ITMO UNIVERSITY

How to Win Coding Competitions: Secrets of Champions

Week 5: Algorithms on Graphs 1
Lecture 4: Depth First Search with Timestamps

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Let's modify DFS to track the time of entering and exiting a vertex

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$G = \langle V, E \rangle$
$T_{\text{in}}, T_{\text{out}} \leftarrow \{\infty\}$
$A(v) = \{u \mid (v, u) \in E\}$
$t \leftarrow 0$
procedure $\mathrm{DFS}(v)$
    $t \leftarrow t + 1$
    $T_{\text{in}}(v) \leftarrow t$
    for $u \in A(v)$ **do**
        if $T_{\text{in}}(u) = \infty$ **then** $\mathrm{DFS}(u)$ **end if**
    end for
    $t \leftarrow t + 1$
    $T_{\text{out}}(v) \leftarrow t$
end procedure

Let's modify DFS to track the time of entering and exiting a vertex

$G = \langle V, E \rangle$
$T_{\text{in}}, T_{\text{out}} \leftarrow \{\infty\}$ ▷ $T_{\text{in}}(v)$: the time of entering v
$A(v) = \{u \mid (v, u) \in E\}$ ▷ $T_{\text{out}}(v)$: the time of exiting v
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  $t \leftarrow t + 1$ ▷ Incrementing time
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Let's modify DFS to track the time of entering and exiting a vertex

```
G = ⟨V, E⟩
T_in, T_out ← {∞}                               ▷ T_in(v): the time of entering v
                                                ▷ T_out(v): the time of exiting v
A(v) = {u | (v, u) ∈ E}
t ← 0
procedure DFS(v)
    t ← t + 1                                   ▷ Incrementing time
    T_in(v) ← t                                 ▷ Marking the time of entering
    for u ∈ A(v) do
        if T_in(u) = ∞ then DFS(u) end if
    end for
    t ← t + 1
    T_out(v) ← t
end procedure
```

Let's modify DFS to track the time of entering and exiting a vertex

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    $T_{\text{in}}(v) \leftarrow t$ ▷ Marking the time of entering
    for $u \in A(v)$ **do**
        if $T_{\text{in}}(u) = \infty$ **then** $\mathrm{DFS}(u)$ **end if** ▷ Means "not previously entered"
    end for
    $t \leftarrow t + 1$
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end procedure

