Summary: Taylor Series

General power series

A **power series** is an infinite series involving positive powers of a variable x:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

The **radius of convergence** R of the power series $\sum_{n=0}^{\infty} a_n x^n$, is a real number $0 \leq R < \infty$ such that

- for $|x| < R$, the power series $\sum_{n=0}^{\infty} a_n x^n$ converges (to a finite number);
- for $|x| > R$, the power series $\sum_{n=0}^{\infty} a_n x^n$ diverges;
- for $|x| = R$, the power series may converge or diverge. But we will mostly ignore what happens at the end points of the interval of convergence.

Examples:

- Geometric series: $1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$, radius of convergence is 1.
- Polynomials: $a_0 + a_1 x + a_2 x^2 + \cdots a_N x^N = \sum_{n=0}^{N} a_n x^n$, radius of convergence ∞. In other words, the sum converges for all x.

Finding the radius of convergence

Given a power series $\sum_{n=0}^{\infty} a_n x^n$, the ratio test implies that the power series converges if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x| < 1.$$

There are 3 possibilities:

1. There is a finite number R such that
   - $|x| < R \Longrightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x| < 1,$
   - $|x| > R \Longrightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x| > 1.$

   We say the radius of convergence is R.

2. For all x $\quad \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x| < 1$. We say the radius of convergence is ∞. (All x satisfy $|x| < \infty$.)

3. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x| > 1$ for all $x \neq 0$. We say the radius of convergence is 0.

Remark: Alternative method using ratio test

(Note that in the method that follows, the $n+1$ term is in the denominator and the n term is in the numerator, which is the opposite of the ratio test.)

Given a power series $\sum_{n=0}^{\infty} a_n x^n$,

if $\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = R \quad$ (where R exists or is ∞),

then the radius of convergence for the power series is R.

Example

Consider $\sum_{n=0}^{\infty} 2^n x^n$, then $\lim_{n\to\infty} \left| \frac{2^n}{2^{n+1}} \right| = \frac{1}{2} \quad \Longrightarrow \quad R = \frac{1}{2}.$

Root test for radius of convergence

Given a power series $\sum_{n=0}^{\infty} a_n x^n$, the root test implies that the power series converges if

$$\lim_{n\to\infty} \sqrt[n]{|a_n x^n|} = \lim_{n\to\infty} \sqrt[n]{|a_n|}|x| < 1.$$

There are 3 possibilities:

1. There is a finite number R such that
   - $|x| < R \Longrightarrow \lim_{n\to\infty} \sqrt[n]{|a_n|}|x| < 1$,
   - $|x| > R \Longrightarrow \lim_{n\to\infty} \sqrt[n]{|a_n|}|x| > 1$.

   We say the radius of convergence is R.
2. For all x $\lim_{n\to\infty} \sqrt[n]{|a_n|}|x| < 1$. We say the radius of convergence is ∞. (All x satisfy $|x| < \infty$.)
3. $\lim_{n\to\infty} \sqrt[n]{|a_n|}|x| > 1$ for all $x \neq 0$. We say the radius of convergence is 0.

Example

Consider $\sum_{n=0}^{\infty} 2^n x^n$, then $\lim_{n\to\infty} \sqrt[n]{|2^n x^n|} = 2|x| < 1$ when $|x| < \frac{1}{2}$. This implies that the radius of convergence is $R = \frac{1}{2}$.

Properties of power series

Add, subtract, multiply, divide, differentiate, and integrate convergent power series as one does for polynomials. We will discuss multiplication and division at a later time.

Consider the power series $\sum_{n=0}^{\infty} a_n x^n$, which converges for $|x| < A$.

- The derivative $\frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$ also converges for $|x| < A$.
- The integral $\int \left(\sum_{n=0}^{\infty} a_n x^n \right) dx = c + \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$ also converges for $|x| < A$. Note that c is the constant of integration.

Consider another power series $\sum_{n=0}^{\infty} b_n x^n$, which converges for $|x| < B$.

- If $A \neq B$, then $\left(\sum_{n=0}^{\infty} a_n x^n\right) \pm \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$ converges for $|x| < \min(A, B)$.
- If $A = B$, then $\left(\sum_{n=0}^{\infty} a_n x^n\right) \pm \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$ has a radius of convergence which is at least A, but it *could* have a larger radius of convergence.

Taylor's formula

Recall that $n! = n(n-1)(n-2)\cdots(3)(2)(1)$ for all integers $n \geq 1$.

We define $0! = 1$. This is a very valuable convention that simplifies many formulas.

Taylor's formula says that

$$\begin{aligned} f(x) &= f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{6} + \cdots \\ &= \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \end{aligned}$$

when $|x| < R$ where R is the radius of convergence of the power series above.

The power series in Taylor's formula is called the **Taylor series** of $f(x)$.

Important examples

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$
- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

We will use Taylor's formula to derive the series for $\sin(x)$ and $\cos(x)$ on the next page.

Notice that the factorial appears in the denominator of all terms in all three power series above.

