ITMO UNIVERSITY

How to Win Coding Competitions: Secrets of Champions

Week 6: Algorithms on Graphs 2
Lecture 4: Single Source Shortest Paths

Maxim Buzdalov
Saint Petersburg 2016

Problem: for every vertex determine a shortest path from v_0

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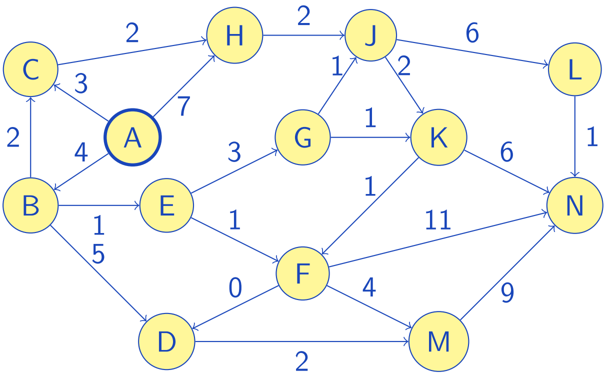

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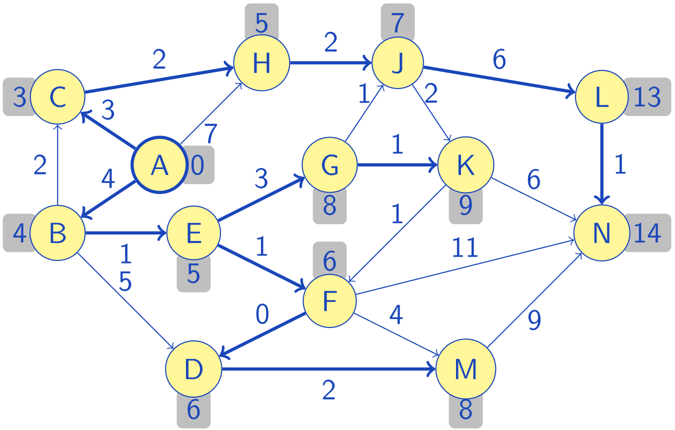

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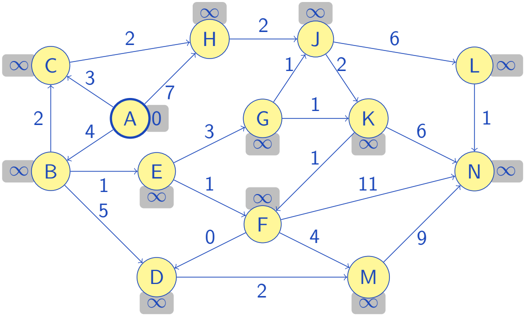

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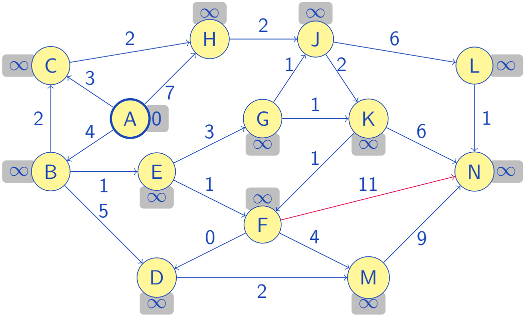

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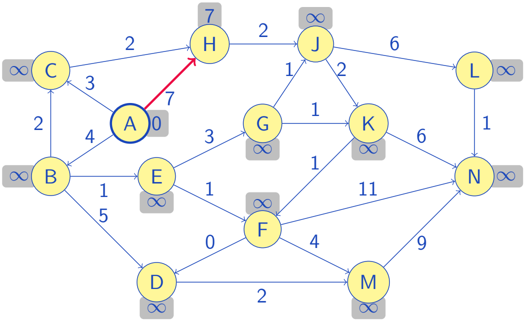

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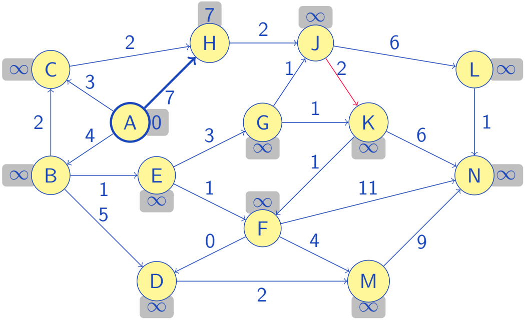

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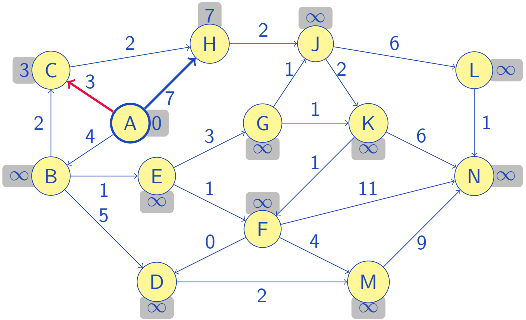

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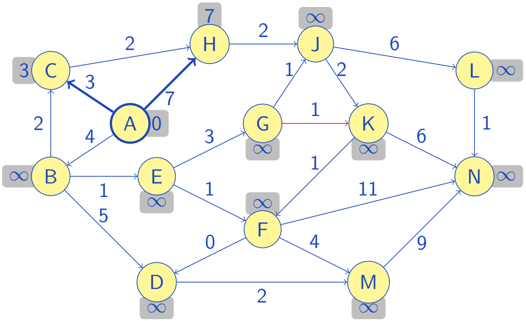

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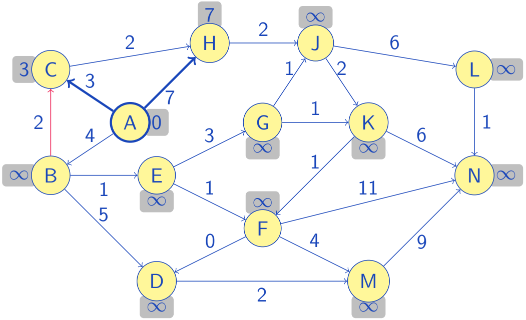

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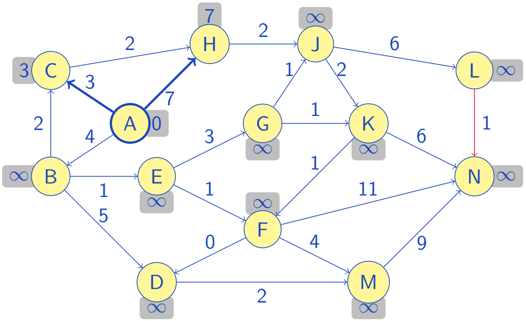

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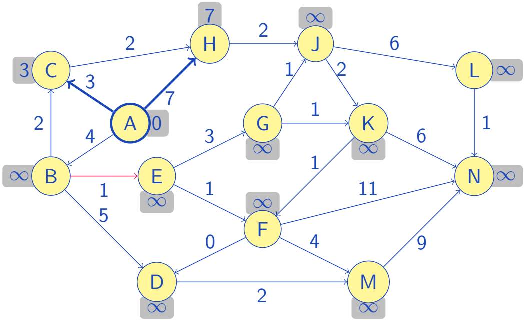

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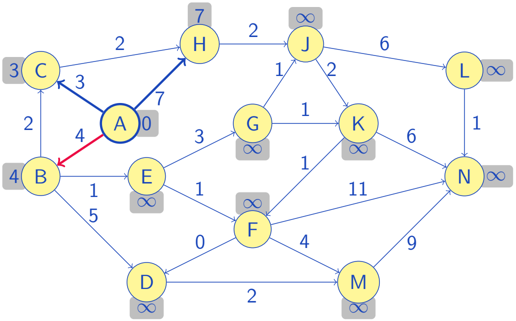

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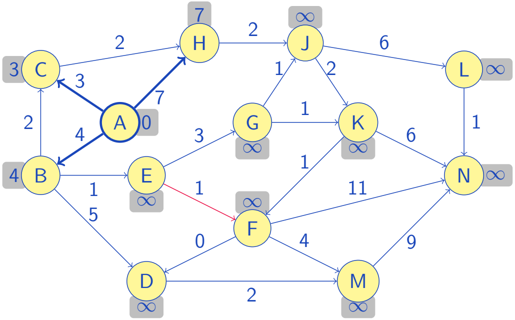

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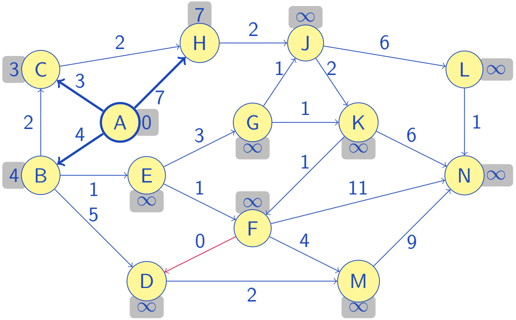

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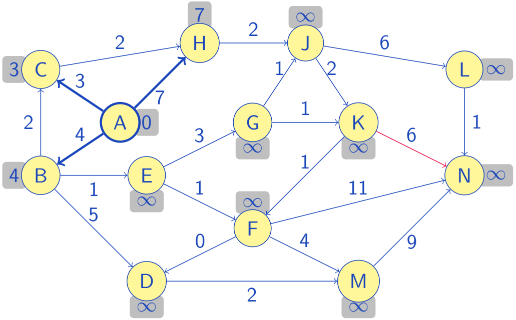

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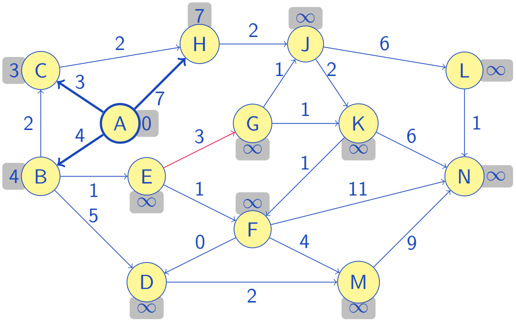

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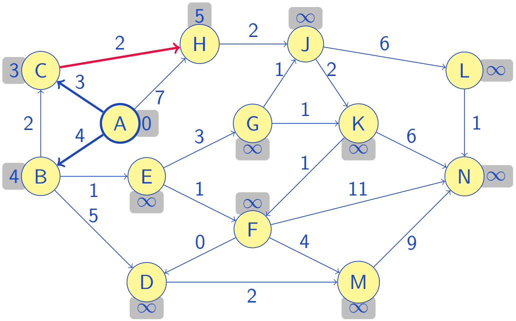

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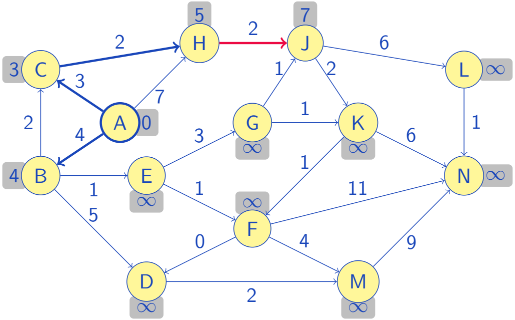

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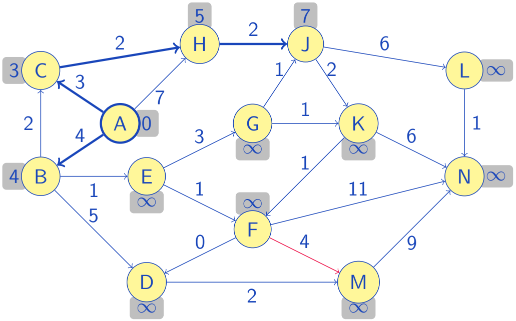

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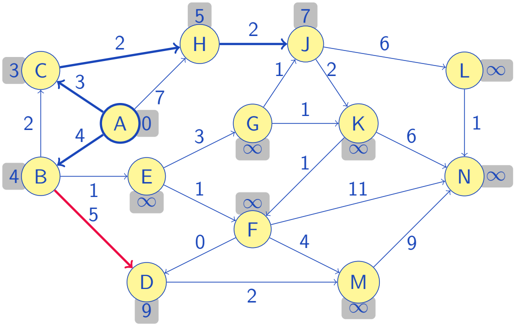

