Summary: Parametric Curves

**Parametric curves**

A parametric curve (in the plane) is a curve defined by two equations

\[
\begin{align*}
x &= x(t), \\
y &= y(t),
\end{align*}
\]

where \( t \) is called a parameter. For each real number \( t \), the point \((x(t), y(t))\) is a point on the curve.

**Eliminating parameters**

To find the underlying curve, try eliminating the parameter using algebra and/or trig identities.

**Tangent lines of parametric curves**

The slope of a parametric curve \( x = x(t), y = y(t) \) is

\[
\frac{dy}{dx} = \left( \frac{dy}{dt} \right) \left( \frac{dx}{dt} \right).
\]

In particular, to find the slope of the tangent line to the curve at \( t = t_0 \), we compute \( \frac{y'(t_0)}{x'(t_0)} \).

**Arc length of parametric curves**

Consider a particle moving along a trajectory. The motion is described by the parametric curve

\[
\begin{align*}
x &= x(t) \\
y &= y(t).
\end{align*}
\]
The speed of the particle is given by

\[ \frac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2} \]

The differential arc length element is given by

\[ ds = \sqrt{(x'(t))^2 + (y'(t))^2} \, dt. \]

Consider the arc length, or distance travelled by the particle from time \( t_0 \) to time \( t_1 \). The letter \( s \) is customarily used to denote arc length. You should think of \( s = s(t) \) as a function of time, where \( s(t) \) is the distance travelled by the particle since some starting time. If \( s_0 = s(t_0) \) and \( s_1 = s(t_1) \), then the distance traveled by the particle from time \( t_0 \) to \( t_1 \) can be calculated by

\[ s_1 - s_0 = \int_{s_0}^{s_1} ds = \int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt \]

A note about notation

\[ ds^2 = dx^2 + dy^2 \quad \text{means} \quad (ds)^2 = (dx)^2 + (dy)^2 \]
\[ ds = \sqrt{dx^2 + dy^2} \quad \text{means} \quad ds = \sqrt{(dx)^2 + (dy)^2} \]

In particular, \( dx^2 = (dx)^2 \). This is the square of a differential, not the differential of the square. The differential of the square is \( d(x^2) = 2x \, dx \), which is not the same.

Position and speed along a parametric curve

We have been thinking of a parametric curve as the description of a particle’s position over time. So what is the velocity? And what about the acceleration?

- The derivative \( x'(t) \) is the velocity in the direction of the \( x \)-axis.
- The derivative \( y'(t) \) is the velocity in the direction of the \( y \)-axis.
- The speed along the curve is given by \( \frac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2} \).
- The notion of velocity along the curve requires considering \( x' \) and \( y' \) together in what is known as a vector. Both velocity and acceleration are vectors and you will see them in multivariable calculus.
Figure 1: Rotating a curve about the $y$-axis.

**Surface area**

Consider the parametric curve

$$x = x(t)$$
$$y = y(t).$$

Consider the surface formed by rotating the curve about the $y$-axis. The differential surface area element is given by

$$dA = 2\pi x \, ds = 2\pi x(t) \sqrt{(x'(t))^2 + (y'(t))^2} \, dt.$$ 

The surface area from time $t_0$ to time $t_1$ is given by the integral:

$$\text{Surface area of parametric curve} = \int_{t_0}^{t_1} 2\pi x(t) \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$$

Consider the surface formed by rotating the curve about the $x$-axis. The differential surface area element is given by

$$dA = 2\pi y \, ds = 2\pi y(t) \sqrt{(x'(t))^2 + (y'(t))^2} \, dt.$$ 

The surface area from time $t_0$ to time $t_1$ is given by the integral:

$$\text{Surface area of parametric curve} = \int_{t_0}^{t_1} 2\pi y(t) \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$$
Figure 2: Rotating a curve about the $x$-axis.

The (signed) area under a curve $y = f(x)$ between $x = a$ and $x = b$ is given by 
\[ \int_{a}^{b} f(x) \, dx. \]

Suppose that this curve is parameterized by the equations
\[
\begin{align*}
x &= x(t) \\
y &= y(t).
\end{align*}
\]

Moreover suppose that $x(t_0) = a$ and $x(t_1) = b$. Then the (signed) area under the curve is also equal to
\[
\int_{a}^{b} f(x) \, dx = \int_{t_0}^{t_1} f(x(t)) x'(t) \, dt.
\]

by change of variables (or by substitution and applying the chain rule).

But note that
\[
\begin{align*}
y &= f(x) \\
y &= y(t) \\
\implies y(t) &= f(x(t))
\end{align*}
\]

hence
\[
\int_{a}^{b} f(x) \, dx = \int_{t_0}^{t_1} y(t) x'(t) \, dt.
\]

This formula for the (signed) area under a parametric curve holds in general!
General result for parametric curves

Given a parametric curve:

\[ x = x(t) \]
\[ y = y(t), \]

The signed area of the region bounded between the curve and the \( x \)-axis for \( t_0 < t < t_1 \) is given by the integral

\[ \text{Signed Area} = \int_{t_0}^{t_1} y(t) \cdot x'(t) \, dt. \]