Using the Taylor series of e^x, we find a formula for the number e as the rapidly converging series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Known Maclaurin series

So far, we have used Taylor's formula to obtain the following Taylor series:

- $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ for $|x| < 1$.
- $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
- $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

We have also integrated the geometric series to obtain a power series for $\ln(1-x)$:

- $-\ln(1-x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$.

Multiplying two power series

We multiply two powers series using the same rule as when we multiply two polynomials.

Consider the power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$, which converges for $|x| < A$, and the power series $\displaystyle\sum_{n=0}^{\infty} b_n x^n$, which converges for $|x| < B$.

- The product of the two power series converges for $|x| < \min(A, B)$, but it *could* have a larger radius of convergence.

Dividing two power series

If

$$F(x) = \frac{G(x)}{H(x)}$$

where $F(x)$, $G(x)$, $H(x)$ are all power series, then we can find $F(x)$ by solving the following equation of power series degree by degree:

$$F(x)H(x) = G(x).$$

The radius of convergence of $F(x)$ is more difficult to track, since $F(x)$ diverges whenever $H(x) = 0$ and $G(x) \neq 0$.

Substitution and Taylor series

We can find the composition of two power series $f(g(x))$ by using similar rules as composition of polynomials. In this course, we will only substitute a polynomial $p(x)$ into a power series $f(x)$.

If the radius of convergence of the power series $f(x)$ is R, then the power series $f(p(x))$ converges whenever $|p(x)| < R$.

In particular, we can find a Taylor series for a function $f(x)$ at $x = 0$ and then substitute in polynomial of the form $p(x) = ax^n$ for x since $p(0) = 0$.

Error function

Recall the **error function** is defined by the following integral formula:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

and cannot be expressed in terms of functions that we already know with algebraic operations such that addition and multiplication.

To obtain the Taylor series of the error function, we replace the integrand e^{-t^2} with its Taylor series:

$$\begin{aligned}
\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \\
&= \frac{2}{\sqrt{\pi}} \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \right) dt \\
&= \frac{2}{\sqrt{\pi}} \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} \right).
\end{aligned}$$

Taylor polynomials

If the Taylor series of $f(x)$ is

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad (\text{for } |x| < R),$$

then the polynomial

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots a_n x^n$$

is called the **Taylor polynomial** of degree n of $f(x)$. In other words, a Taylor polynomial is the polynomial obtained by truncating the Taylor series to degree n.

The degree n Taylor polynomial $P_n(x)$ is the best fit degree n polynomial of $f(x)$ at $x = 0$, in the sense that

$$\frac{d^k}{dx^k} P_n(x) = \frac{d^k}{dx^k} f(x) \bigg|_{x=0} \qquad \text{for } 0 < k \leq n.$$

Hence, the degree 1 and degree 2 Taylor polynomials for $f(x)$, are the linear and quadratic approximations of $f(x)$ respectively.

Taylor polynomials are especially useful as for approximating functions, functions that cannot be expressed algebraically in terms of the elementary functions that we know, such as the error function. Often, numerical tools, such as the graphing tool on your calculator or computer, use Taylor polynomials to approximate these functions.

We can use Taylor polynomials to approximate a function with arbitrarily high accuracy inside the radius of convergence of the Taylor series. But how do we know the degree of the Taylor polynomial needed to achieve a certain accuracy? The answer is in the **Taylor remainder theorem** below.

Taylor remainder theorem

Suppose the Taylor series of $f(x)$ is

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad \text{for } |x| < R,$$

and let $P_n(x)$ be the degree n Taylor polynomial of $f(x)$:

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

Then, for any $|x| < R$,

- if $f(x)$, is $n+1$ times differentiable on the open interval $(0, x)$, that is, $f^{(n+1)}(x)$ and all lower derivatives f exist on $(0, x)$, and
- if $f^{(n)}$ is continuous on the closed interval $[0, x]$,

then there is a number c in $(0, x)$ such that

$$f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}.$$

This is the **Taylor remainder theorem**.

Note that the $n = 0$ case is the Mean Value Theorem (MVT). As in the MVT, we do not know exactly where c is.

Nevertheless, we can use the Taylor remainder theorem to find upper bounds on the error caused by approximating a function by a Taylor polynomial.

Taylor series centered at $x = a$

Let $g(t) = f(t + a)$. That is, g is the translation of f to the left by a.

Recall the Taylor series of $g(t)$ at $t = 0$ is

$$g(t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n \qquad (|t| < R).$$

Then

$$\begin{aligned} f(t + a) = g(t) &= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n \qquad (|t| < R) \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} t^n \qquad (|t| < R). \end{aligned}$$

Now let $x = t + a$. In terms of x, the above formula becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad (|x-a| < R).$$

This is the Taylor series of $f(x)$ at $x = a$.

Note that the radius of convergence of the Taylor series of $f(x)$ at $x = a$ is the number R such that $f(x)$ converges when $|x - a| < R$, and diverges when $|x - a| > R$.

If $a = 0$, we get the formula for the Taylor series that we started with in this section. This special case of Taylor's formula gives us a power series often referred to as the **Maclaurin series**.