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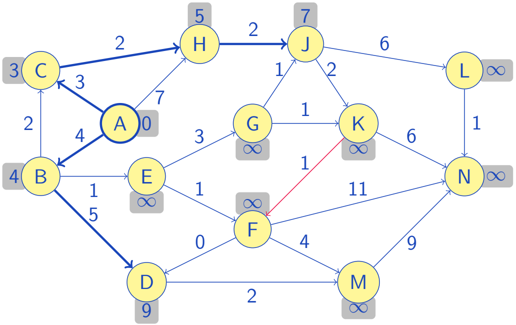

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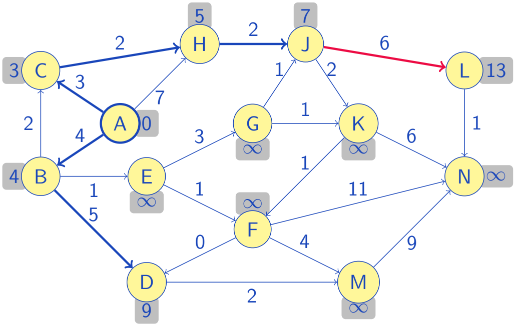

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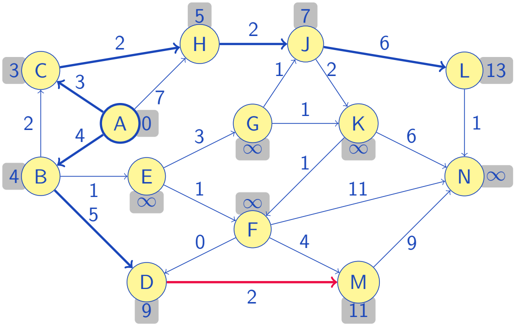

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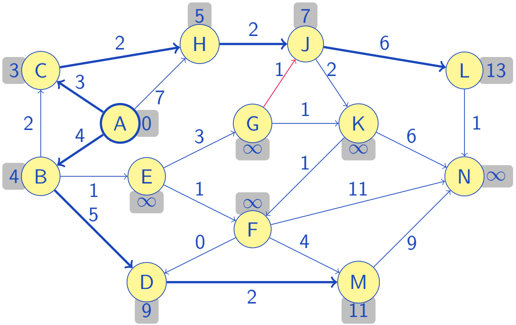

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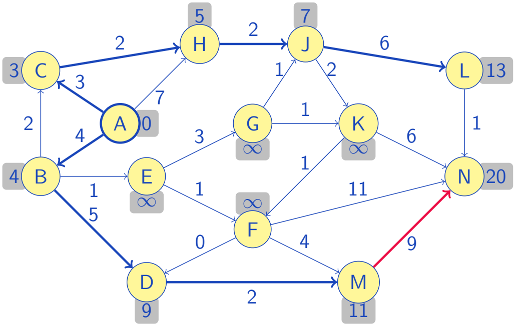

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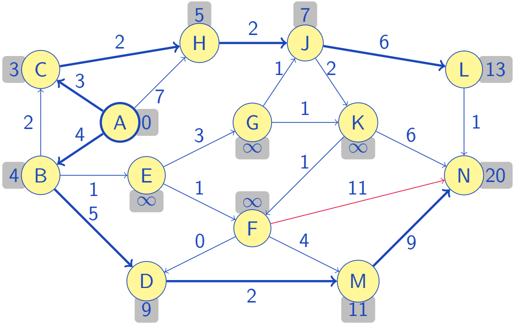

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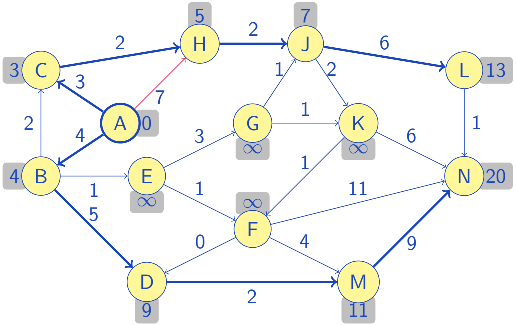

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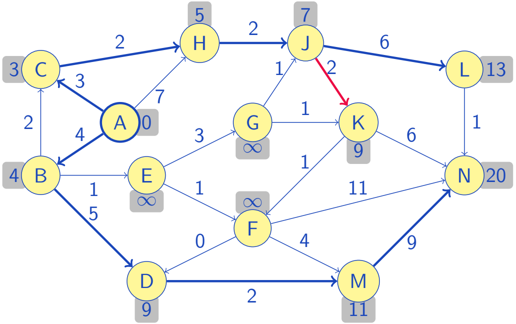

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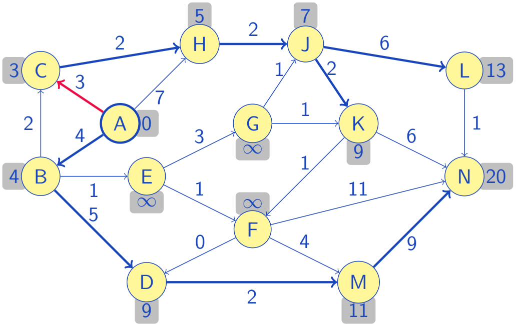

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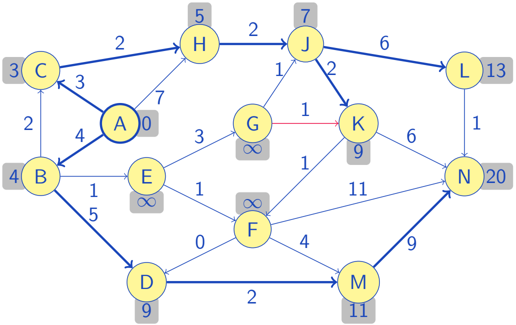

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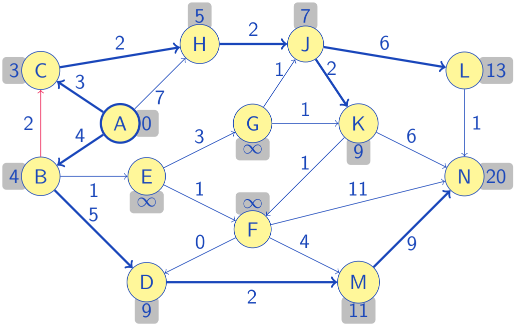

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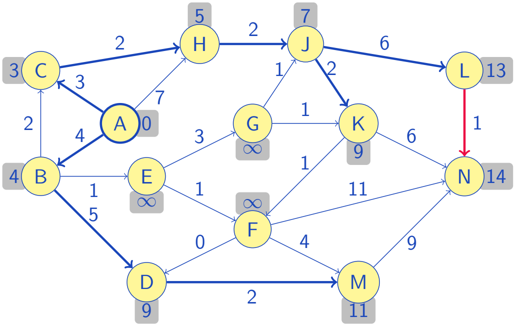

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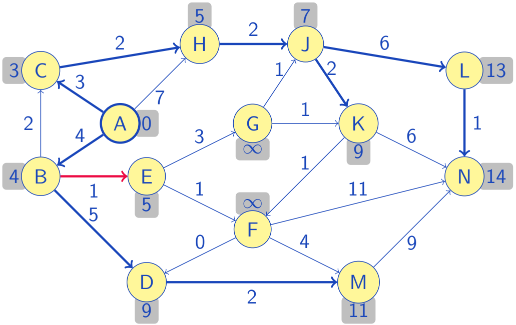

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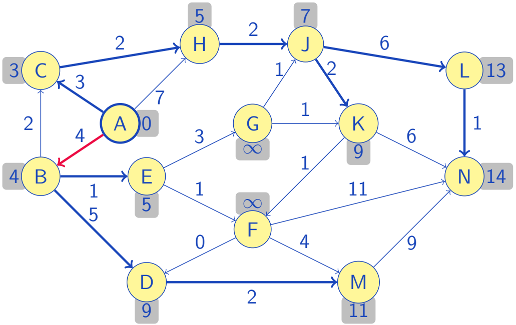

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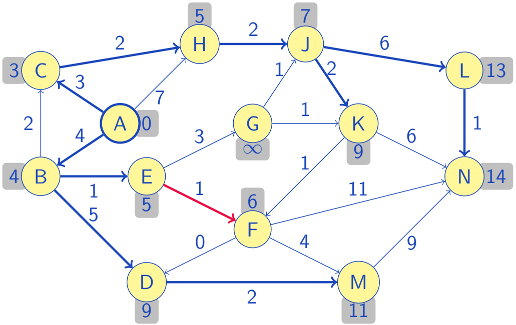

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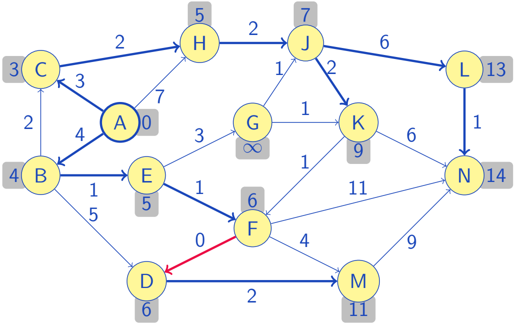

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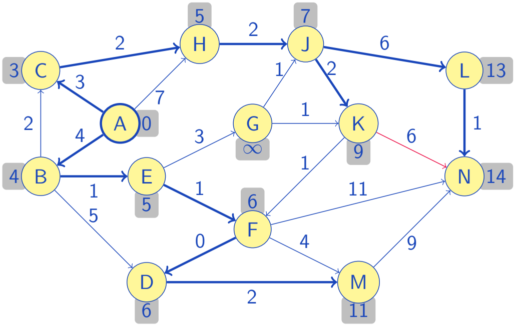

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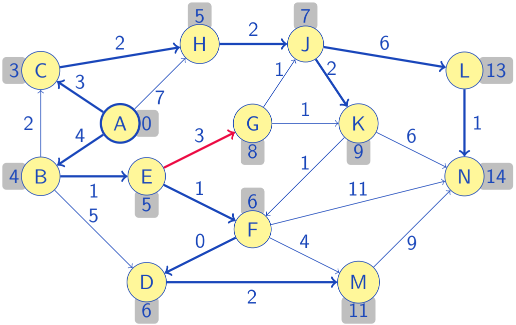

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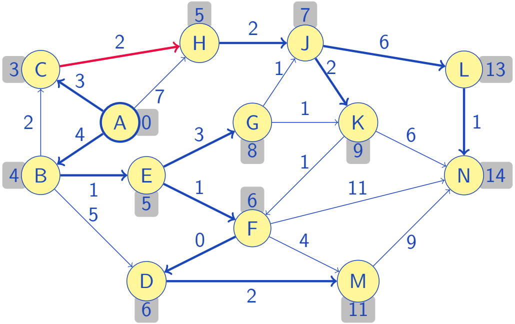

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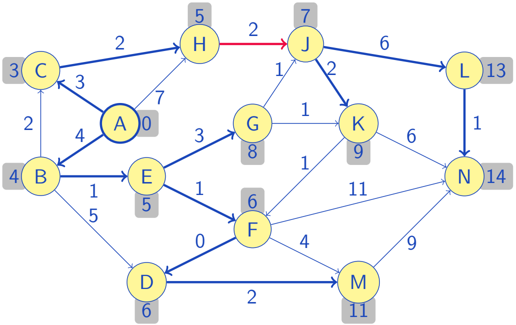

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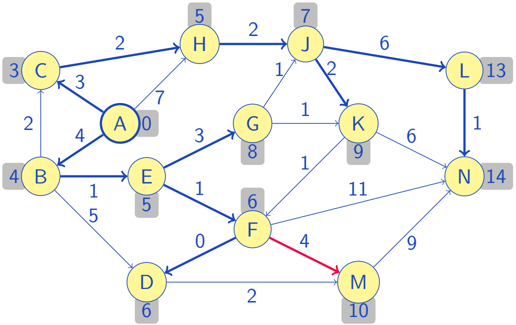

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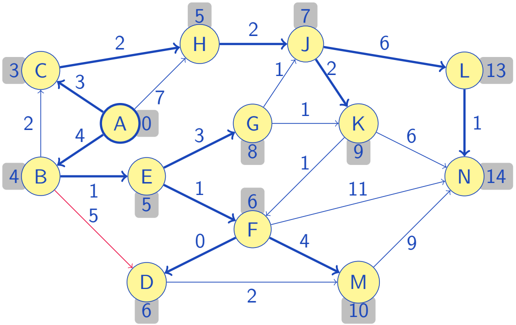