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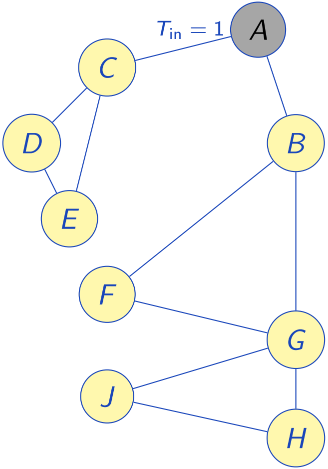
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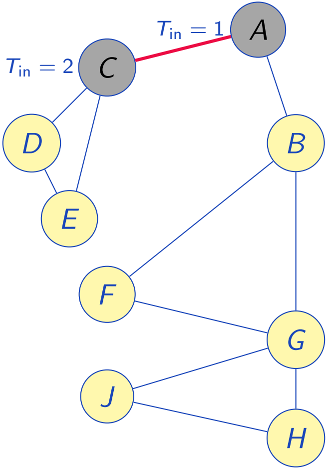
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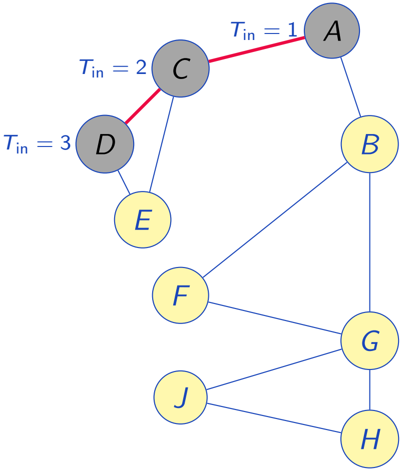
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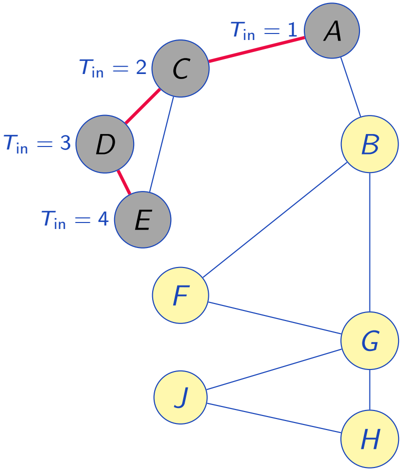
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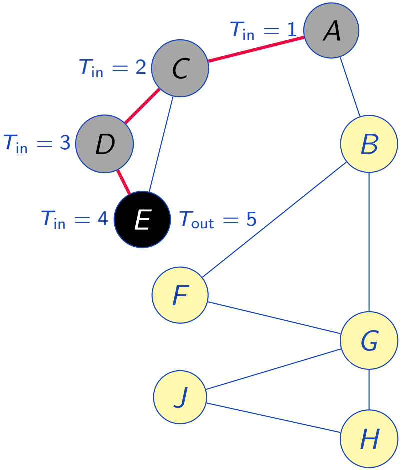
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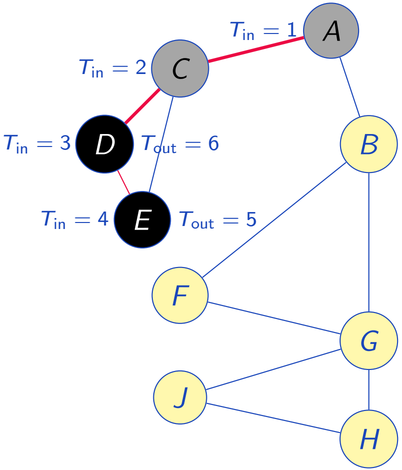
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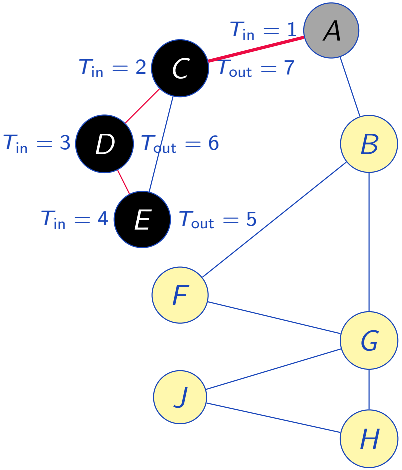
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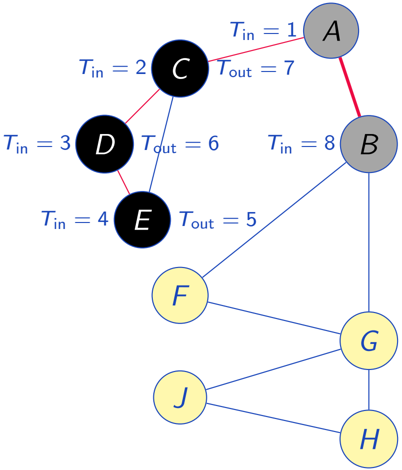
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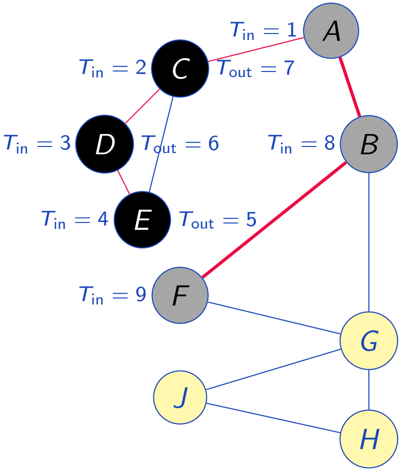
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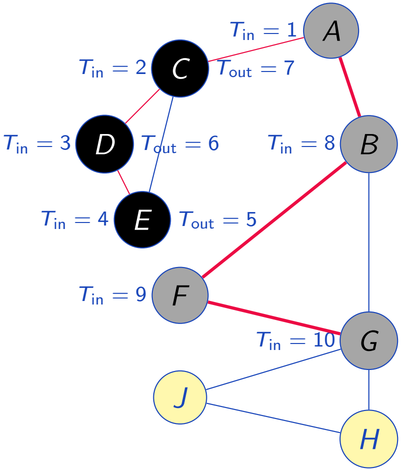
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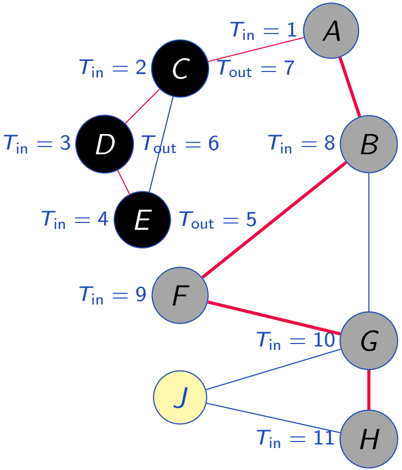
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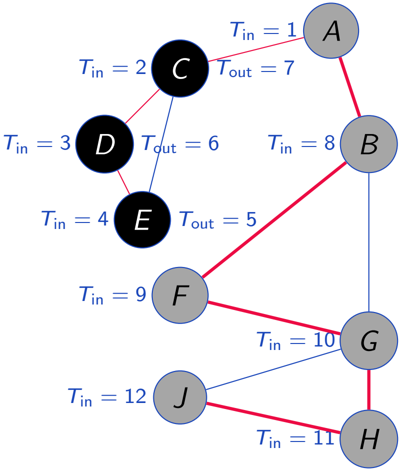
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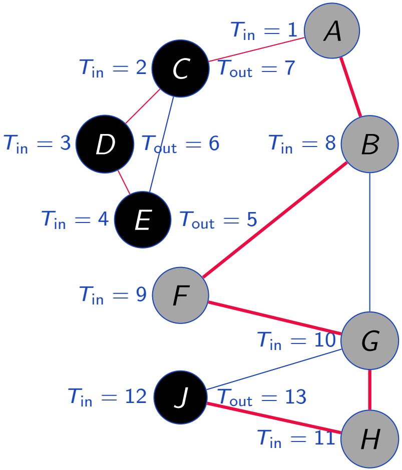
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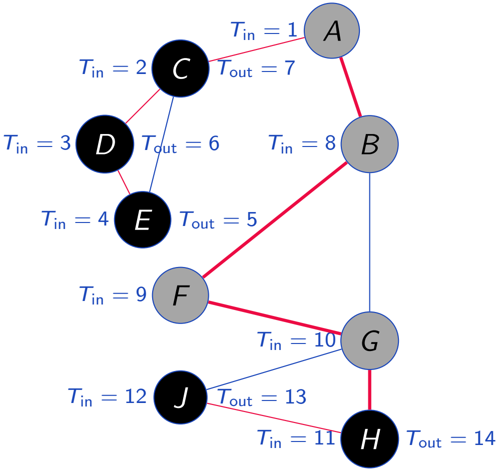
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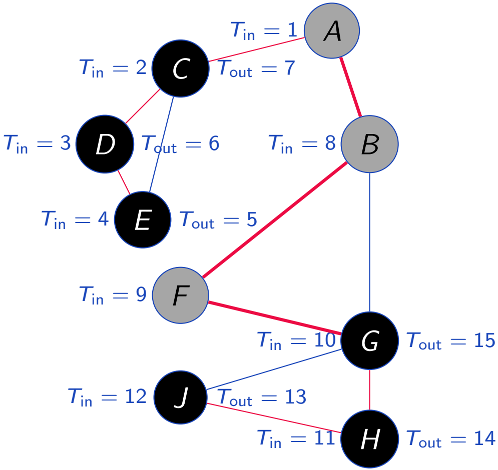
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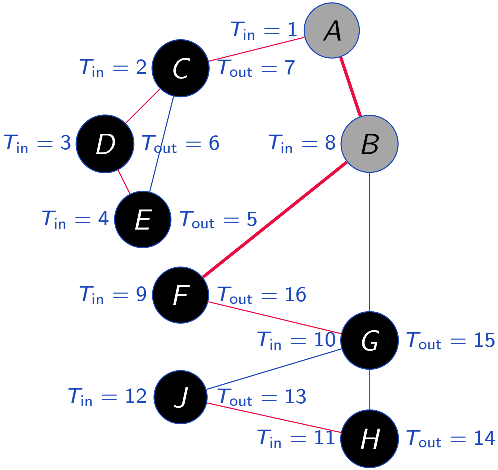
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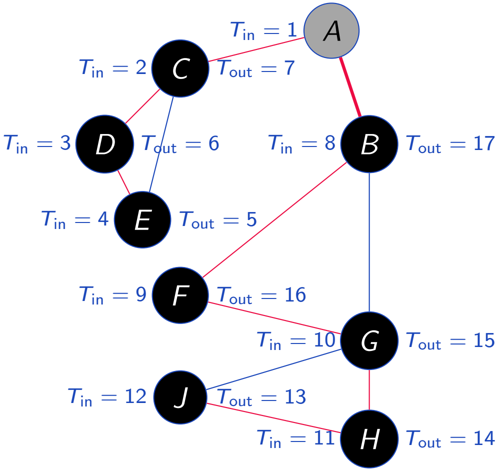
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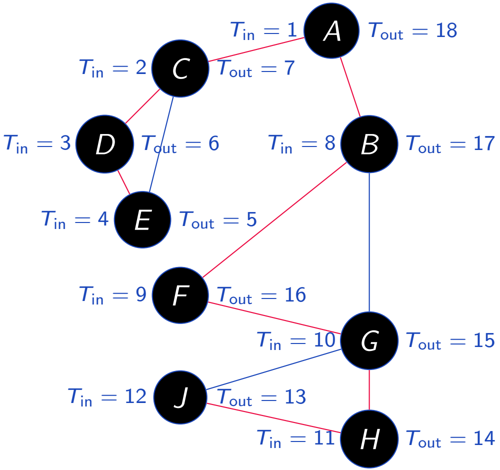
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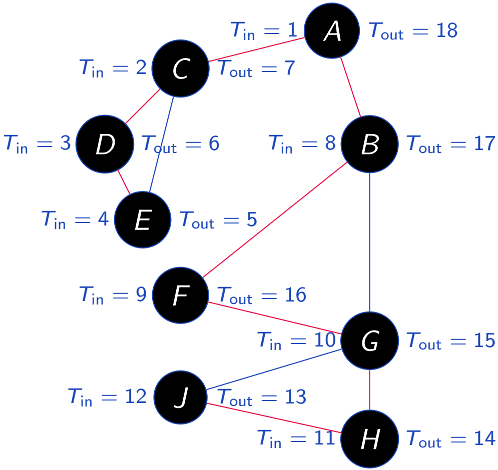
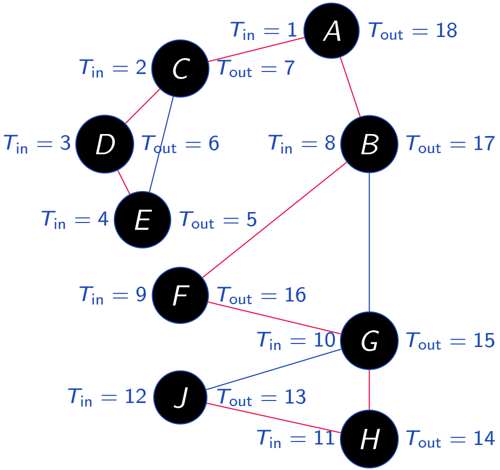
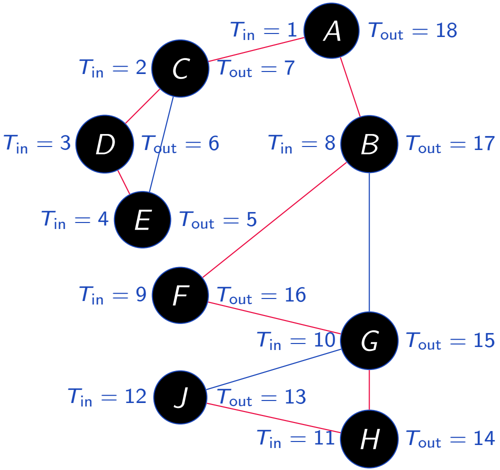


