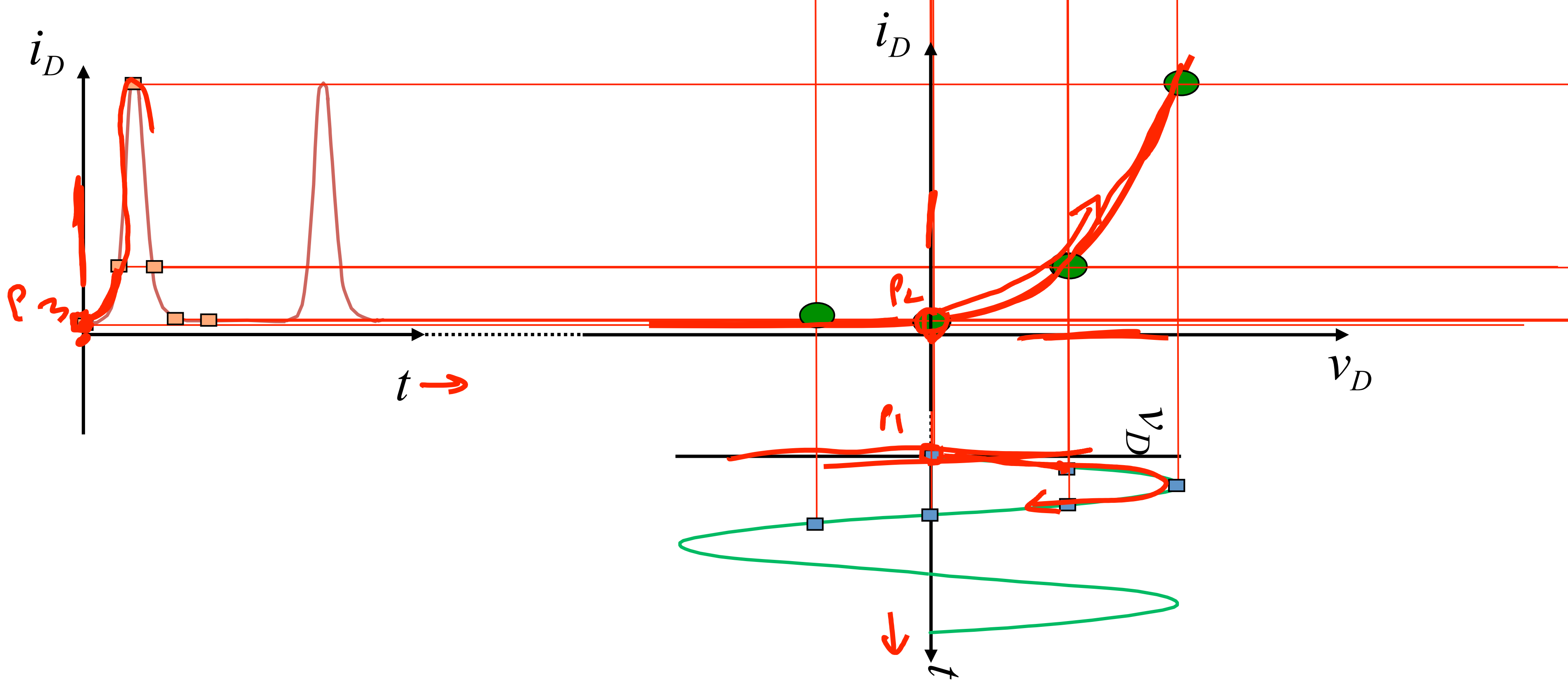




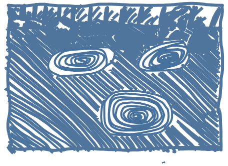
$v_D = v_I$

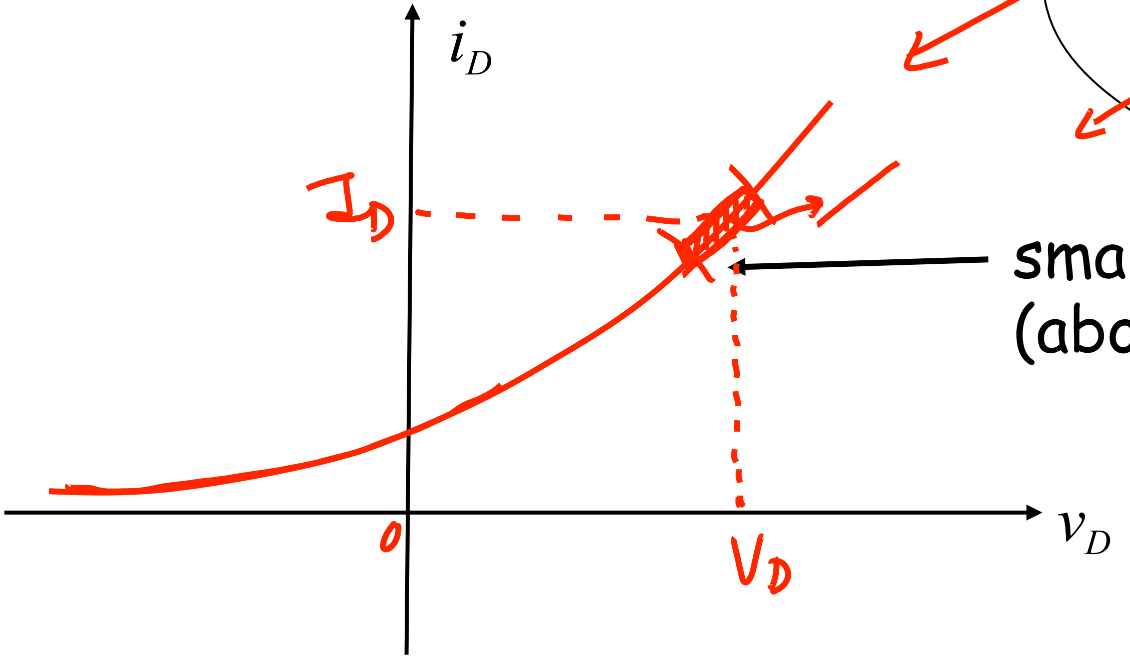


Problem: The LED is Nonlinear