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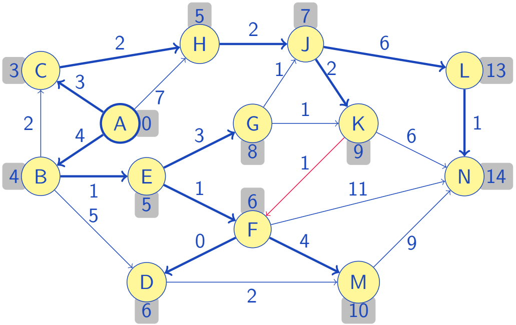

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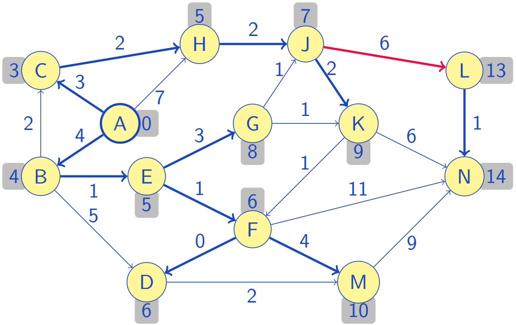

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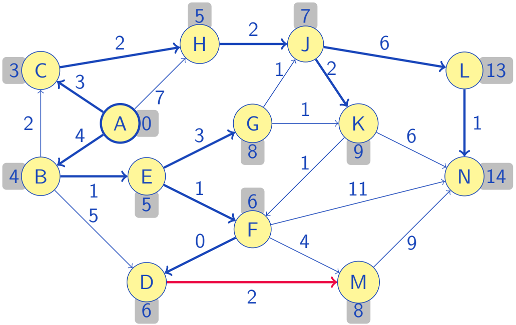

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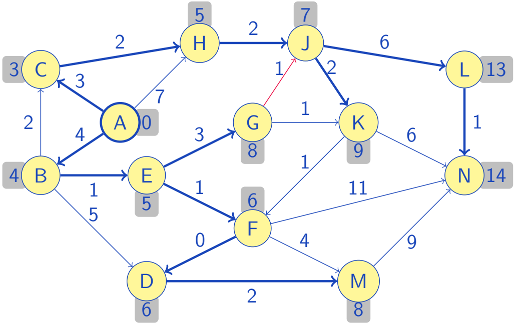

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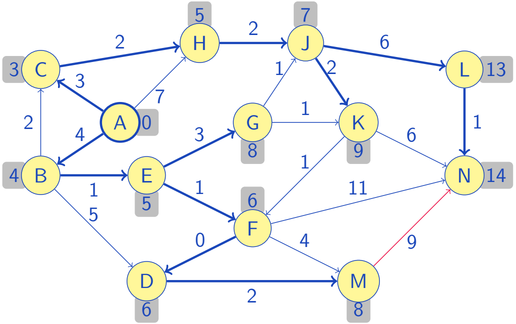

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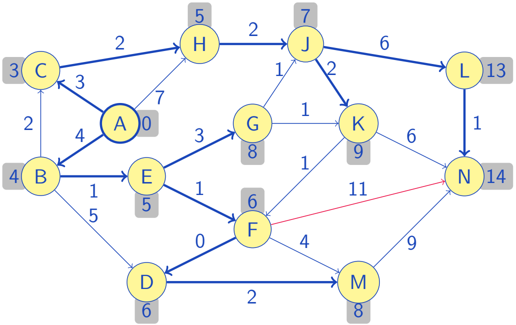

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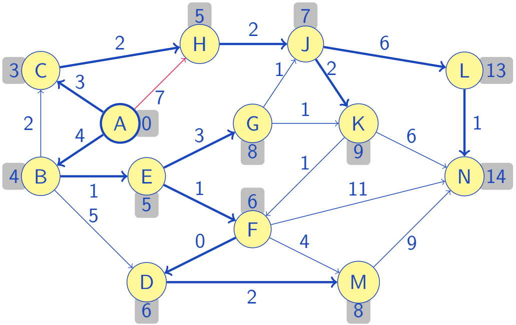

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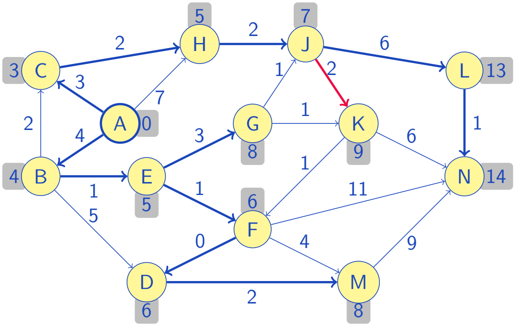

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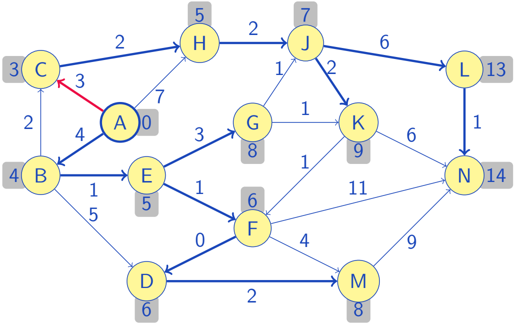

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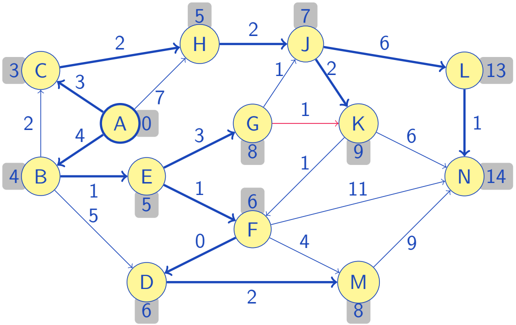

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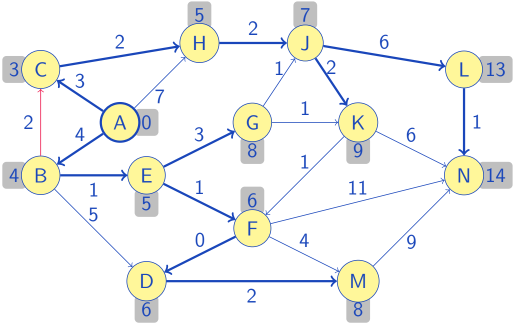

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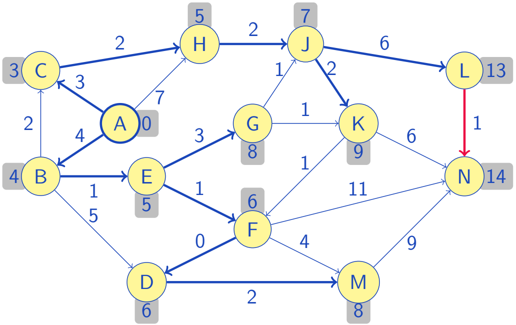

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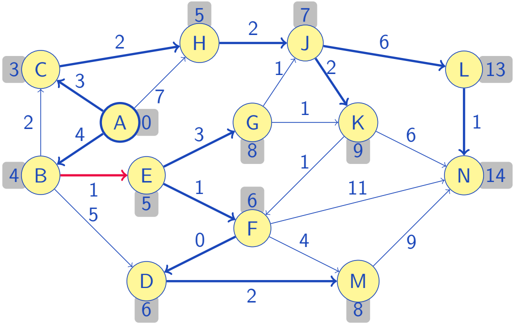

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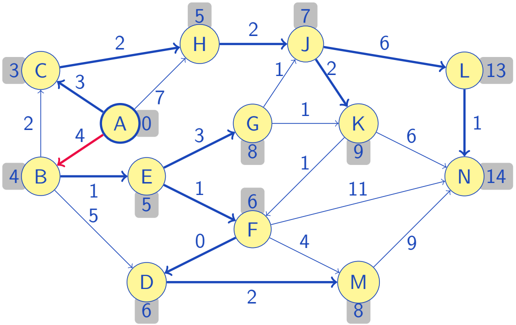

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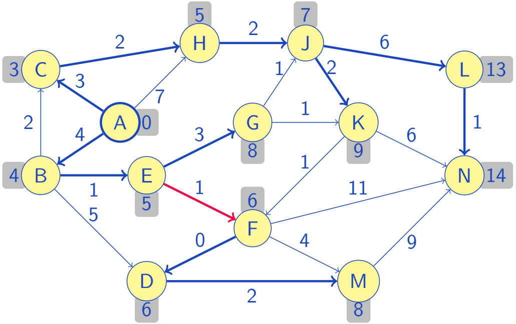

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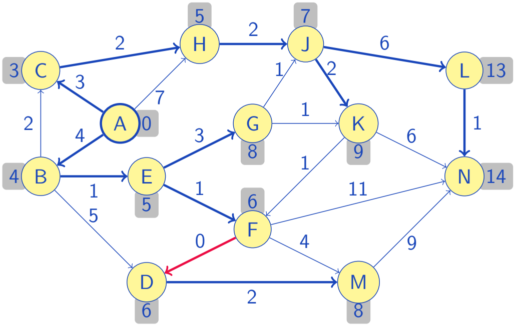

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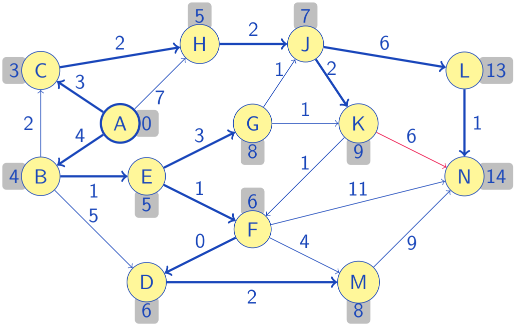

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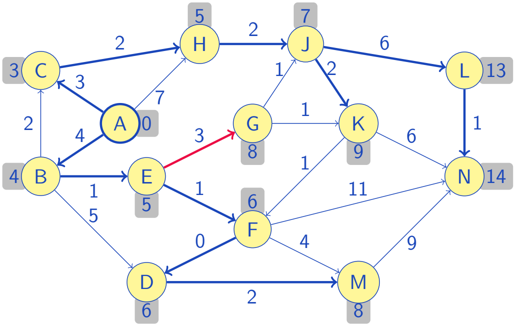

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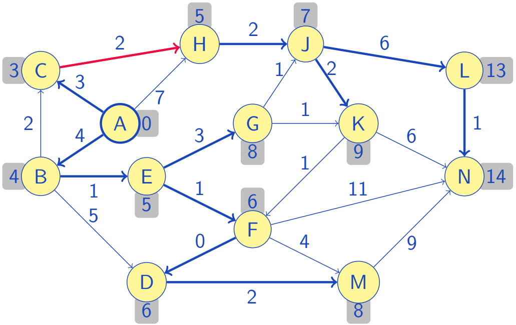

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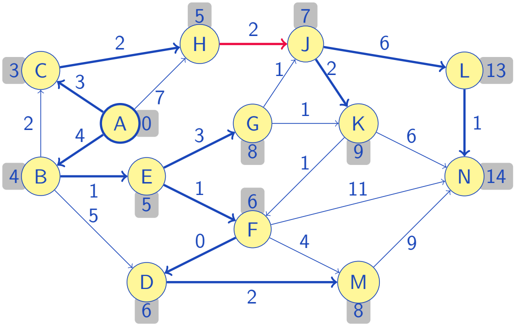

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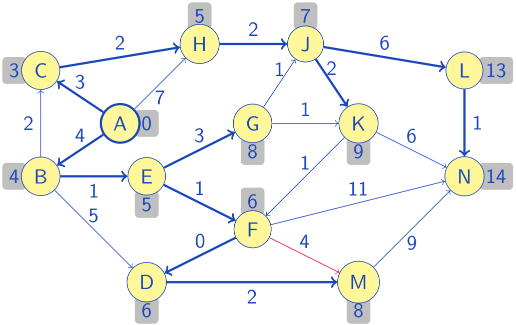