- Important timestamp property:
  A is ancestor of $B \Leftrightarrow$
  $T_{\text{in}}(A) < T_{\text{in}}(B) < T_{\text{out}}(B) < T_{\text{out}}(A)$

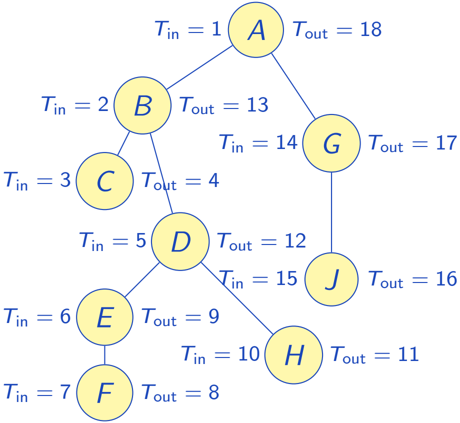


- Important timestamp property: A is ancestor of $B \Leftrightarrow$ $T_{\text{in}}(A) < T_{\text{in}}(B) < T_{\text{out}}(B) < T_{\text{out}}(A)$
- This is a fast way to determine whether a vertex is an ancestor of another one

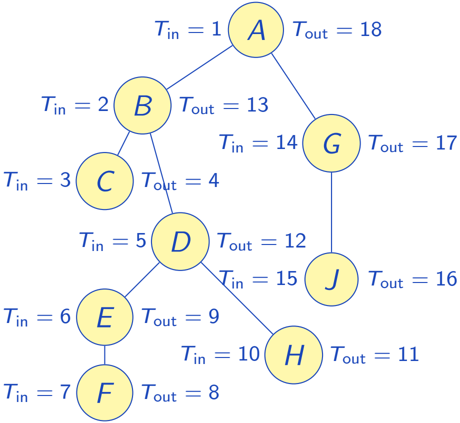


- Important timestamp property:
  A is ancestor of $B \Leftrightarrow$
  $T_{in}(A) < T_{in}(B) < T_{out}(B) < T_{out}(A)$
- This is a fast way to determine whether a vertex is an ancestor of another one
- Some examples follow where this idea is crucial

Example of working with timestamps: finding Least Common Ancestors in trees

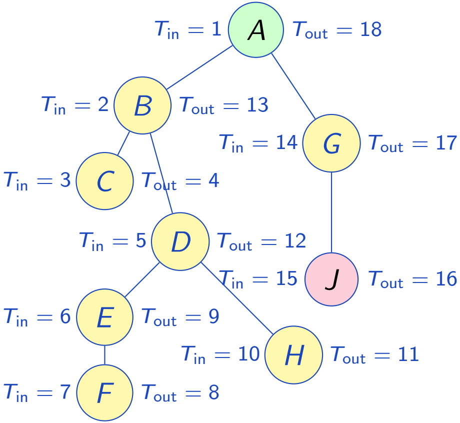


Example of working with timestamps: finding Least Common Ancestors in trees



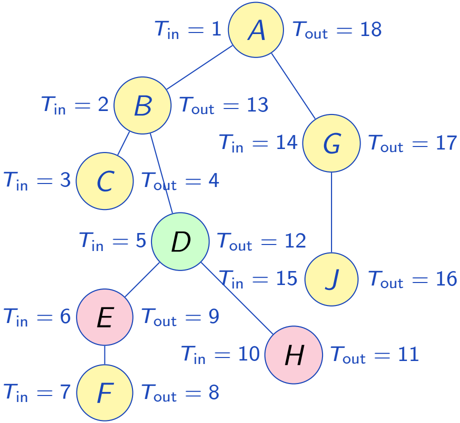
- Examples:

Example of working with timestamps: finding Least Common Ancestors in trees



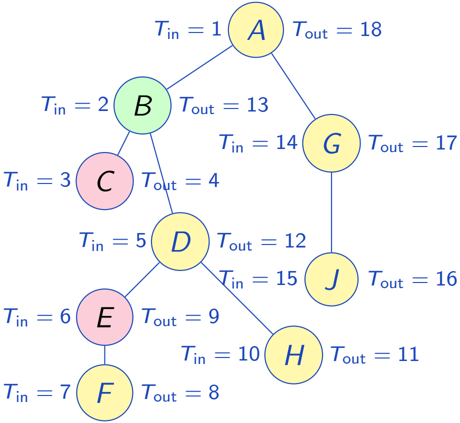
- Examples:
  - $\text{LCA}(A, J) = A$

Example of working with timestamps: finding Least Common Ancestors in trees



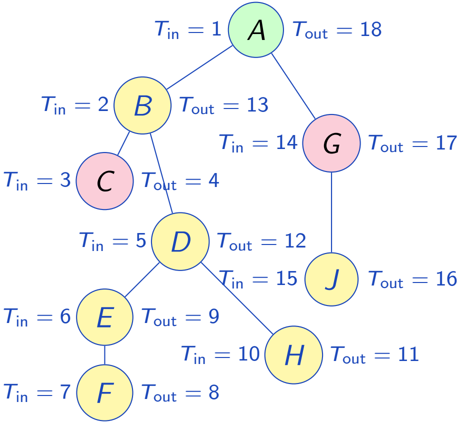
- Examples:
  - $\text{LCA}(A, J) = A$
  - $\text{LCA}(E, H) = D$

Example of working with timestamps: finding Least Common Ancestors in trees



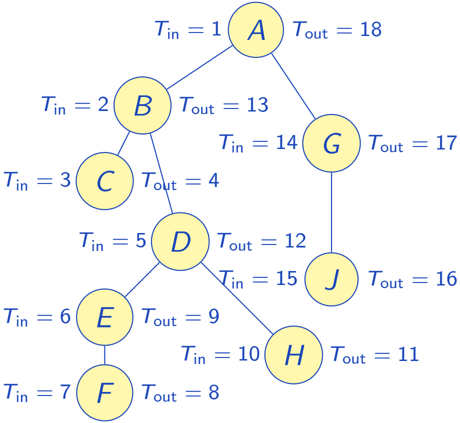
- Examples:
  - $\text{LCA}(A, J) = A$
  - $\text{LCA}(E, H) = D$
  - $\text{LCA}(C, E) = B$