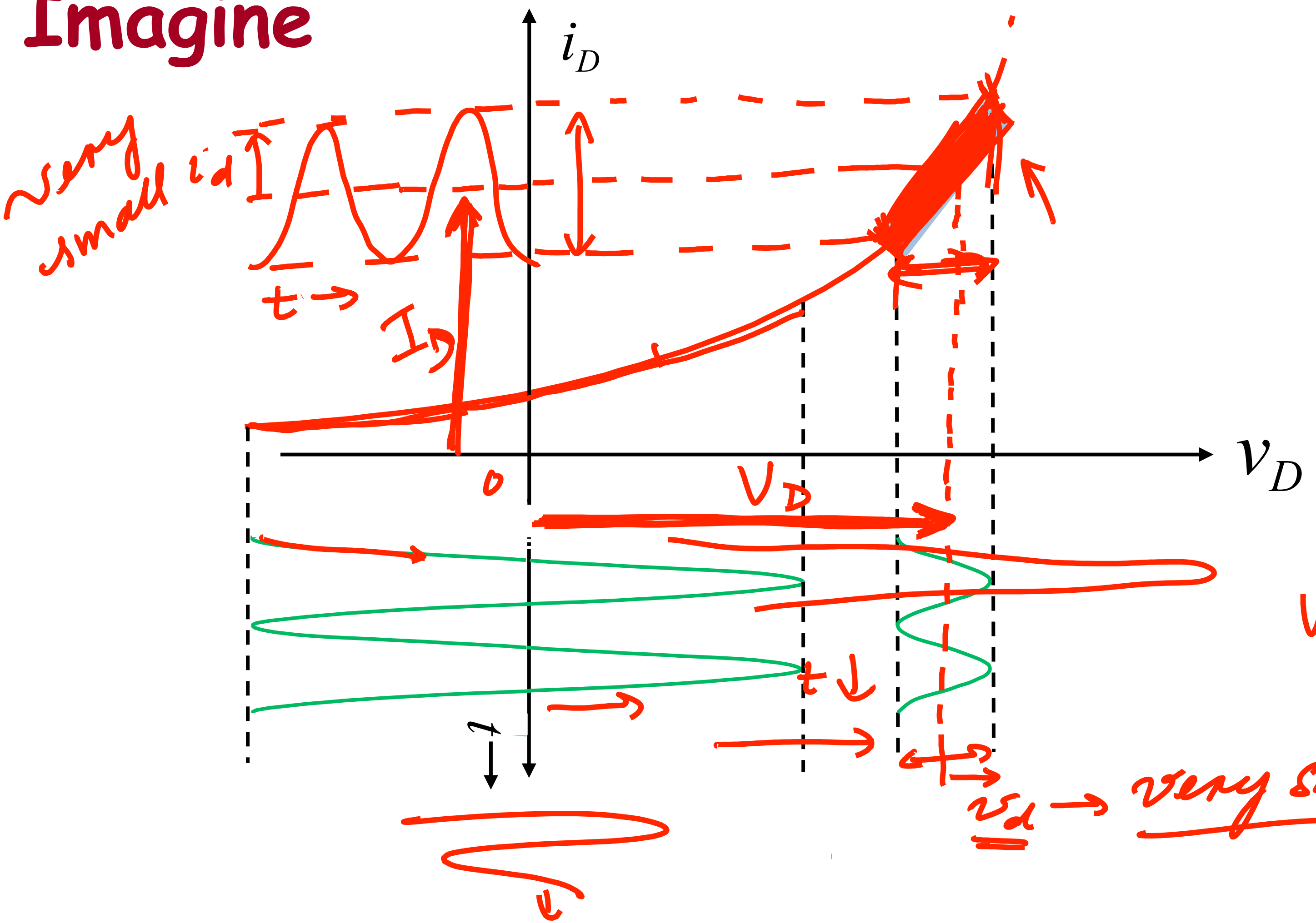
Insight

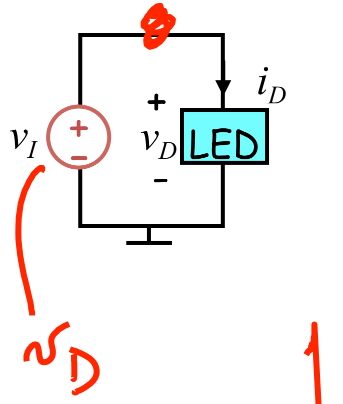




small region looks linear
(about some given V_D, I_D)