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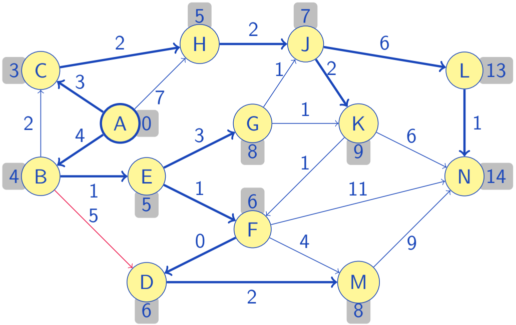

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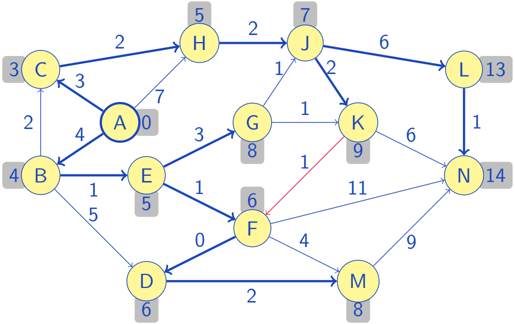

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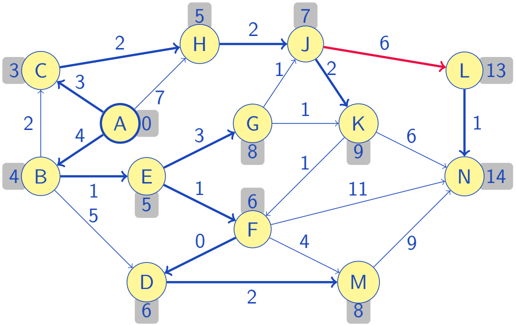

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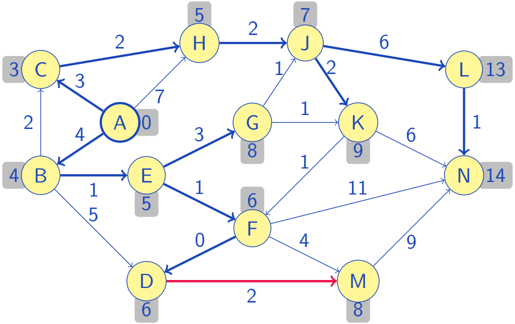

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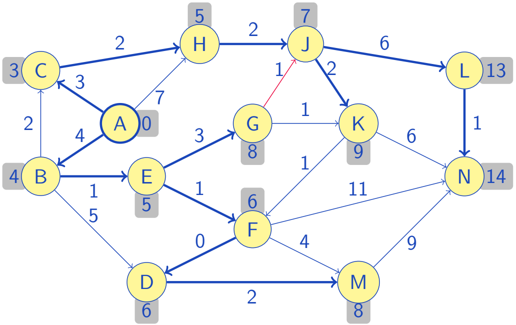

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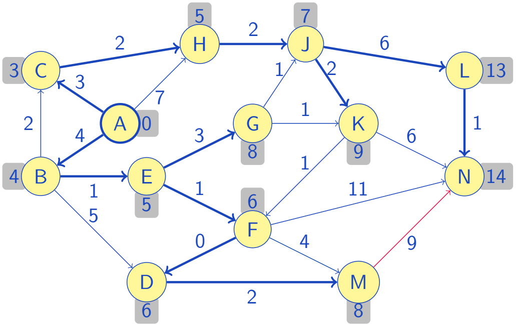

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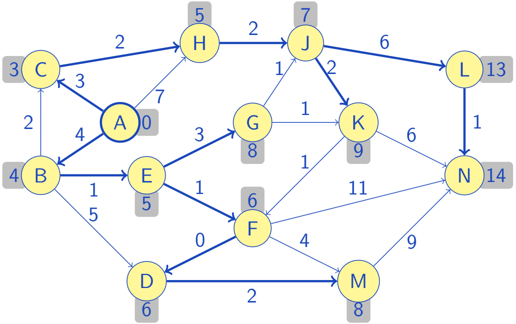
Example. Iteration 3. Nothing changed. We may stop



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The Bellman-Ford algorithm can detect negative cycles. How?

- Update shortest distances along all edges once more
- If a shortest distance to v changes, then v is reachable from a negative cycle

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- Idea: Maintain a set of vertices S with determined shortest distance from v_0
  - Initially, $S = \{v_0\}$
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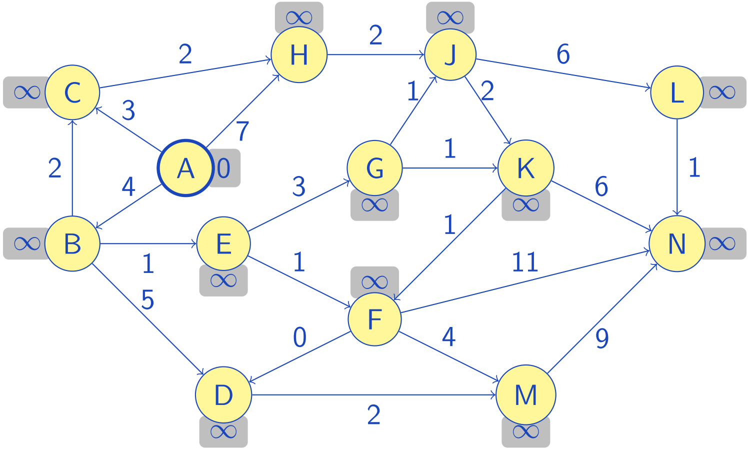
  - But edge lengths are non-negative $\rightarrow$ contradiction
- How to update S:
  - Choose $v \notin S$ with the smallest $D'[v]$
  - Set $D[v] = D'[v]$
  - $S \leftarrow S \cup \{v\}$
  - For all edges $(v, v') \in E$, update $D'[v'] \leftarrow \min(D'[v'], D[v] + L(v, v'))$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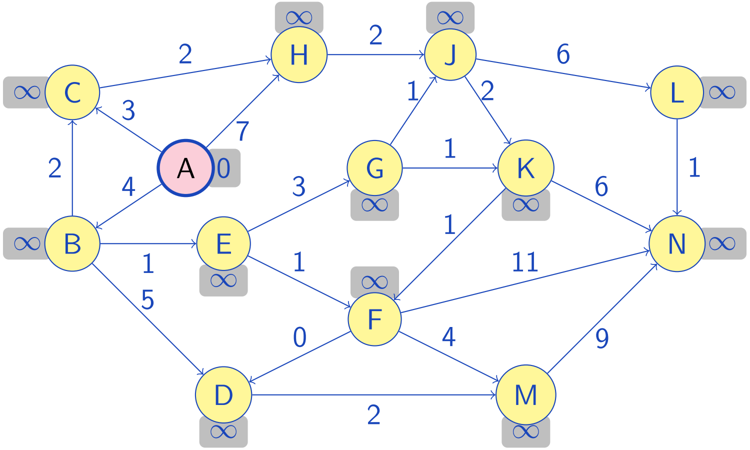