Example of working with timestamps: finding Least Common Ancestors in trees



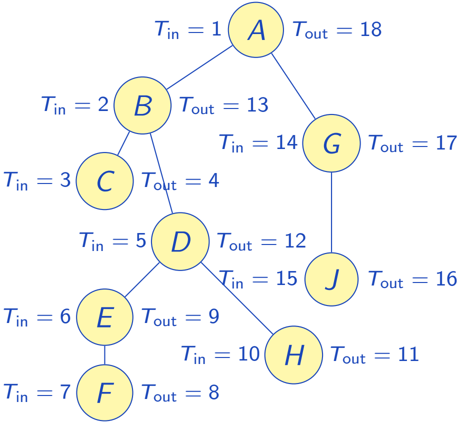
- Examples:
  - $\text{LCA}(A, J) = A$
  - $\text{LCA}(E, H) = D$
  - $\text{LCA}(C, E) = B$
  - $\text{LCA}(C, G) = A$

Example of working with timestamps: finding Least Common Ancestors in trees



- Examples:
  - $\text{LCA}(A, J) = A$
  - $\text{LCA}(E, H) = D$
  - $\text{LCA}(C, E) = B$
  - $\text{LCA}(C, G) = A$
- Algorithm for answering $\text{LCA}(x, y)$:
  - b: the best ancestor (initially: root)
  - For every vertex z, test if it is an ancestor for both x and y
  - If it is, and b is an ancestor of z, then $b \leftarrow z$

Example of working with timestamps: finding Least Common Ancestors in trees



- Examples:
  - $\text{LCA}(A, J) = A$
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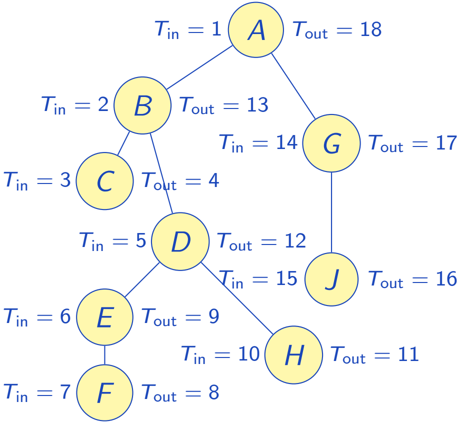
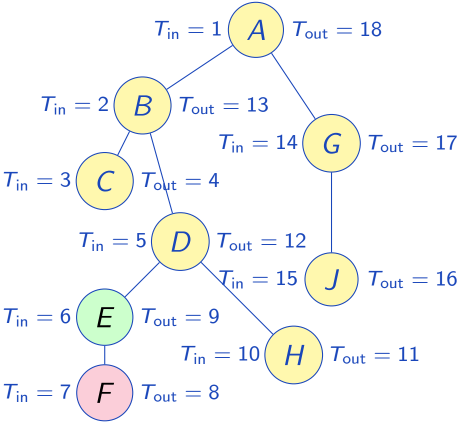
  - $\text{LCA}(C, E) = B$
  - $\text{LCA}(C, G) = A$
- Algorithm for answering $\text{LCA}(x, y)$:
  - b: the best ancestor (initially: root)
  - For every vertex z, test if it is an ancestor for both x and y
  - If it is, and b is an ancestor of z, then $b \leftarrow z$
- Runtime: $\Theta(|V|)$. Can we do it faster?

Example of working with timestamps: finding Least Common Ancestors in trees



- Path compression ("binary hops"):
  - $d[v][0]$ = parent of v
  - $d[v][i]$ = 2^i-th vertex towards root

Example of working with timestamps: finding Least Common Ancestors in trees



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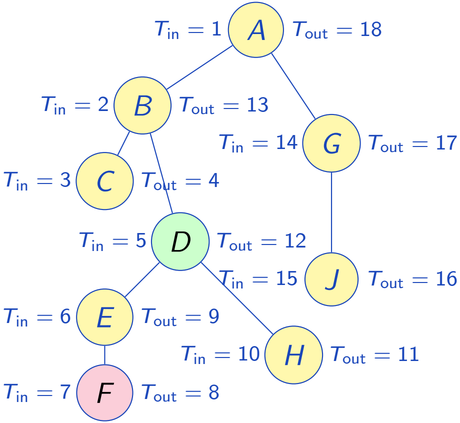

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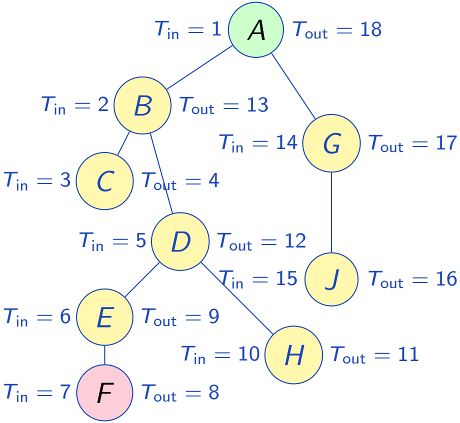

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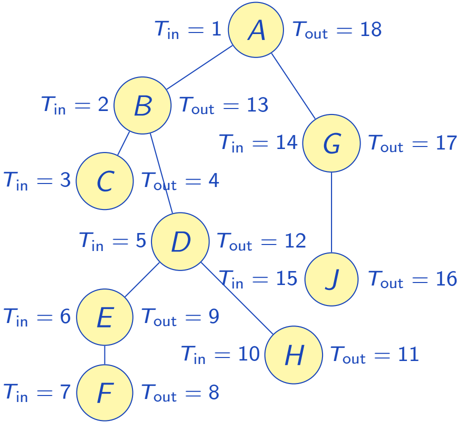

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procedure FillHops(V)
  for $v \in V$ **do**
    $d[v][0]$ = parent of v
  end for
  for $i \in [1; \log_2 |V|]$ **do**
    for $v \in V$ **do**
      $d[v][i] = d[d[v][i-1]][i-1]$
    end for
  end for
end procedure

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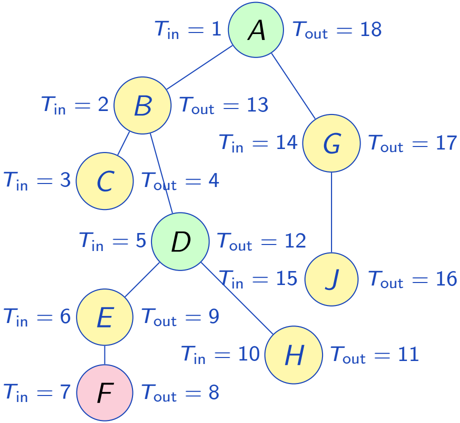

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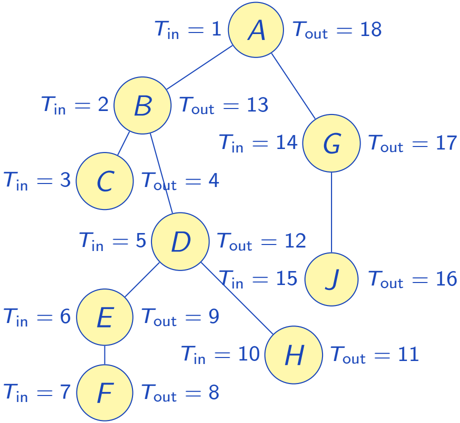

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  if $\textsc{IsAncestor}(b, a)$ **then return** b **end if**
  for i **from** $\log_2 |V|$ **down to** 1 **do**
    if not $\textsc{IsAncestor}(\mathrm{d}[a][i], b)$ **then**
      $a \leftarrow \mathrm{d}[a][i]$
    end if
  end for
  return $\mathrm{d}[a][0]$
end procedure