Imagine





V_D Boost

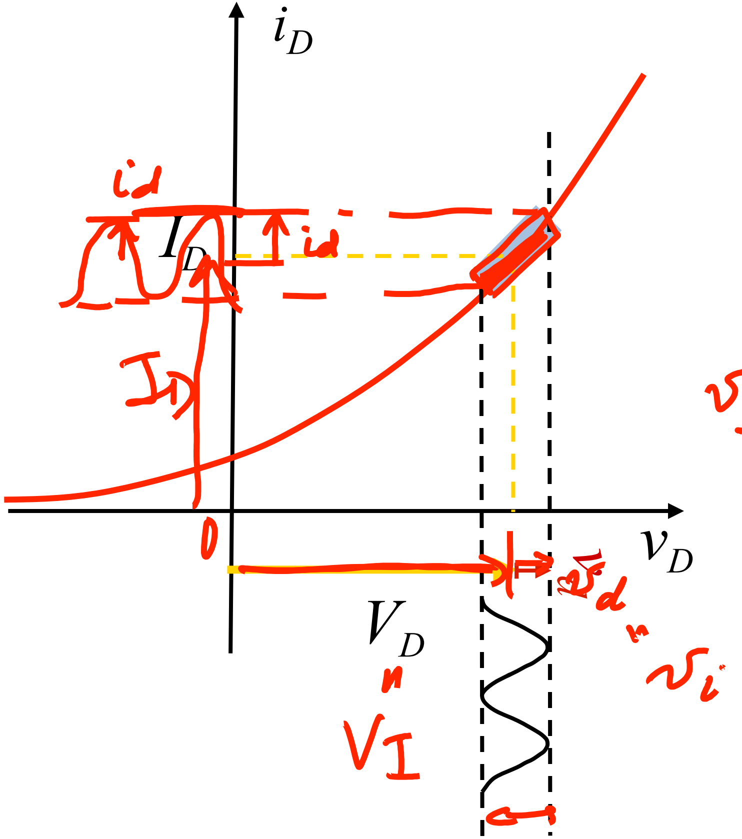
and

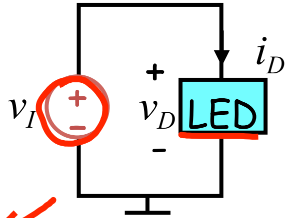
Shrink

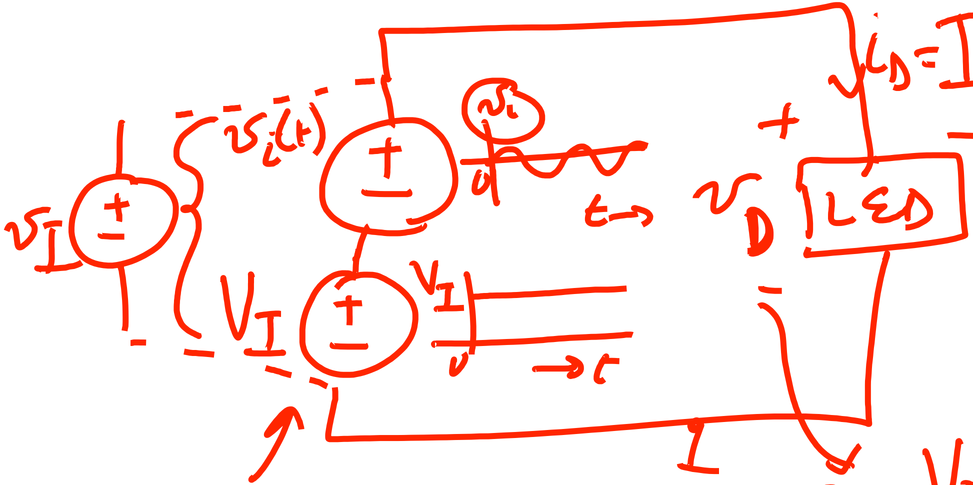
v_d

$v_d \rightarrow$ very small

How to implement this trick







$i_D = I_D + i_d$

$v_D = V_D + v_d$

$v_I = V_I + v_i$

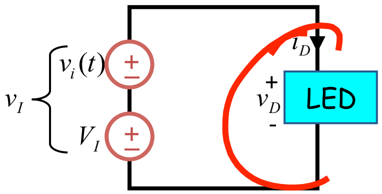
"Boost and Shrink"