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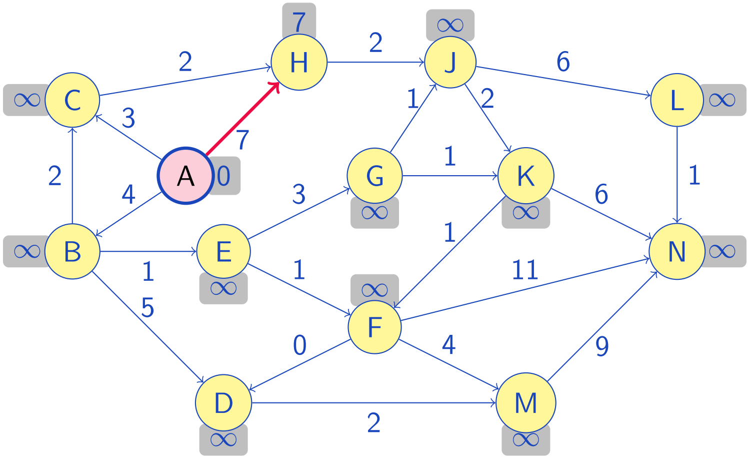
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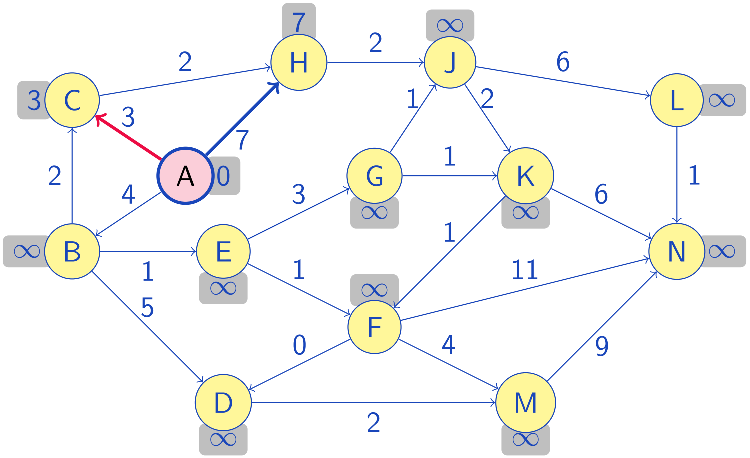
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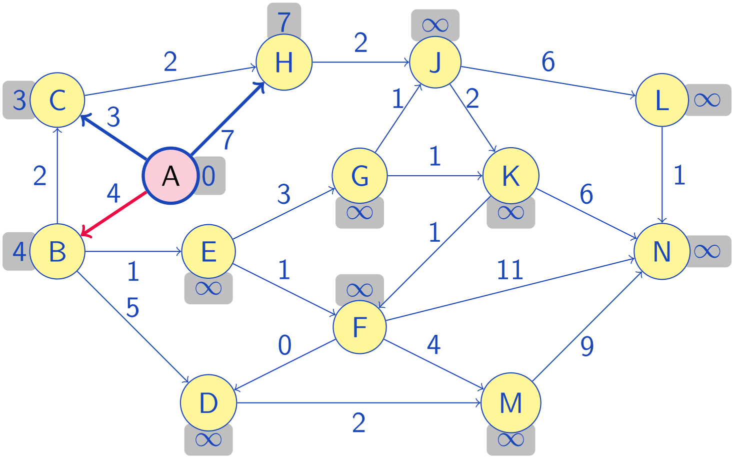
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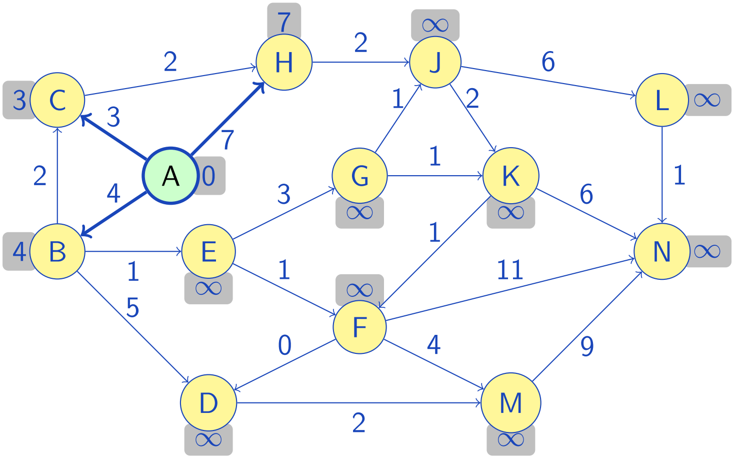
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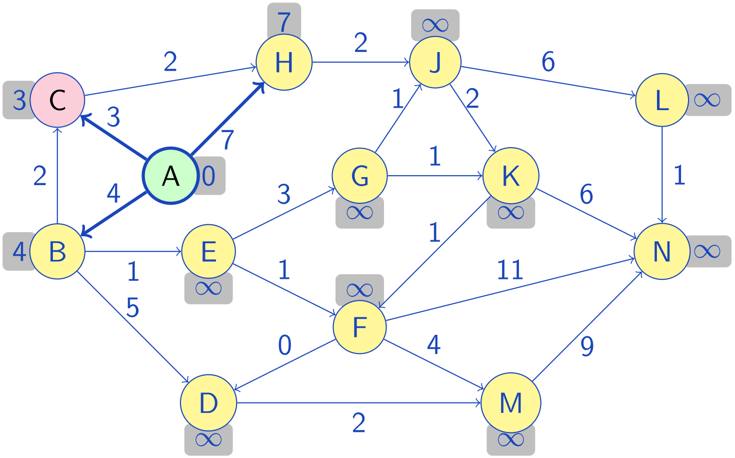
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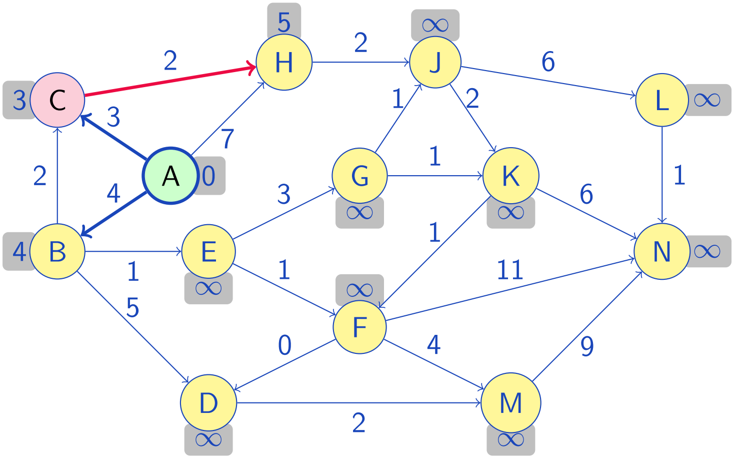
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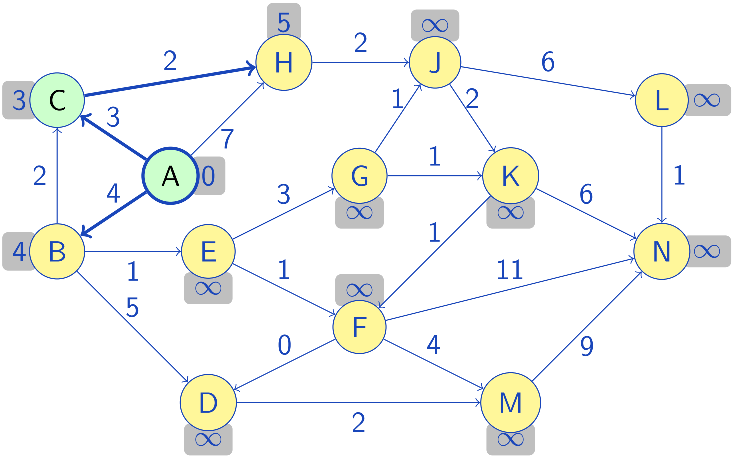
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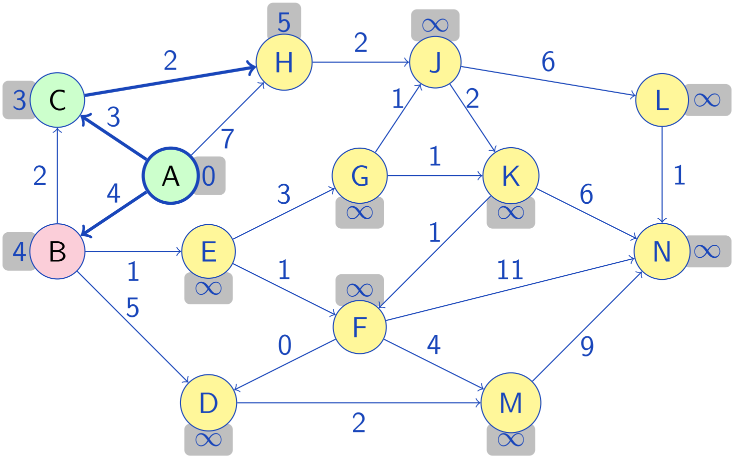
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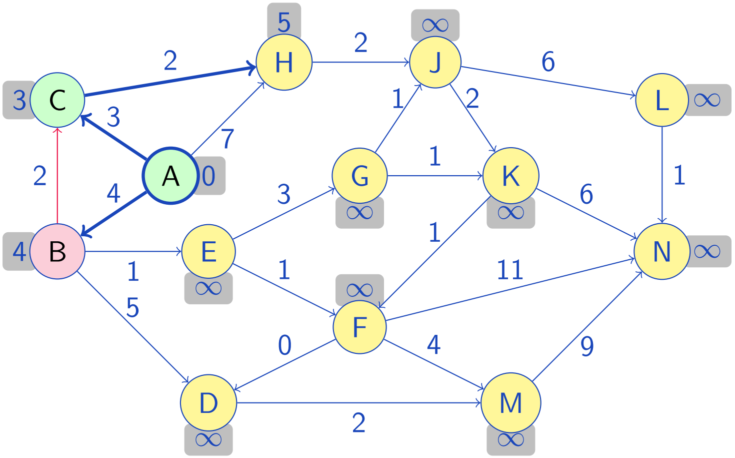
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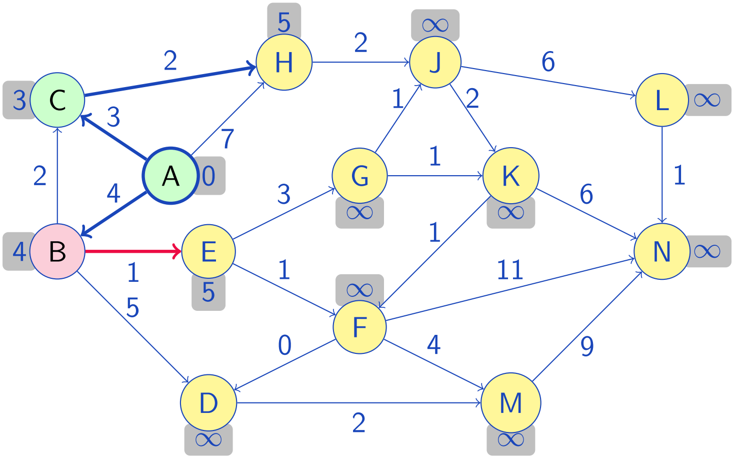
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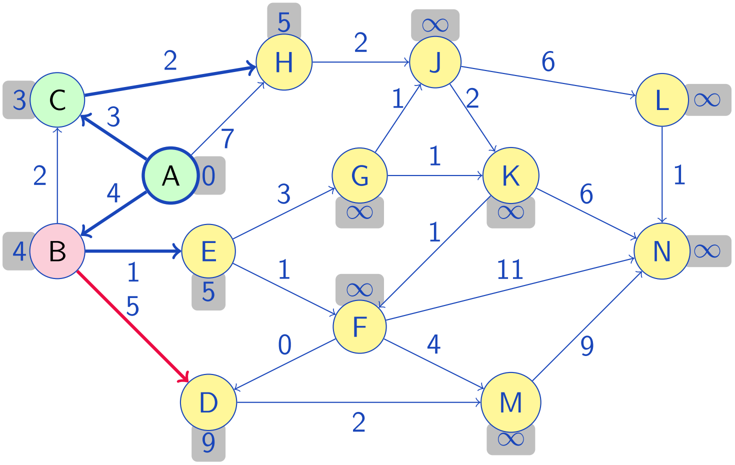
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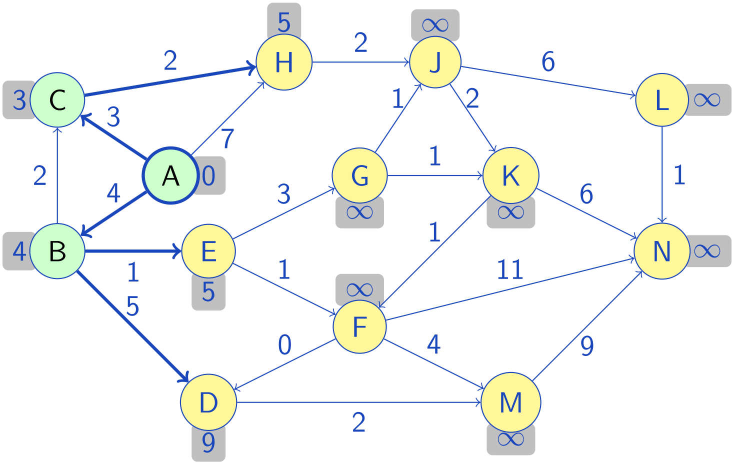