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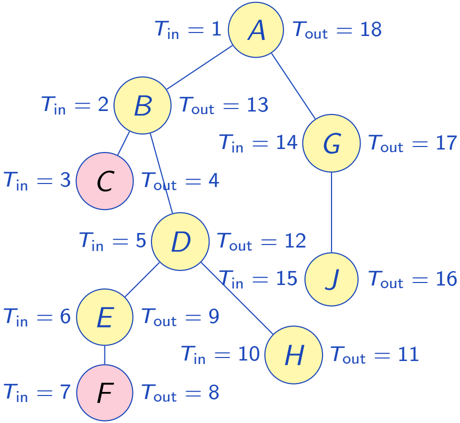

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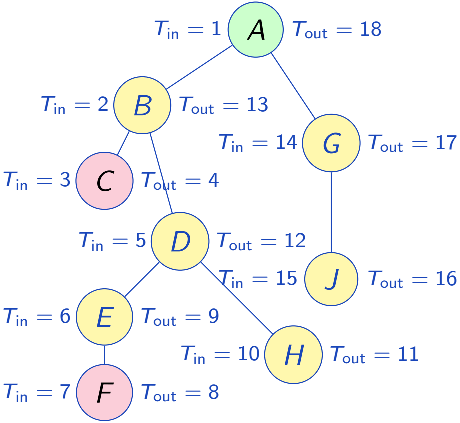

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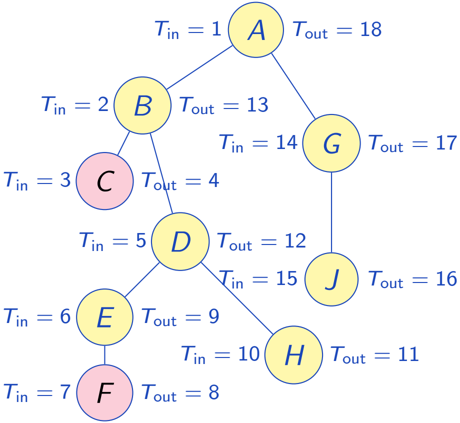

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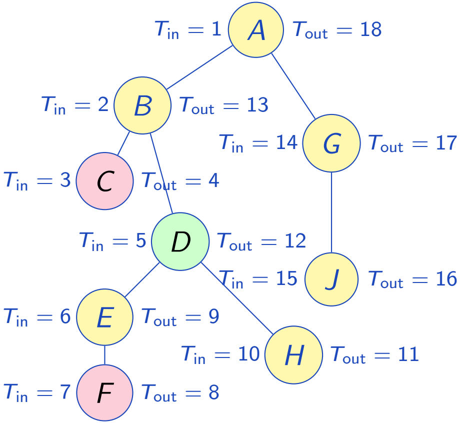

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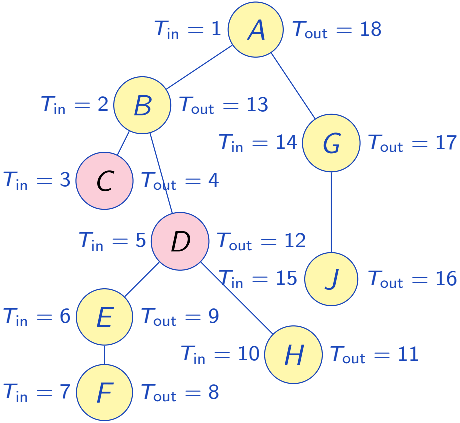

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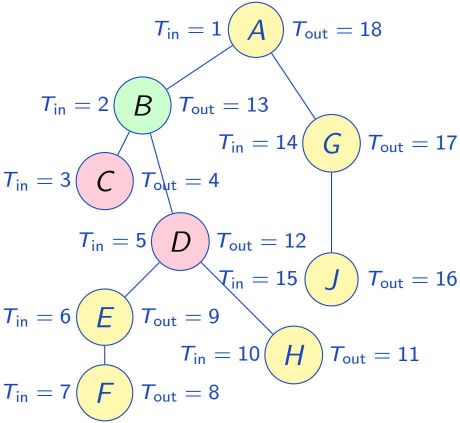

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            a ← d[a][i]
        end if
    end for
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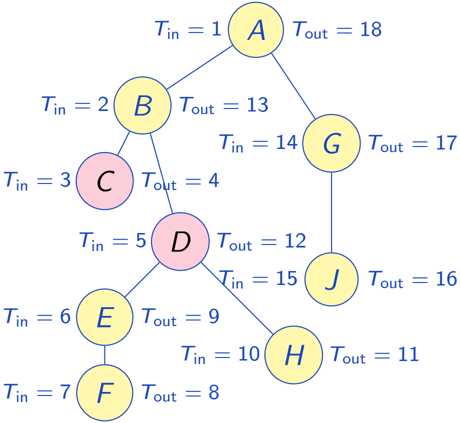

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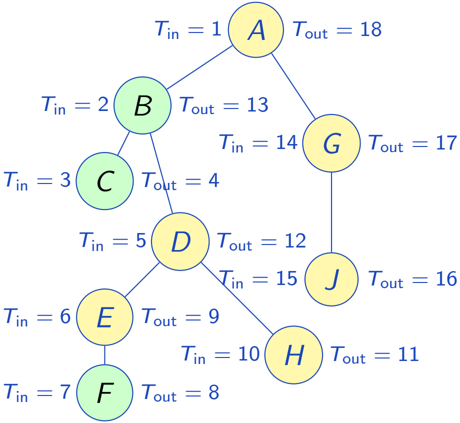

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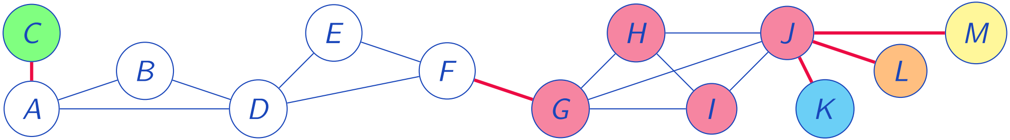

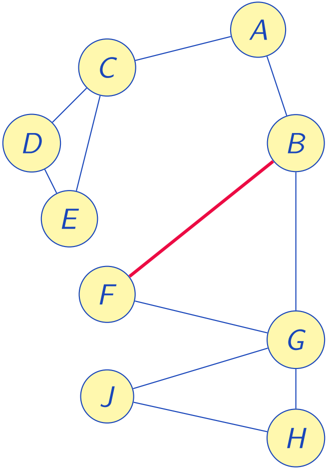

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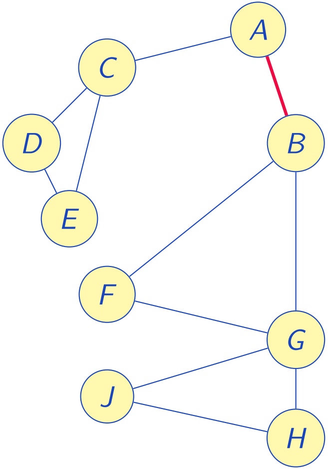

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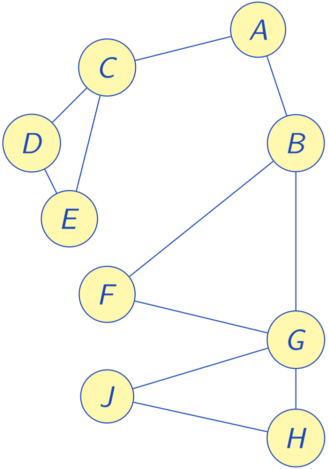