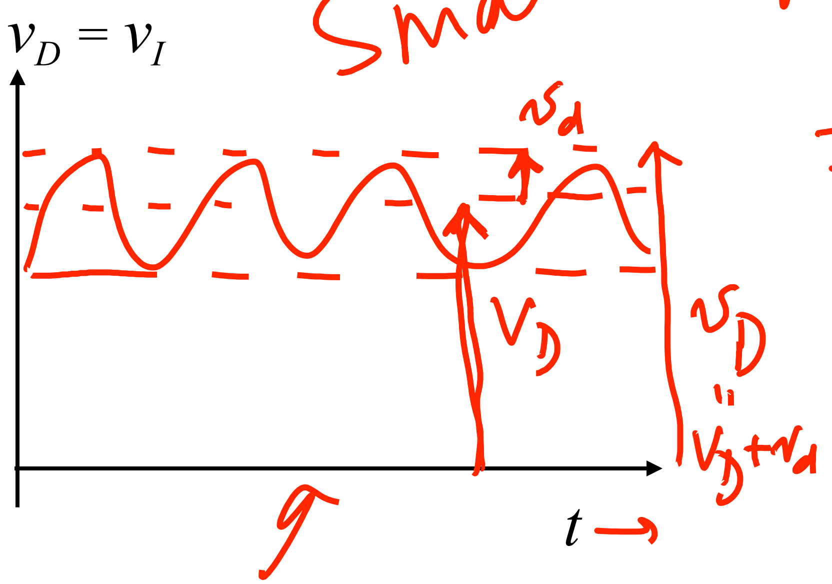
→ Shrink signal of interest v_i

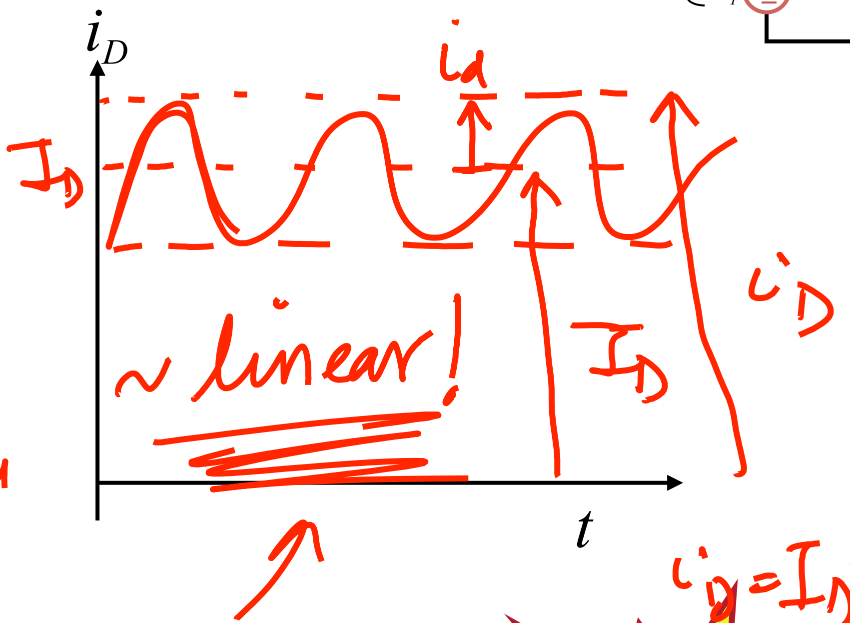
→ Add DC offset to it V_I

Result

Small signal trick







So, this is really a very clever and disciplined way of using a circuit so that we can get a more or less linear response out of a nonlinear circuit

The incremental method:
(or small signal method)

1. Operate at some DC offset or bias point V_D, I_D.
2. Superimpose small signal v_d (music) on top of V_D.
3. Response i_d to small signal v_d is approximately linear.

Notation

$i_D = I_D + i_d$

total variable

DC offset

small superimposed signal

What does this mean mathematically?

Or, why is the small signal response linear?

$i_D = f(v_D)$ ← nonlinear

we replaced

$v_D = V_D + \Delta v_D$ ← v_d

large DC

small increment about V_D

Using Taylor's Expansion to expand $f(v_D)$ near $v_D = V_D$:

$$i_D = f(V_D) + \left.\frac{df(v_D)}{dv_D}\right|_{v_D=V_D} \boxed{\Delta v_D} + \left.\frac{1}{2!}\frac{d^2 f(v_D)}{dv_D^2}\right|_{v_D=V_D} \cdot \Delta v_D^2 + \ldots$$

Neglect higher order terms → Δv_D small

Why is the small signal response linear?

LED

$i_D = f(v_D)$

$$i_D \approx \underbrace{f(V_D)}_{\text{constant w.r.t. } \boxed{\Delta v_D}} + \underbrace{\left.\frac{df(v_D)}{dv_D}\right|_{v_D=V_D}}_{\substack{\text{constant w.r.t. } \boxed{\Delta v_D} \\ \text{slope at } V_D,\, I_D}} \boxed{\Delta v_D} + \text{higher order terms}$$

const + const . Δv_D

slope at V_D, I_D

$\otimes$: $I_D + \Delta i_D \approx f(V_D) + \left.\dfrac{d f(v_D)}{d v_D}\right|_{v_D=V_D} \cdot \Delta v_D$

const

equate DC & time varying component

$I_D = f(V_D)$

→ operating point

$\Delta i_D = \left.\dfrac{d f(v_D)}{d v_D}\right|_{v_D=V_D} \Delta v_D$

const. wrt Δv_D

Why is the small signal response linear?

$$\text{X}: \; I_D + \Delta i_D \approx f(V_D) + \left.\frac{df(v_D)}{dv_D}\right|_{v_D=V_D} \cdot \Delta v_D$$

slope at V_D, I_D

DC

$I_D = f(V_D)$

operating point

time varying part

$$\Delta i_D = \underbrace{\left.\frac{df(v_D)}{dv_D}\right|_{v_D=V_D}}_{\text{const. wrt } \Delta v_D} \cdot \Delta v_D$$

so $\Delta i_D \propto \Delta v_D$

$\Delta i_D \rightarrow i_d$

$\Delta v_D \rightarrow v_d$

$\propto$

Incremental response is linear

In our example

$$(X): I_D + \Delta i_D \approx f(V_D) + \boxed{\left.\frac{d f(v_D)}{d v_D}\right|_{v_D = V_D}} \cdot \Delta v_D$$

$$i_D = f(v_D) = a e^{b v_D}$$

From (X): $I_D + i_d \approx a e^{b V_D} + a \cdot b e^{b V_D} \cdot v_d \rightarrow I_D!$

$$\approx a e^{b V_D} + a e^{b V_D} \cdot b \cdot v_d$$

Equate DC and incremental terms

DC

$$\boxed{I_D = a e^{b V_D}}$$

↓

operating point
aka DC offset
aka bias point

Incremental terms

$$i_d = \underbrace{a e^{b V_D}}\cdot b \cdot v_d$$

$$\boxed{i_d = \underbrace{I_D}_{\text{constant}} \cdot b \cdot v_d}$$

small signal response is linear!