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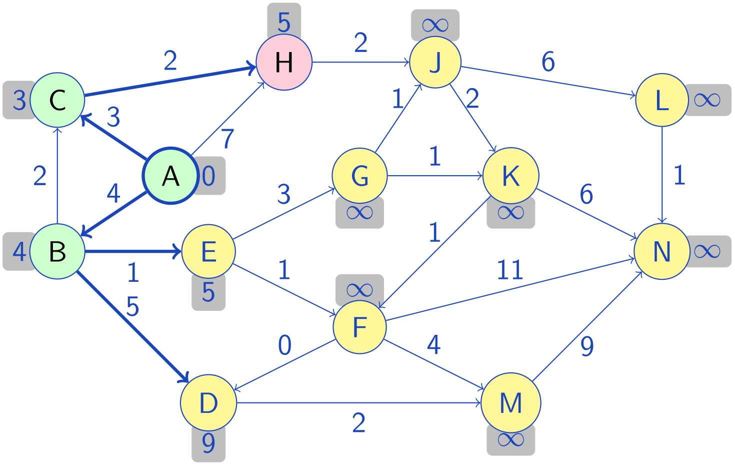
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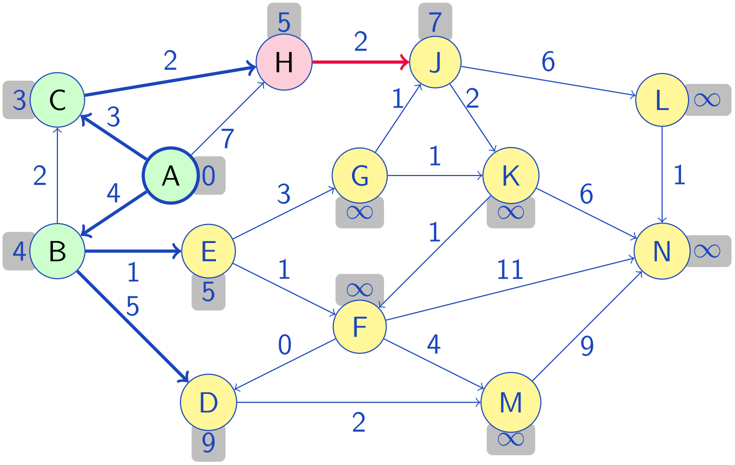
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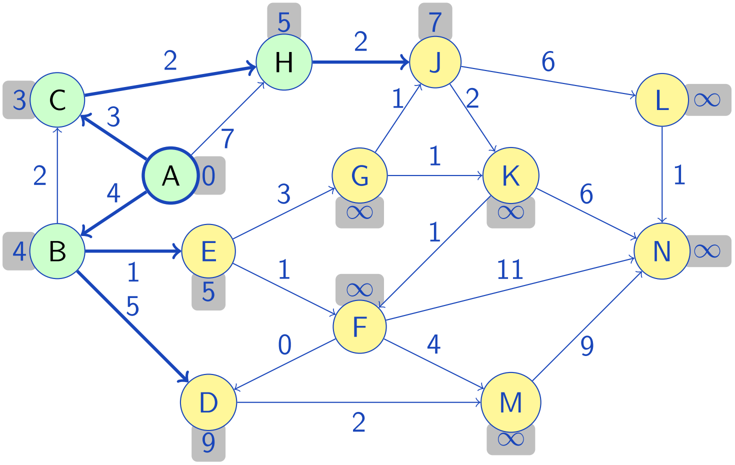
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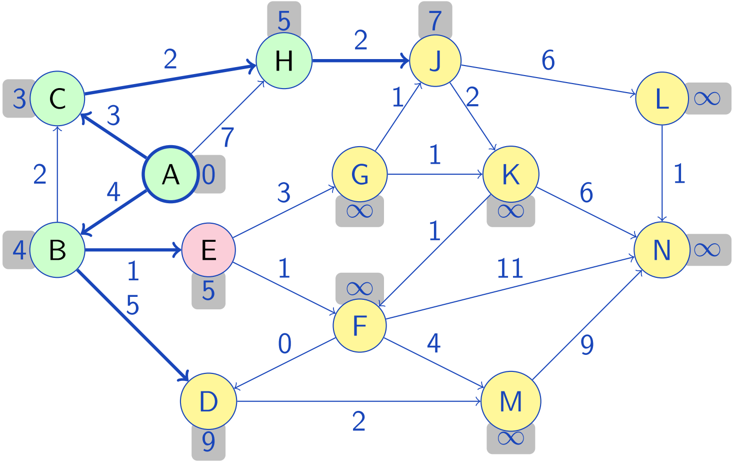
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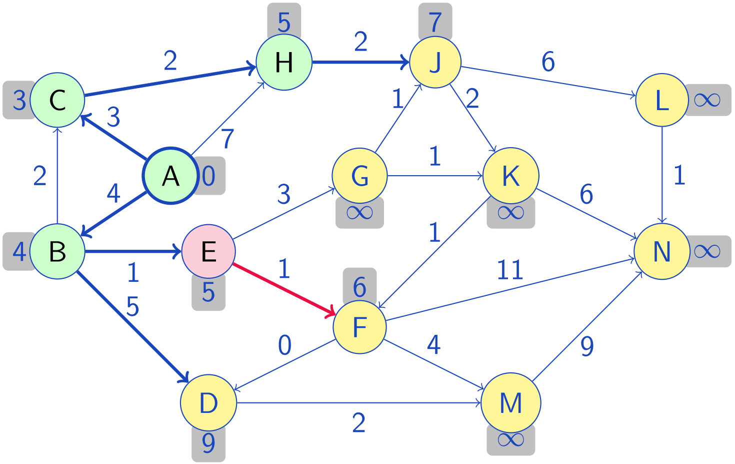
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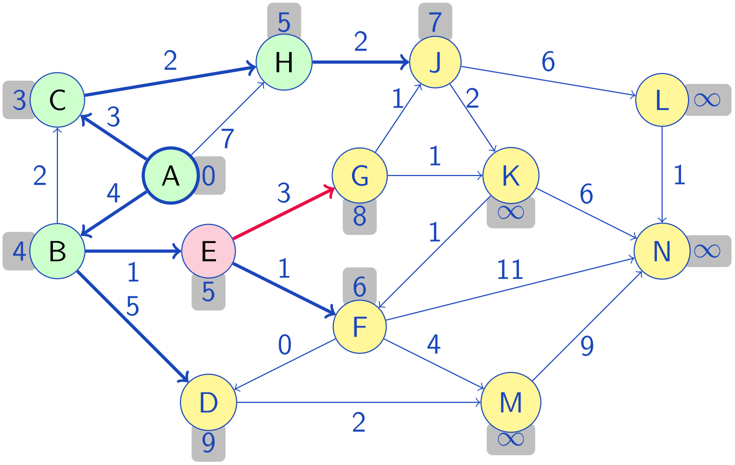
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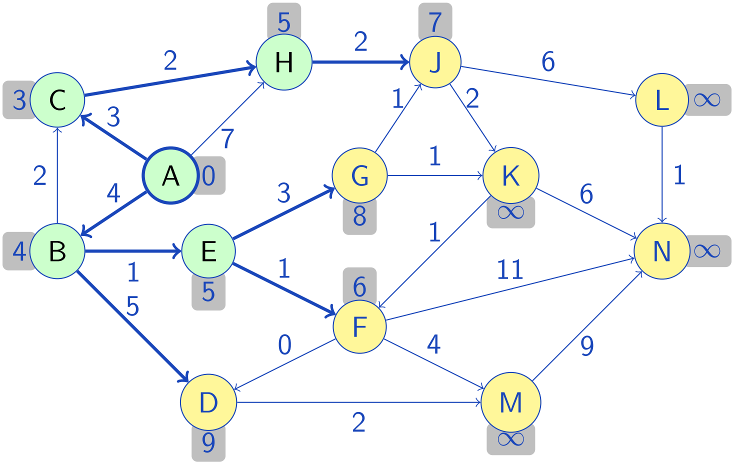
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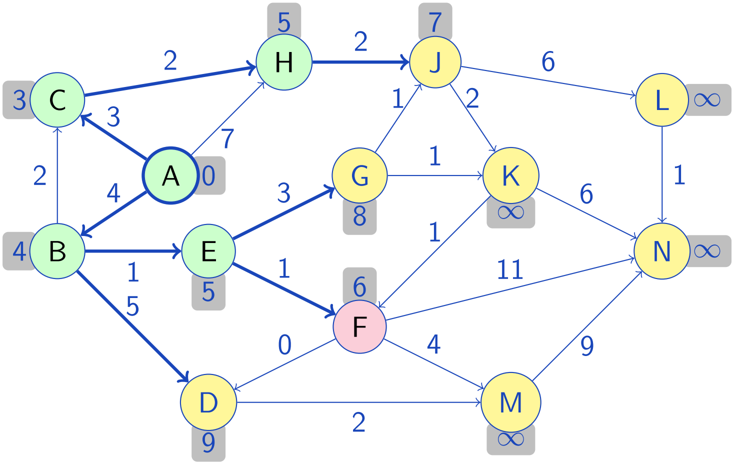
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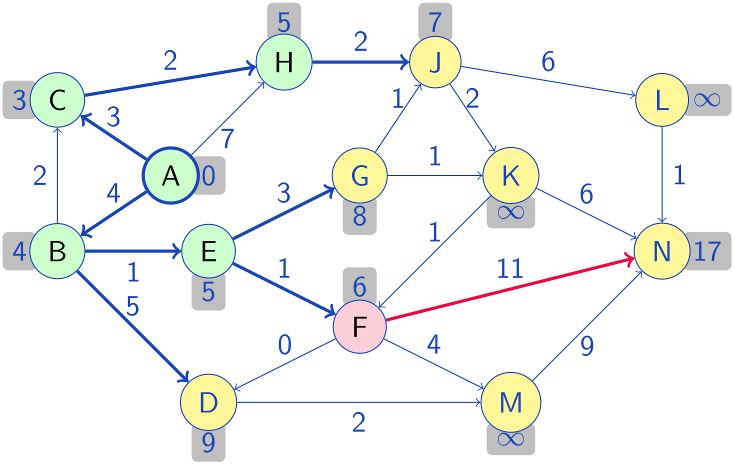
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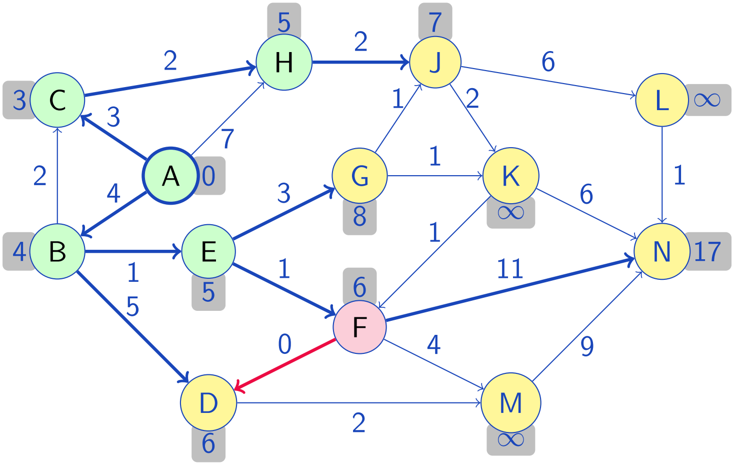
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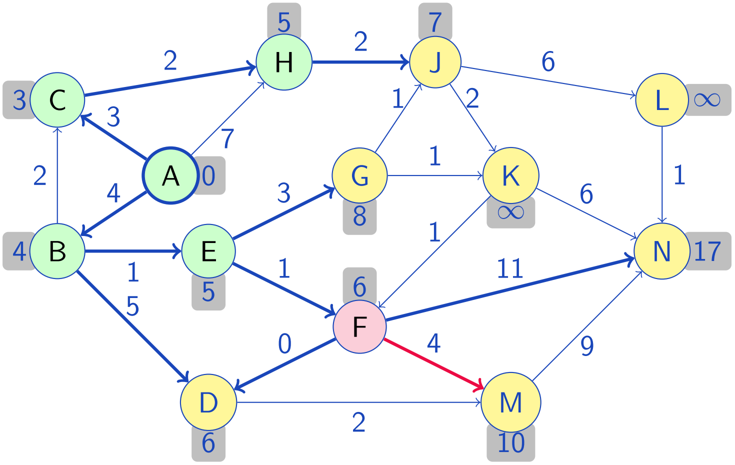