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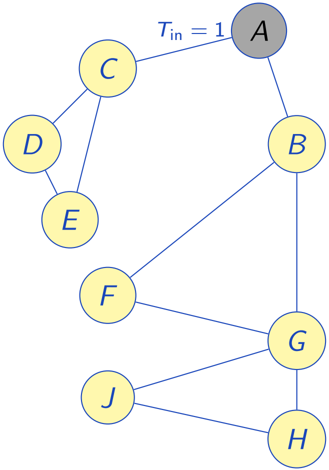

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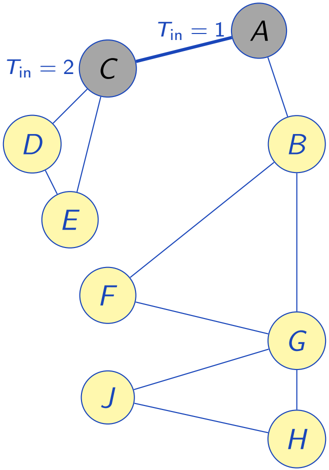

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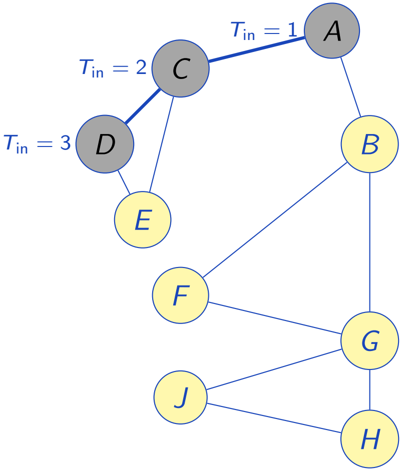

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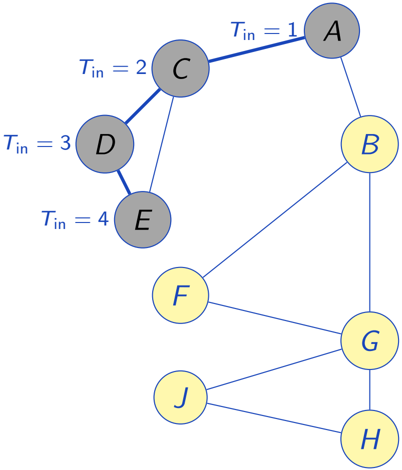

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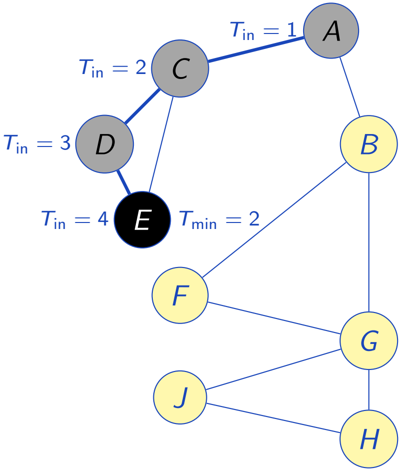

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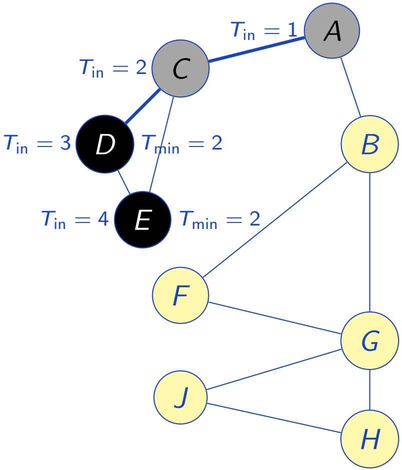

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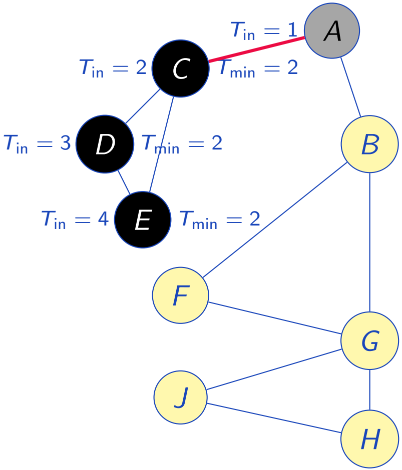

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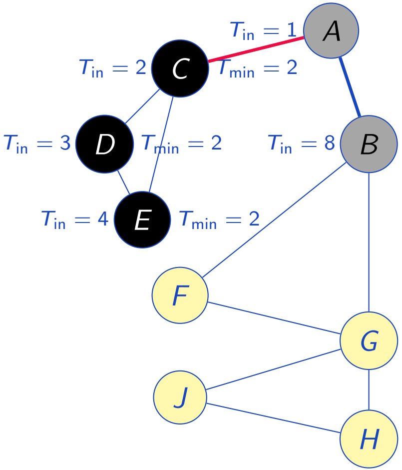

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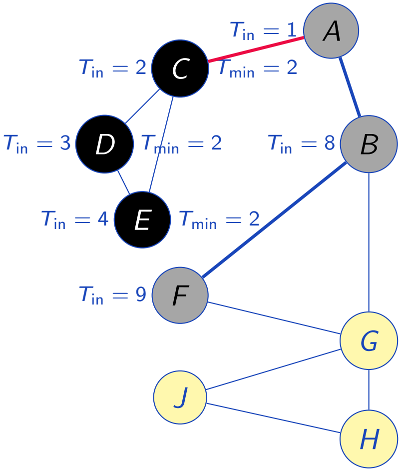

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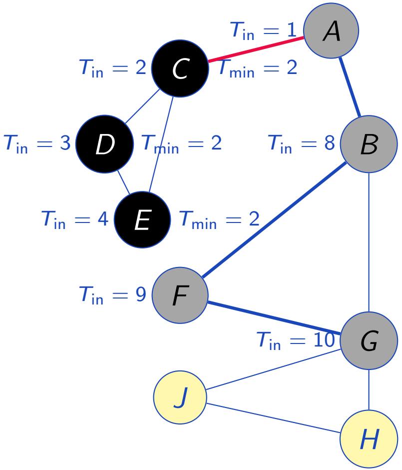

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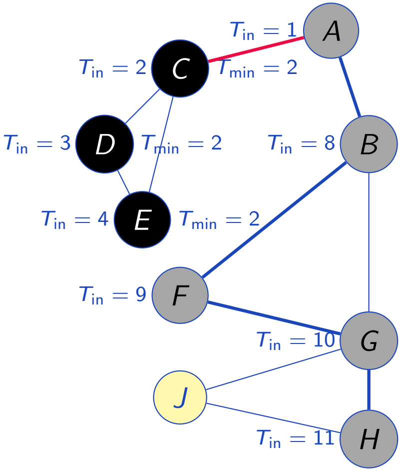

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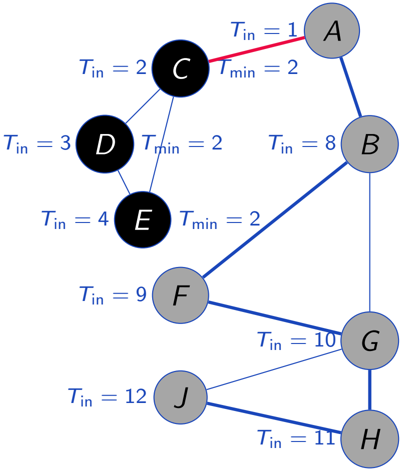

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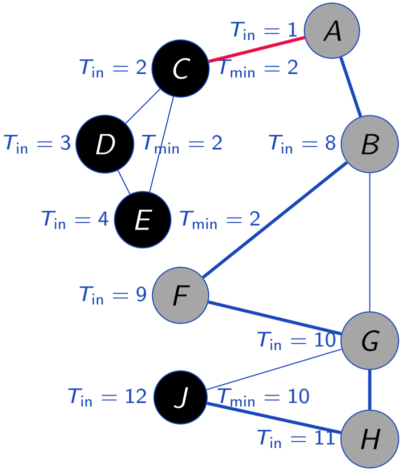