Graphical interpretation

$\rightarrow \quad I_D = a\, e^{bV_D}$ ➞ operating point

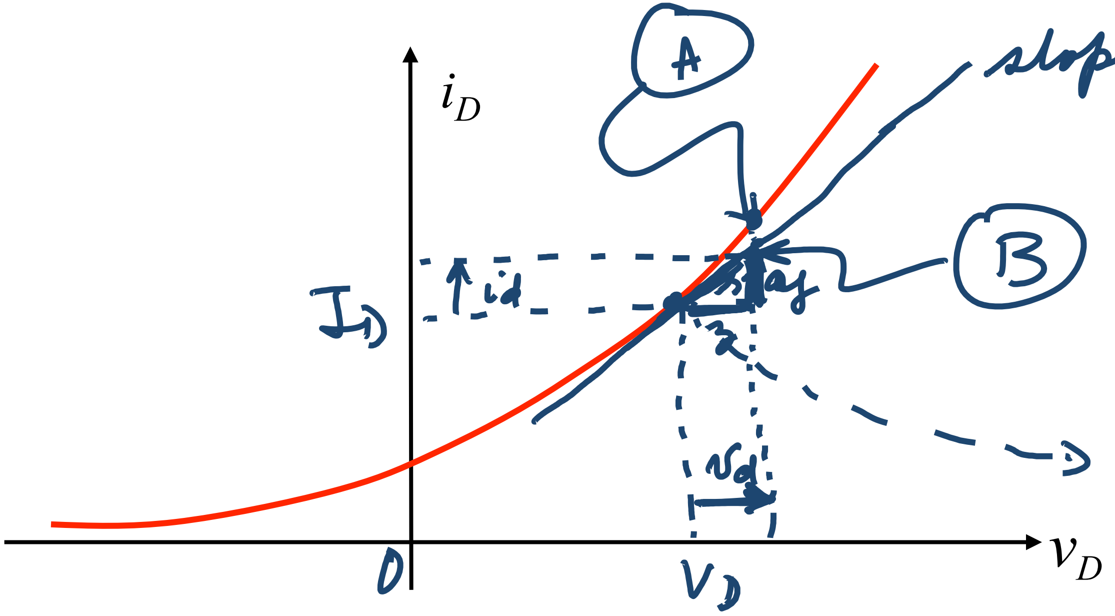
$\rightarrow \quad i_d = I_D \cdot b \cdot v_d$

slope at V_D, I_D

$i_D = f(v_D)$

$i_D = a\, e^{b v_D}$

$I_D \qquad V_D$



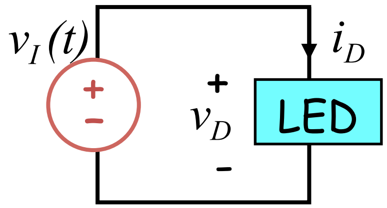
operating point

$s \Delta y$, z

$s = \frac{y}{z}$ → i_d, v_d

$y = s \cdot z$

we are approximating (A) with (B)



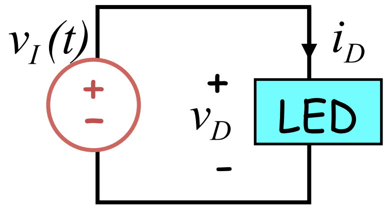
$$v_D = f(v_D) = a\, e^{bV_D}$$

$$I_D = a\, e^{bV_D}$$

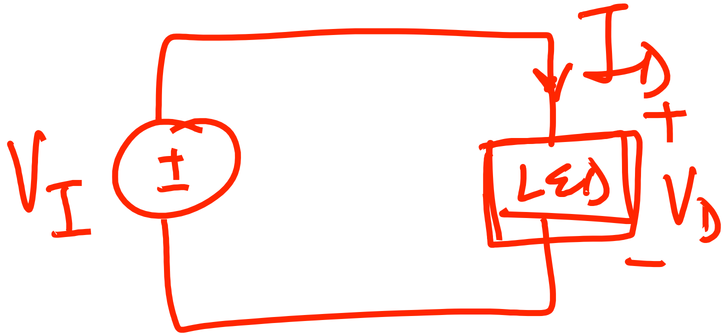
$$i_d = I_D \cdot b \cdot v_d$$

We studied the small signal → graphically
→ mathematically
→ Next, circuit view

A circuit view of the small signal model



Large signal circuit:



$\boxed{I_D = a e^{bV_D}}$

solve nonlinear system

$v_D = f(v_D) = a\, e^{bV_D}$

$\Rightarrow I_D = a\, e^{bV_D}$

$\Rightarrow i_d = I_D \cdot b \cdot v_d$

Small signal reponse:

$\rightarrow i_d = I_D \cdot b \; v_d$

constant

$R = \frac{1}{I_D \cdot b}$

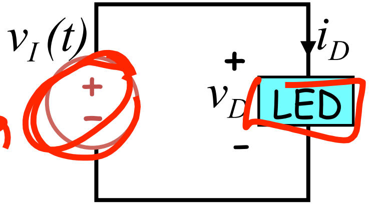
circuit interpretation of the equation

Where did you see
i = constant x v
before?

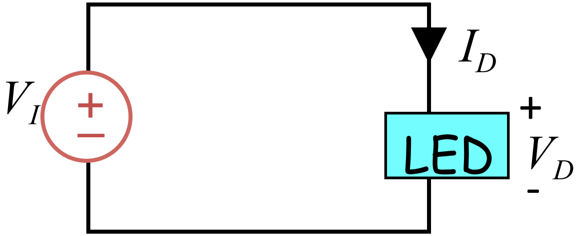
$$R = \frac{1}{\left.\frac{d f(v_D)}{d v_D}\right|_{v_D = V_D}}$$

For small signals, device behaves like a resistor!