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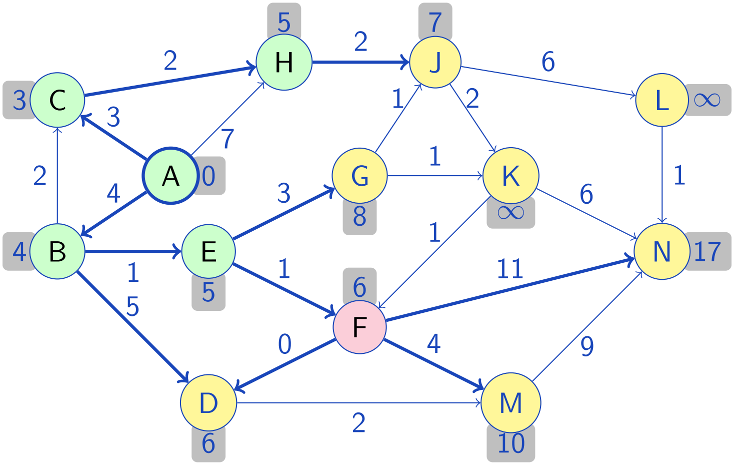
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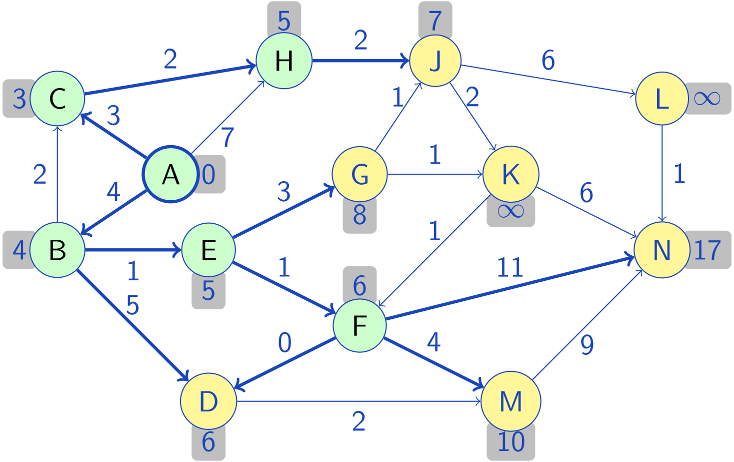
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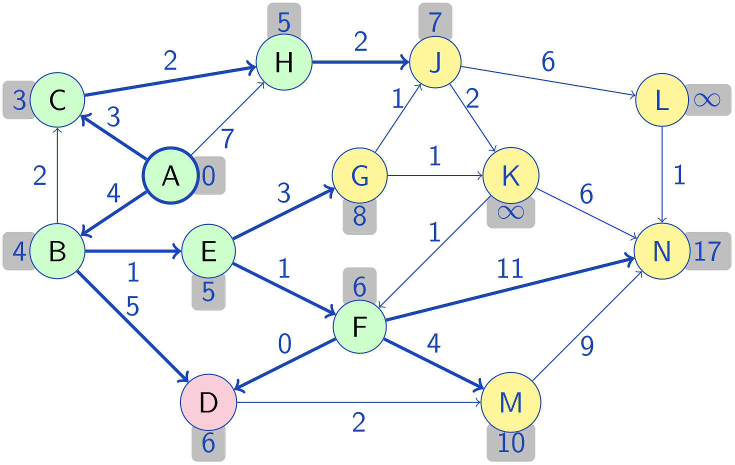
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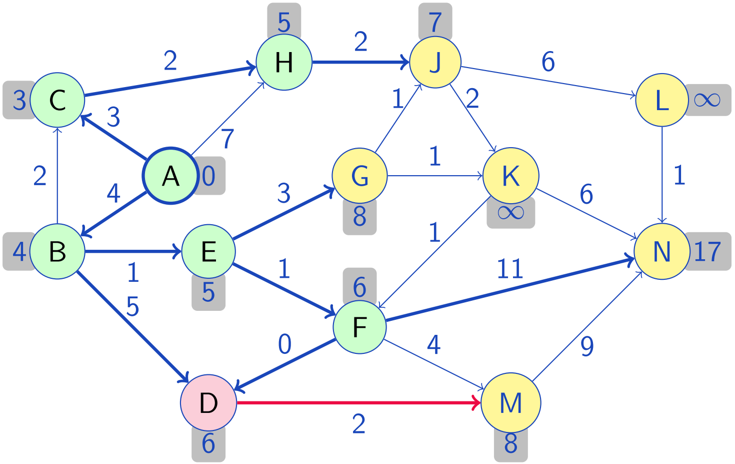
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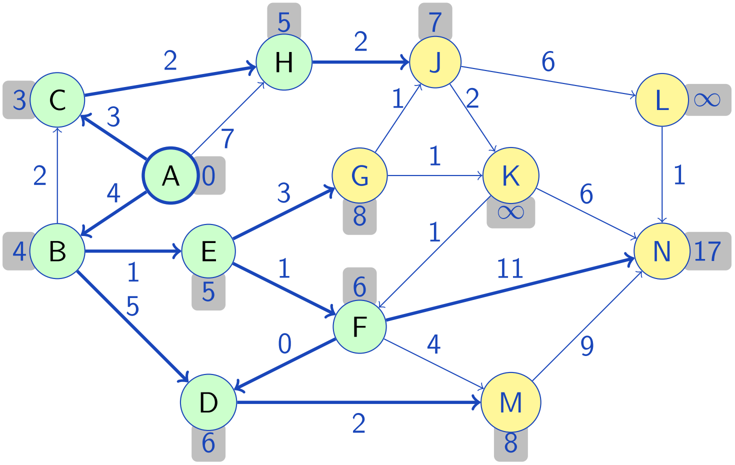
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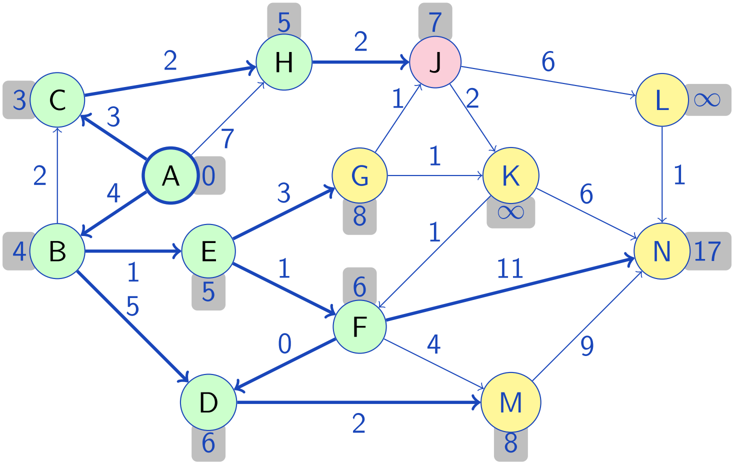
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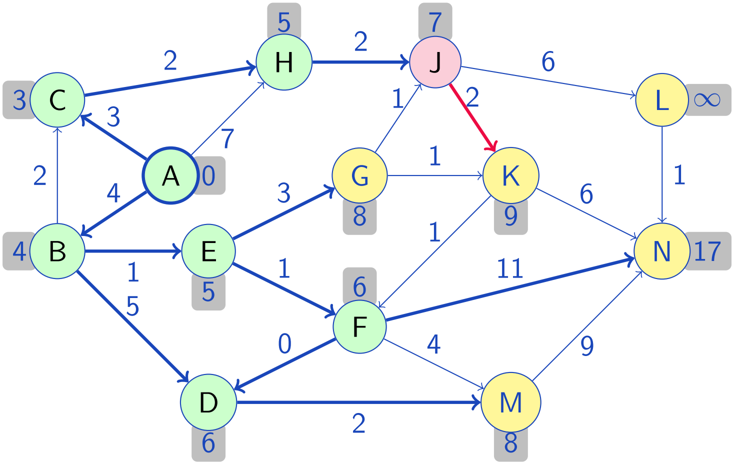
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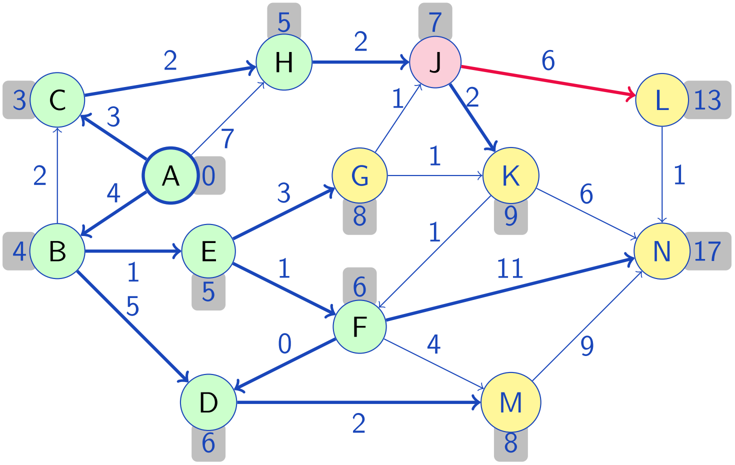
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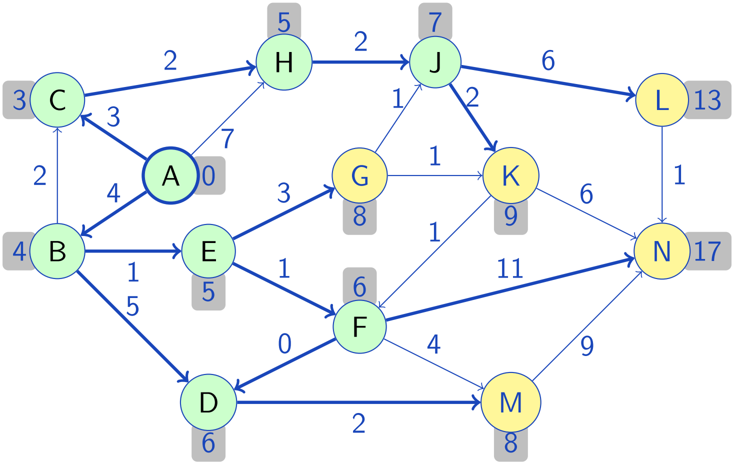
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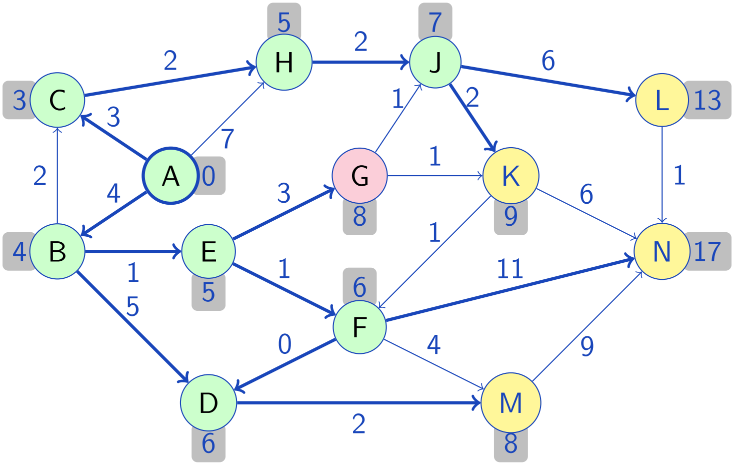
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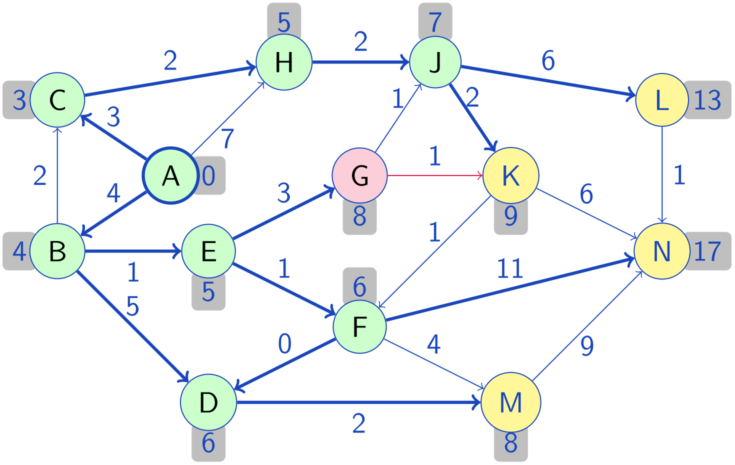
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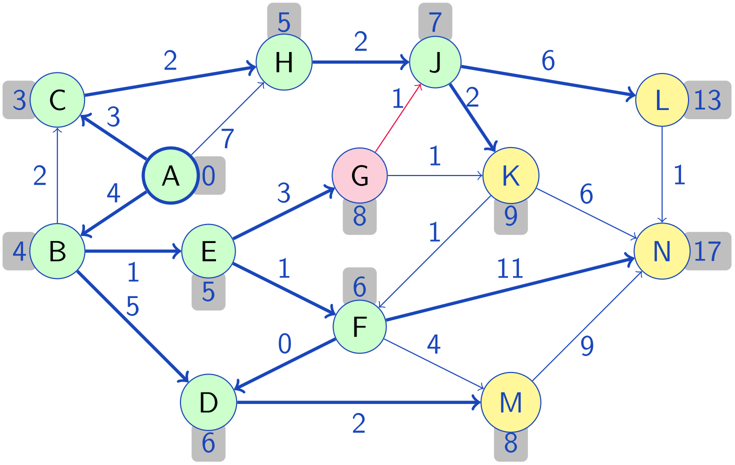