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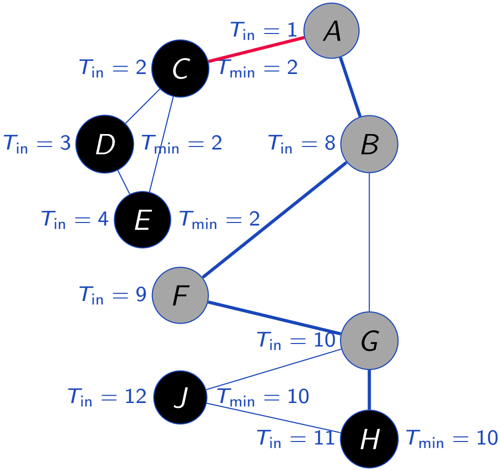

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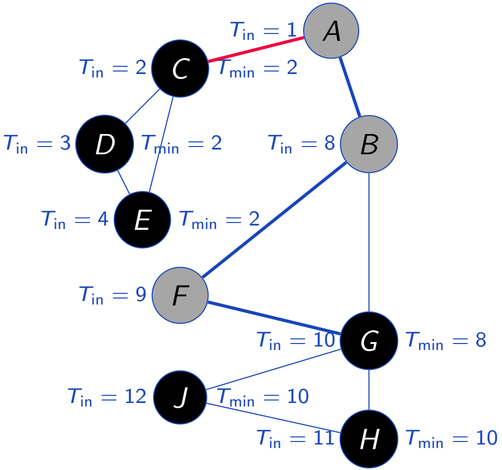
  - $T_{min}(u) > T_{in}(v)$: (v, u) is a bridge



- Consider an edge BF
  - B is reachable from F without this edge: BF is not a bridge
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  - A is not reachable from B without this edge: AB is a bridge
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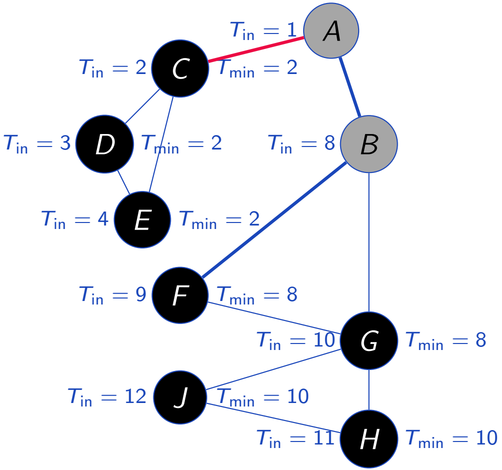

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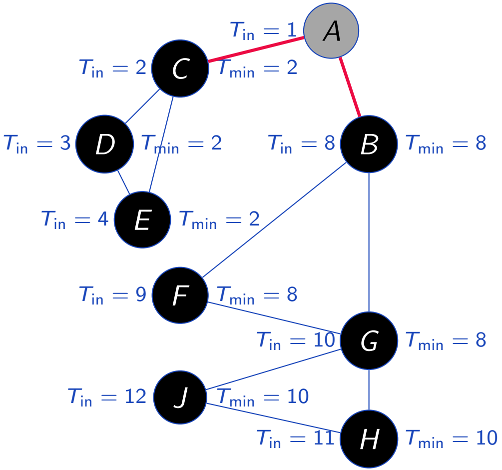

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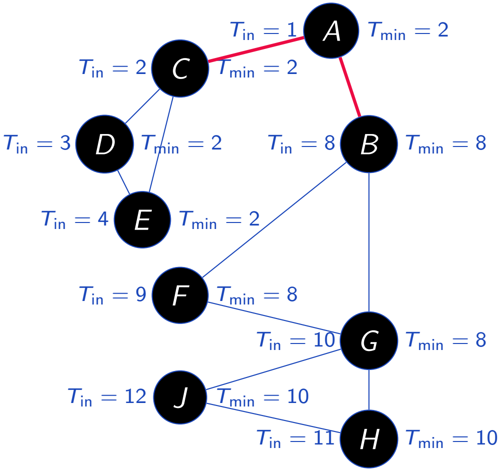

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$G = \langle V, E \rangle$
$T_{\text{in}}, T_{\text{min}} \leftarrow \{\infty\}$
$A(v) = \{u \mid (v, u) \in E\}$
$t \leftarrow 0$
procedure $\textsc{Bridges}(v,\ p = -1)$
    $t \leftarrow t + 1;\ T_{\text{in}}(v) \leftarrow t,\ T_{\text{min}}(v) \leftarrow t$
    for $u \in A(v)$ **do**
        if $p = u$ **then continue end if**
        if $T_{\text{in}}(u) = \infty$ **then**
            $\textsc{Bridges}(u,\ v)$
            $T_{\text{min}}(v) \leftarrow \min(T_{\text{min}}(v), T_{\text{min}}(u))$
            if $T_{\text{min}}(u) > T_{\text{in}}(v)$ **then**
                $\textsc{ReportBridge}(v,\ u)$
            end if
        else
            $T_{\text{min}}(v) \leftarrow \min(T_{\text{min}}(v), T_{\text{min}}(u))$
        end if
    end for
end procedure

$G = \langle V, E \rangle$
$T_{\text{in}}, T_{\text{min}} \leftarrow \{\infty\}$
$A(v) = \{u \mid (v, u) \in E\}$
$t \leftarrow 0$
procedure BRIDGES($v,\ p = -1$)
    $t \leftarrow t + 1;\ T_{\text{in}}(v) \leftarrow t,\ T_{\text{min}}(v) \leftarrow t$
    for $u \in A(v)$ **do**
        if $p = u$ **then continue end if**
        if $T_{\text{in}}(u) = \infty$ **then**
            BRIDGES($u,\ v$)
            $T_{\text{min}}(v) \leftarrow \min(T_{\text{min}}(v), T_{\text{min}}(u))$
            if $T_{\text{min}}(u) > T_{\text{in}}(v)$ **then**
                REPORTBRIDGE($v,\ u$)
            end if
        else
            $T_{\text{min}}(v) \leftarrow \min(T_{\text{min}}(v), T_{\text{min}}(u))$
        end if
    end for
end procedure