So, We Can Build a Small Signal Circuit



Large signal circuit:



$$\boxed{I_D = a\, e^{bV_D}}$$

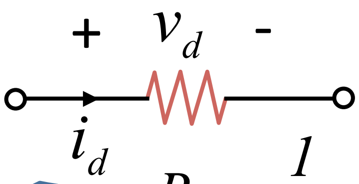
$$I_D = a\, e^{bV_D}$$

$$i_d = I_D \cdot b \cdot v_d$$

$$i_d = \frac{v_i}{R} = v_i \cdot I_D b$$

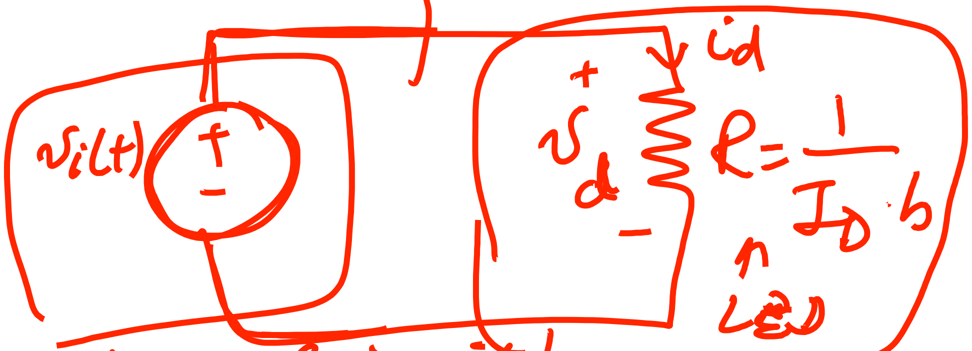
Small signal reponse:

$$i_d = I_D\, b\, v_d$$



$$R = \frac{1}{I_D\, b}$$

We can build a small signal circuit:



Linear Circuit!

For small signals, device behaves like a resistor!

See pages 222 – 228 of textbook

Small Signal Circuit Method

1. Find operating point using DC bias inputs from large signal circuit

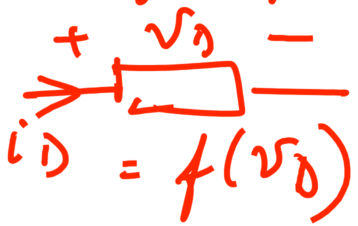
typically involves a nonlinear analysis

2. Develop small signal (linearized) models for each of the elements around the operating point.

incremental

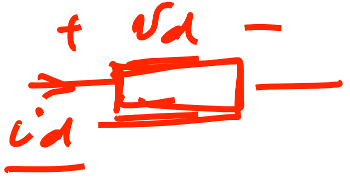
Key: we can use superposition and other linear circuit tools with linearized circuit!

Large Signal



$i_D = f(v_D)$

small signal



$$i_d = \frac{\partial f(v_D)}{\partial v_D}\Bigg|_{v_D = V_D} \cdot v_d$$

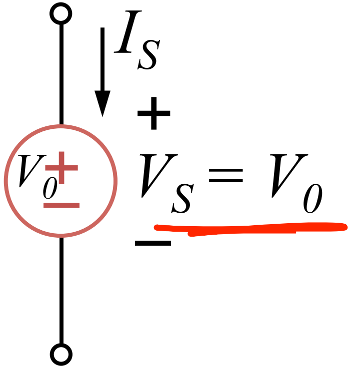
Replace original elements with small signal element models.

3. Analyze resulting linearized circuit to obtain small signal response...

superposition, Thevenin etc apply

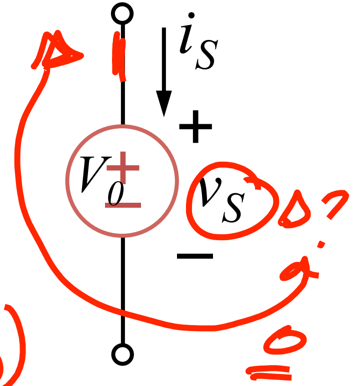
Step 2: Voltage Sources and DC Supply V_0

large signal



$v_s = V_0$

$v_s = f(i_D)$

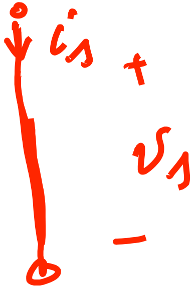


1- op pt I_D, V_D

2- lin. mod each ele

3- analyze lin ckt

Small signal



$$v_s = \left.\frac{\partial V_0}{\partial i_s}\right|_{i_s = I_s} \cdot i_s$$

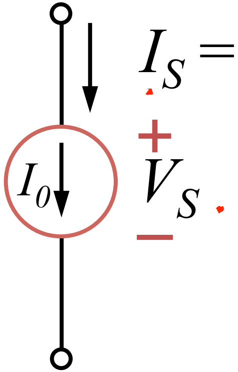
$v_s = 0$

DC voltage source behaves as short to small signals.
DC current source behaves as open to small signals.

See page 416 of textbook

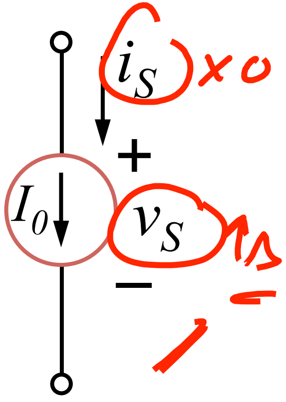
Current Sources

large signal



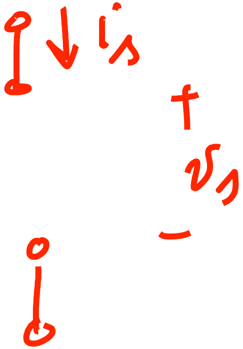
$$\boxed{I_S = I_0}$$

$i_S = I_0$



$i_S = f(v_S)$
$= I_0$

Small signal



$$\boxed{i_s = 0}$$

$$i_s = \left.\frac{\partial f(v_S)}{\partial v_S}\right|_{v_S = V_S} \cdot v_s$$