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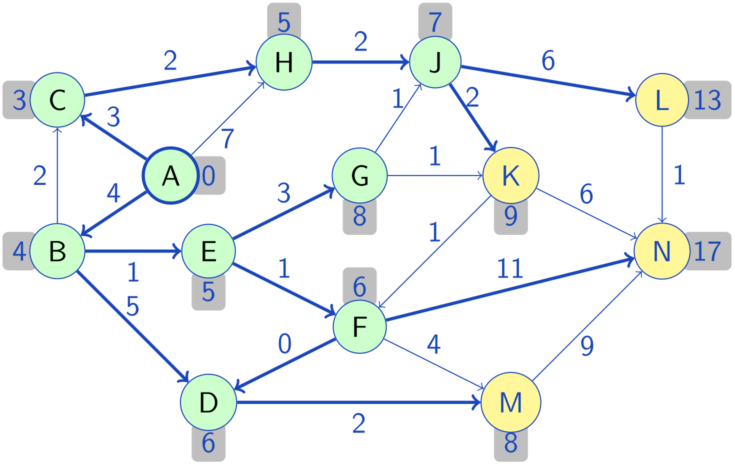
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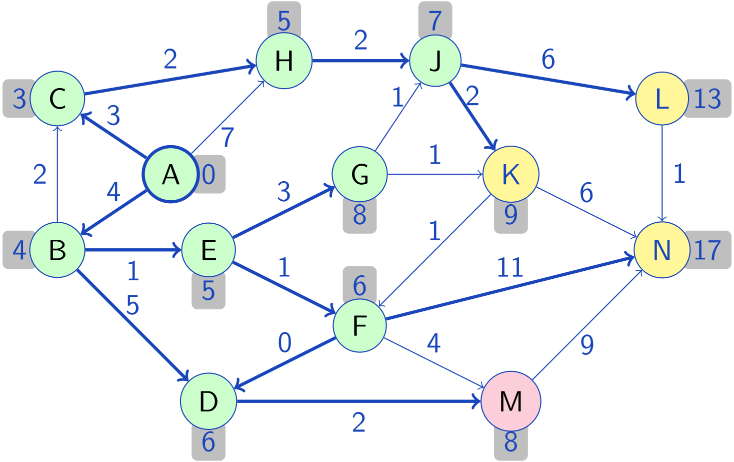
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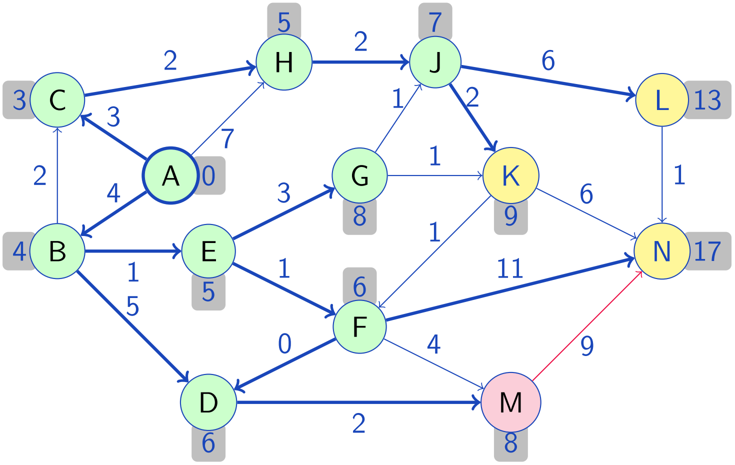
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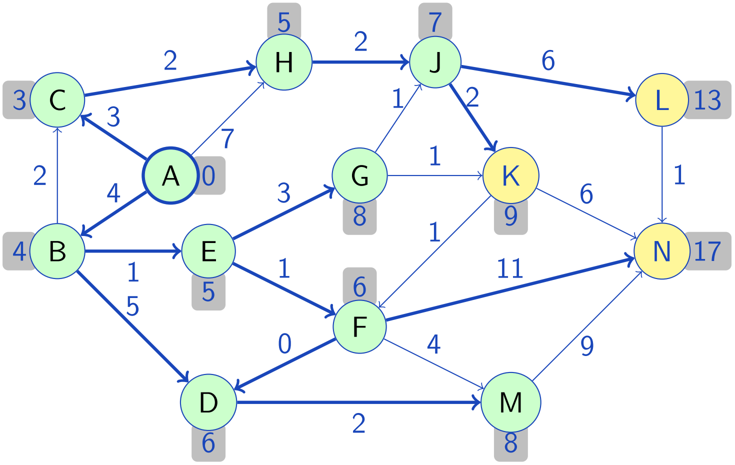
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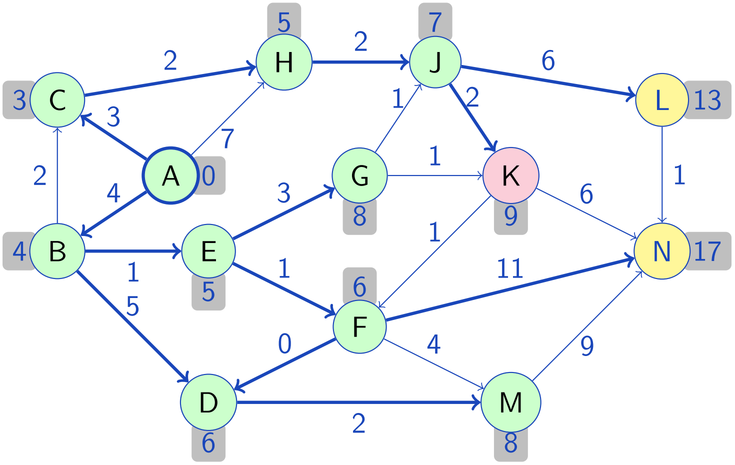
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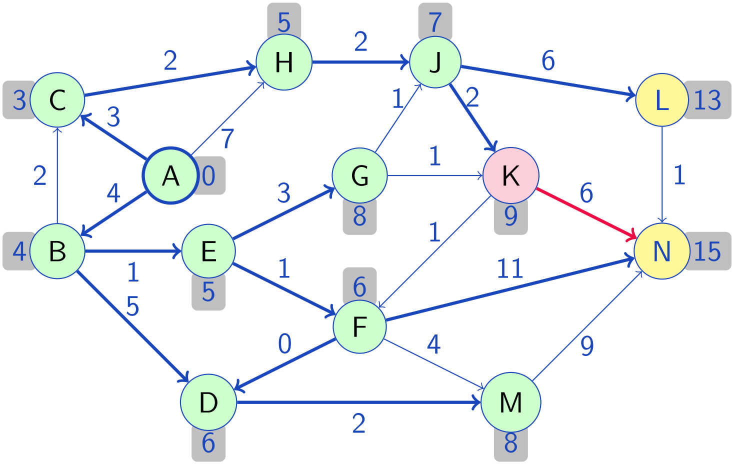
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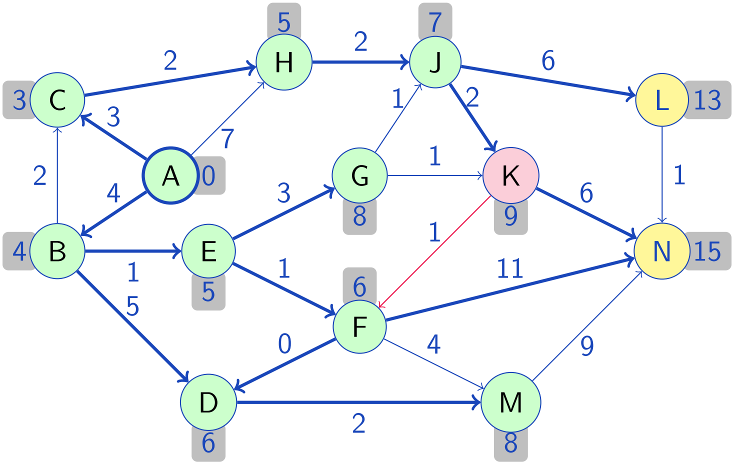
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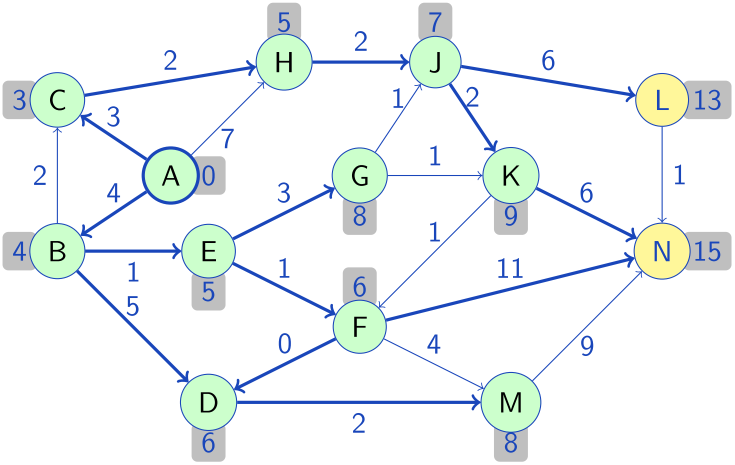
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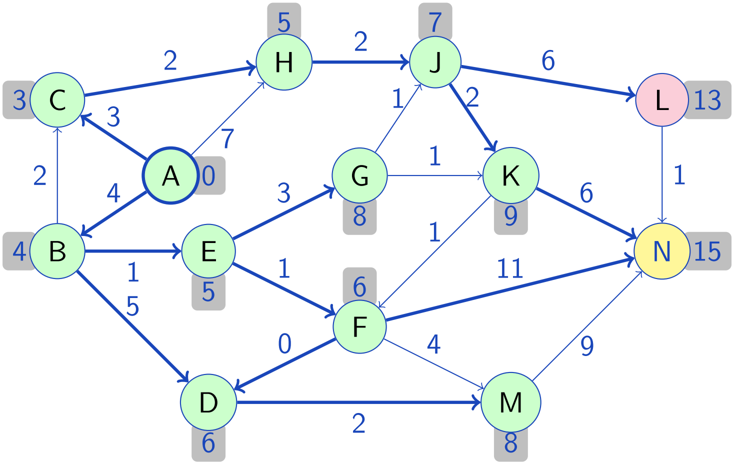
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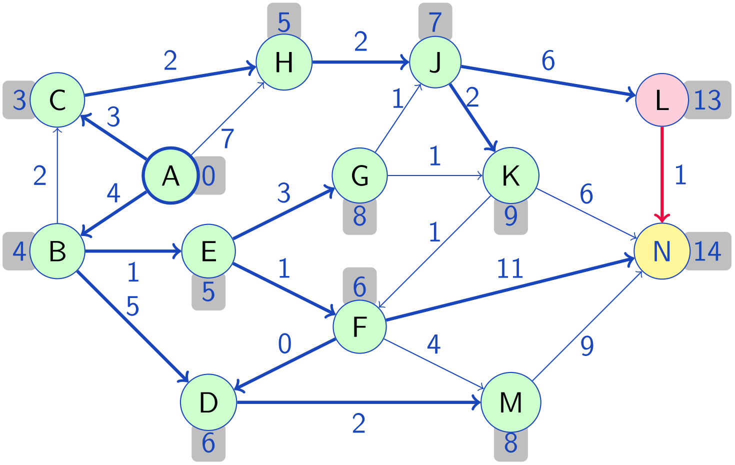
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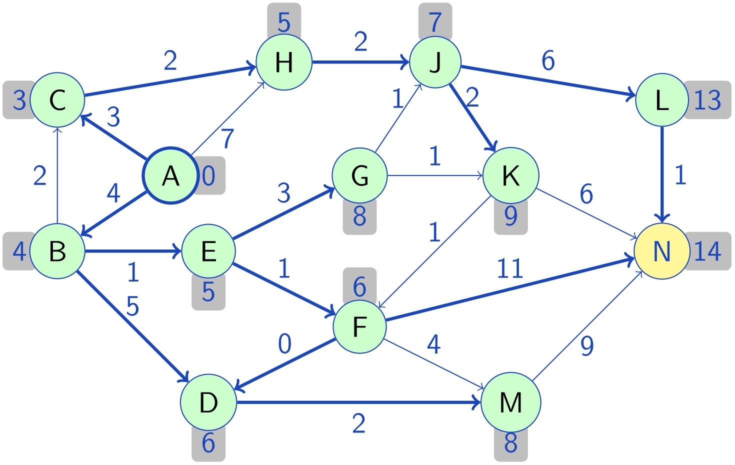
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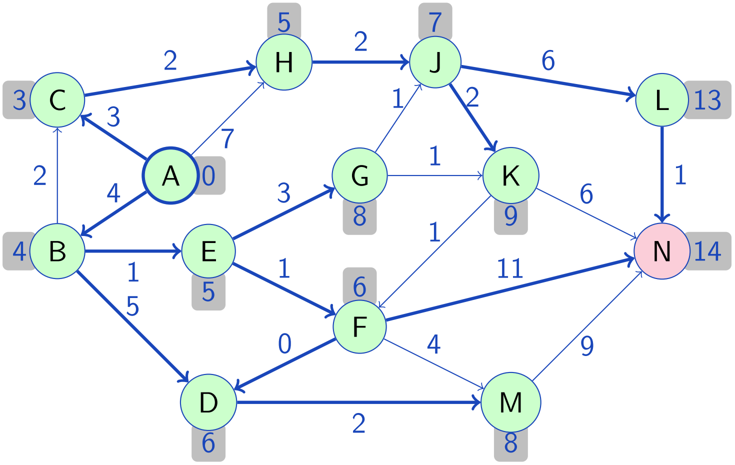
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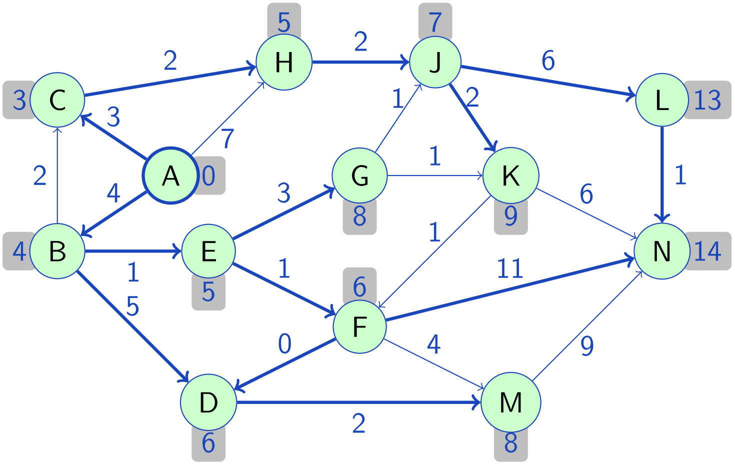
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- Recall: How to update S
  - Choose $v \notin S$ with the smallest $D'[v]$
  - Set $D[v] = D'[v]$
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  3. Using Fibonacci heap: "decrease key" in amortized $O(1)$ time
     - Total running time: $O(|V| \log |V| + |E|)$. However, impractical :(

Contest trick: How to implement Dijkstra with heap using standard libraries?

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