▷ Tracking T_{min} instead of T_{out}

```
G = ⟨V, E⟩
T_in, T_min ← {∞}                                   ▷ Tracking T_min instead of T_out
A(v) = {u | (v, u) ∈ E}
t ← 0
procedure BRIDGES(v, p = −1)                        ▷ Extra parameter: the parent of v
    t ← t + 1; T_in(v) ← t, T_min(v) ← t
    for u ∈ A(v) do
        if p = u then continue end if
        if T_in(u) = ∞ then
            BRIDGES(u, v)
            T_min(v) ← min(T_min(v), T_min(u))
            if T_min(u) > T_in(v) then
                REPORTBRIDGE(v, u)
            end if
        else
            T_min(v) ← min(T_min(v), T_min(u))
        end if
    end for
end procedure
```

```
G = ⟨V, E⟩
T_in, T_min ← {∞}                                    ▷ Tracking T_min instead of T_out
A(v) = {u | (v, u) ∈ E}
t ← 0
procedure Bridges(v, p = −1)                          ▷ Extra parameter: the parent of v
    t ← t + 1; T_in(v) ← t, T_min(v) ← t
    for u ∈ A(v) do
        if p = u then continue end if
        if T_in(u) = ∞ then
            Bridges(u, v)
            T_min(v) ← min(T_min(v), T_min(u))        ▷ Updating T_min by T_min of a descendant
            if T_min(u) > T_in(v) then
                ReportBridge(v, u)
            end if
        else
            T_min(v) ← min(T_min(v), T_min(u))
        end if
    end for
end procedure
```

```
G = ⟨V, E⟩
T_in, T_min ← {∞}                                  ▷ Tracking T_min instead of T_out
A(v) = {u | (v, u) ∈ E}
t ← 0
procedure Bridges(v, p = −1)                       ▷ Extra parameter: the parent of v
    t ← t + 1; T_in(v) ← t, T_min(v) ← t
    for u ∈ A(v) do
        if p = u then continue end if
        if T_in(u) = ∞ then
            Bridges(u, v)
            T_min(v) ← min(T_min(v), T_min(u))     ▷ Updating T_min by T_min of a descendant
            if T_min(u) > T_in(v) then
                ReportBridge(v, u)
            end if
        else
            T_min(v) ← min(T_min(v), T_min(u))     ▷ Updating T_min by T_min of other vertex
        end if
    end for
end procedure
```

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- If any vertex is removed, the graph will remain connected

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A graph can be decomposed into vertex-biconnected components, connected by articulation points.

How to do it faster than in $\Theta(|V| \cdot |E|)$?

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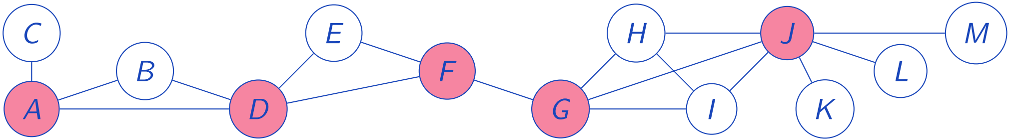

```
G = ⟨V, E⟩
T_in, T_min ← {∞}
A(v) = {u | (v, u) ∈ E}
t ← 0
procedure ARTICULATION(v, p = −1)
    t ← t + 1; T_in(v) ← t; T_min(v) ← t; ch ← 0
    for u ∈ A(v) do
        if p = u then continue end if
        if T_in(u) = ∞ then
            ch ← ch + 1
            ARTICULATION(u, v)
            T_min(v) ← min(T_min(v), T_min(u))
            if T_min(u) ≥ T_in(v) and p ≠ −1 then
                REPORTARTICULATION(v)
            end if
        else
            T_min(v) ← min(T_min(v), T_min(u))
        end if
    end for
    if p = −1 and ch > 1 then REPORTARTICULATION(v) end if
end procedure
```

$G = \langle V, E \rangle$
$T_{in}, T_{min} \leftarrow \{\infty\}$
$A(v) = \{u \mid (v, u) \in E\}$
$t \leftarrow 0$

```
procedure ARTICULATION(v, p = −1)
    t ← t + 1; T_in(v) ← t; T_min(v) ← t; ch ← 0          ▷ Now we also track children count
    for u ∈ A(v) do
        if p = u then continue end if
        if T_in(u) = ∞ then
            ch ← ch + 1
            ARTICULATION(u, v)
            T_min(v) ← min(T_min(v), T_min(u))
            if T_min(u) ≥ T_in(v) and p ≠ −1 then
                REPORTARTICULATION(v)
            end if
        else
            T_min(v) ← min(T_min(v), T_min(u))
        end if
    end for
    if p = −1 and ch > 1 then REPORTARTICULATION(v) end if
end procedure
```

$G = \langle V, E \rangle$
$T_{\text{in}}, T_{\text{min}} \leftarrow \{\infty\}$
$A(v) = \{u \mid (v, u) \in E\}$
$t \leftarrow 0$
procedure $\textsc{Articulation}(v,\ p = -1)$
    $t \leftarrow t + 1$; $T_{\text{in}}(v) \leftarrow t$; $T_{\text{min}}(v) \leftarrow t$; $ch \leftarrow 0$ ▷ Now we also track children count
    for $u \in A(v)$ **do**
        if $p = u$ **then continue end if**
        if $T_{\text{in}}(u) = \infty$ **then**
            $ch \leftarrow ch + 1$ ▷ ...and incrementing it on every child
            $\textsc{Articulation}(u,\ v)$
            $T_{\text{min}}(v) \leftarrow \min(T_{\text{min}}(v), T_{\text{min}}(u))$
            if $T_{\text{min}}(u) \geq T_{\text{in}}(v)$ **and** $p \neq -1$ **then**
                $\textsc{ReportArticulation}(v)$
            end if
        else
            $T_{\text{min}}(v) \leftarrow \min(T_{\text{min}}(v), T_{\text{min}}(u))$
        end if
    end for
    if $p = -1$ **and** $ch > 1$ **then** $\textsc{ReportArticulation}(v)$ **end if**
end procedure

$G = \langle V, E \rangle$
$T_{\text{in}}, T_{\text{min}} \leftarrow \{\infty\}$
$A(v) = \{u \mid (v, u) \in E\}$
$t \leftarrow 0$
procedure ARTICULATION(v, $p = -1$)
    $t \leftarrow t + 1$; $T_{\text{in}}(v) \leftarrow t$; $T_{\text{min}}(v) \leftarrow t$; $ch \leftarrow 0$ ▷ Now we also track children count
    for $u \in A(v)$ **do**
        if $p = u$ **then continue end if**
        if $T_{\text{in}}(u) = \infty$ **then**
            $ch \leftarrow ch + 1$ ▷ ...and incrementing it on every child
            ARTICULATION(u, v)
            $T_{\text{min}}(v) \leftarrow \min(T_{\text{min}}(v), T_{\text{min}}(u))$
            if $T_{\text{min}}(u) \geq T_{\text{in}}(v)$ **and** $p \neq -1$ **then** ▷ Now inequality is non-strict, and root is not considered
                REPORTARTICULATION(v)
            end if
        else
            $T_{\text{min}}(v) \leftarrow \min(T_{\text{min}}(v), T_{\text{min}}(u))$
        end if
    end for
    if $p = -1$ **and** $ch > 1$ **then** REPORTARTICULATION(v) **end if**
end procedure

$G = \langle V, E \rangle$
$T_{\text{in}}, T_{\text{min}} \leftarrow \{\infty\}$
$A(v) = \{u \mid (v, u) \in E\}$
$t \leftarrow 0$
procedure $\textsc{Articulation}(v,\ p = -1)$
    $t \leftarrow t + 1;\ T_{\text{in}}(v) \leftarrow t;\ T_{\text{min}}(v) \leftarrow t;\ ch \leftarrow 0$ ▷ Now we also track children count
    for $u \in A(v)$ **do**
        if $p = u$ **then continue end if**
        if $T_{\text{in}}(u) = \infty$ **then**
            $ch \leftarrow ch + 1$ ▷ . . . and incrementing it on every child
            $\textsc{Articulation}(u,\ v)$
            $T_{\text{min}}(v) \leftarrow \min(T_{\text{min}}(v), T_{\text{min}}(u))$
            if $T_{\text{min}}(u) \geq T_{\text{in}}(v)$ **and** $p \neq -1$ **then** ▷ Now inequality is non-strict, and root is not considered
                $\textsc{ReportArticulation}(v)$
            end if
        else
            $T_{\text{min}}(v) \leftarrow \min(T_{\text{min}}(v), T_{\text{min}}(u))$
        end if
    end for
    if $p = -1$ **and** $ch > 1$ **then** $\textsc{ReportArticulation}(v)$ **end if** ▷ A root is AP iff $ch > 1$
end procedure