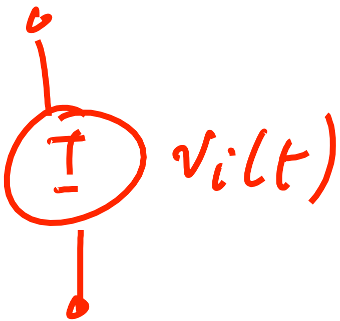
$$= \underbrace{\left.\frac{\partial I_0}{\partial v_S}\right|_{v_S = V_S}}_{0} \cdot v_s$$

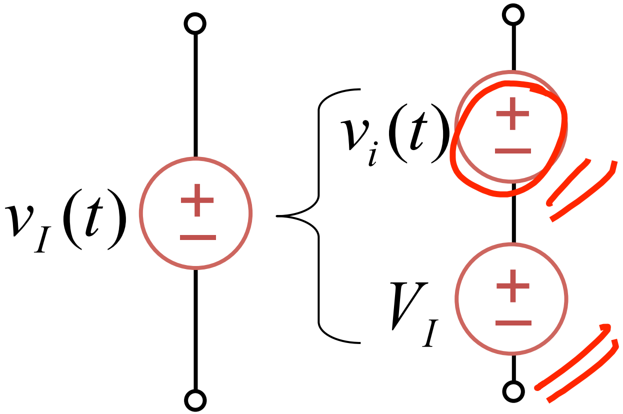
DC voltage source behaves as short ~~to small signals.~~
DC current source behaves as open to small signals.

Voltage Source Containing Both DC and Small Signal

large signal

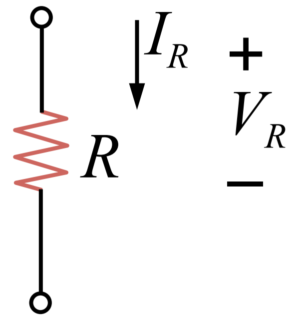
Small signal



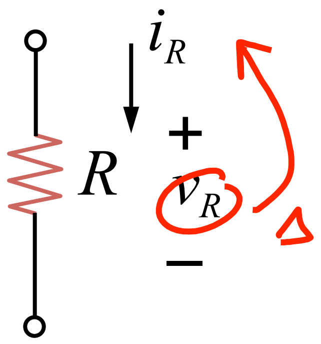


Similarly, R

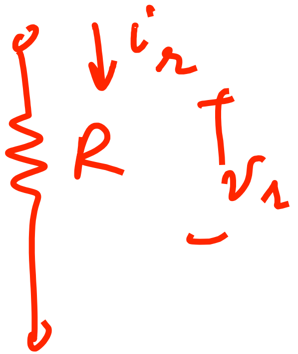
large signal



$$\boxed{V_R = R\,I_R}$$



small signal



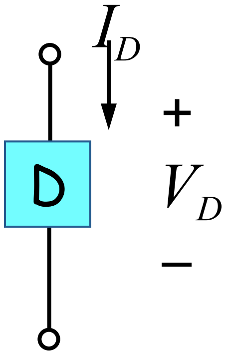
$$v_r = R\,i_r$$

$$v_r = \left.\frac{\partial (R\,i_R)}{\partial i_R}\right|_{i_R = I_R} \cdot i_r$$

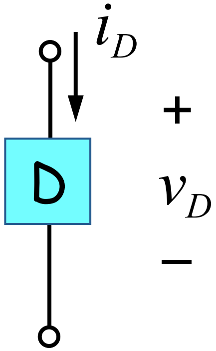
$$v_r = R \cdot i_r$$

For Non-Linear Device D

large signal



$$I_D = a\,e^{bV_D}$$



$$i_D = a\,e^{bv_D}$$

$$i_D = f(v_D)$$

small signal

$$i_d = \left.\frac{\partial \overbrace{f(v_D)}^{a e^{bv_D}}}{\partial v_D}\right|_{v_D=V_D} \cdot v_d$$

$$= \left. a\,e^{bv_D}\, b \right|_{v_D=V_D} \cdot v_d$$

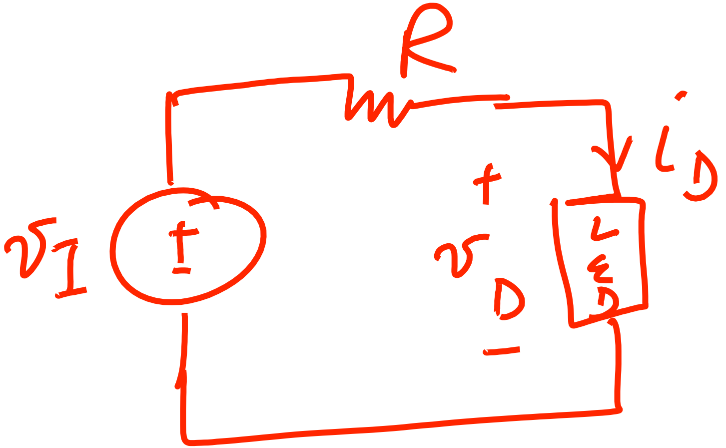
$$= \underbrace{a\,e^{bV_D}} \cdot b v_d$$

$$\boxed{i_d = I_D\, b\, v_d}$$

$$R = \frac{1}{I_D b}$$

We will visit small signal circuits again shortly...

Small signal circuit analysis example



$i_D = \cancel{\ } \, a e^{b v_D}$

1. large sig analysis
2. linearized models for elements
3. analyze SS ckt

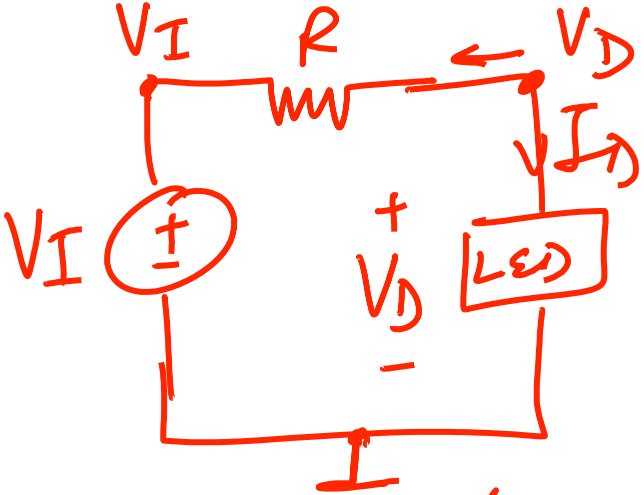
Find i_d for v_i

assume $R = 1\Omega$, $a = \frac{1}{4}A$, $b = 1V^{-1}$

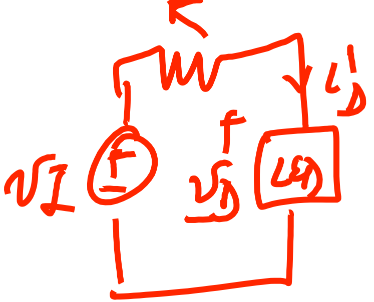
also assume that the bias point set at $V_I = 1V$

a

Step 1



$I_D = a e^{bV_D}$



For $R = 1\Omega$

$a = \frac{1}{4} A$

$b = 1V^{-1}$

at $V_I = 1V$

$i_D = f(v_D) = a e^{bv_D}$

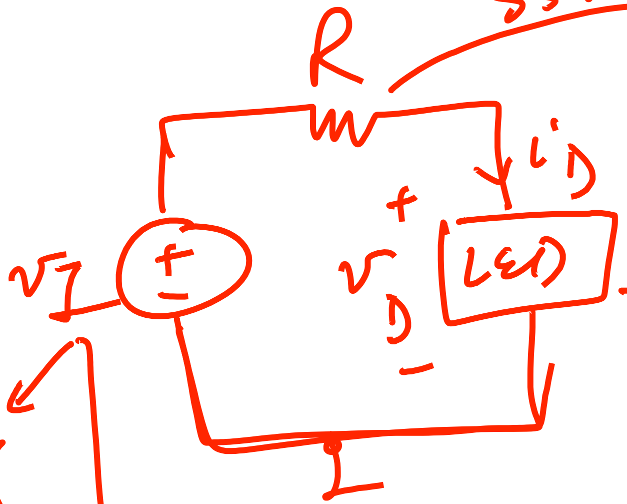
Analytical method

$$\frac{V_D - V_I}{R} + a e^{bV_D} = 0$$

$$V_D - 1 + \frac{1}{4} e^{V_D} = 0$$

$$\boxed{V_D = 0.561V, \; I_D = 0.44A}$$

Step 2



SS model

✓

$$i_D = a e^{b v_D}$$

$f(v_D)$

SS ?

$$R_D = \left.\frac{1}{\frac{\partial f(v_D)}{\partial v_D}}\right|_{v_D = V_D}$$

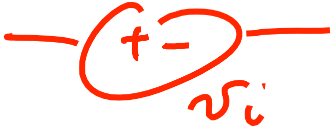
$$= \left.\frac{1}{a\, b\, e^{b v_D}}\right|_{v_D = V}$$

$$R_D = \frac{1}{a\, e^{b V_D}\, b}$$

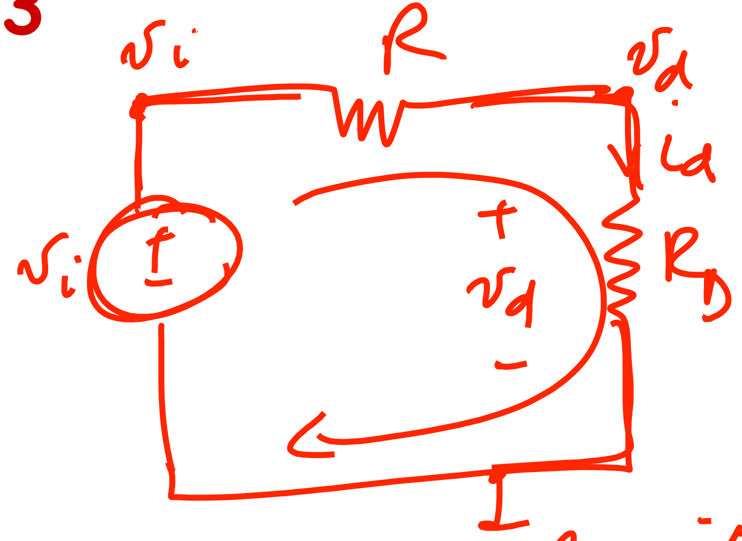
✓ $R_D = \dfrac{1}{I_D\, b}$

SS eqvt

✓



Step 3



Small Signal Circuit

$R_D = \frac{1}{I_D b}$

i_d ?

$$\boxed{v_d = \frac{v_i}{\underline{R} + \underline{R_D}}}$$

$R = 1$

$\frac{1}{I_D b}$

$b \rightarrow 1$

from large signal analysis

$I_D = 0.44A$