ITMO UNIVERSITY

How to Win Coding Competitions: Secrets of Champions

Week 6: Algorithms on Graphs 2
Lecture 2: Hamiltonian paths and Hamiltonian tours

Maxim Buzdalov
Saint Petersburg 2016

A Hamiltonian path is a path in a graph that contains each **vertex** of the graph exactly once

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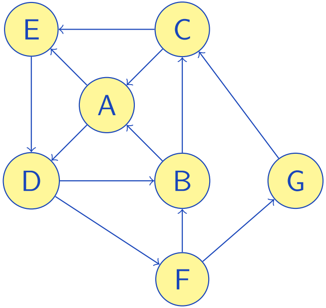

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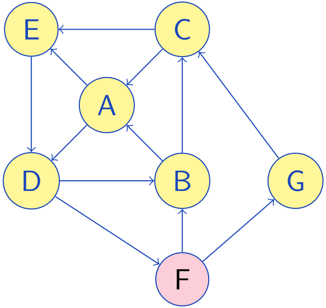

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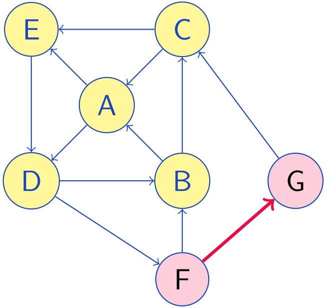

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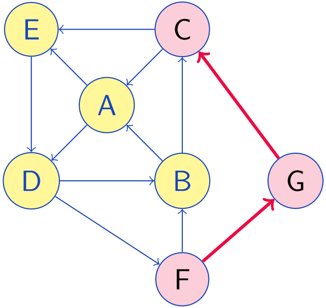

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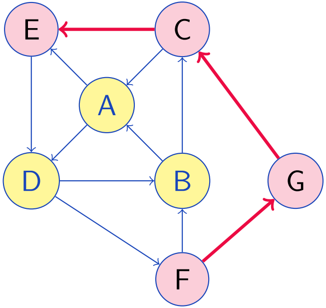

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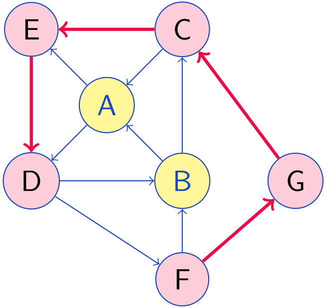

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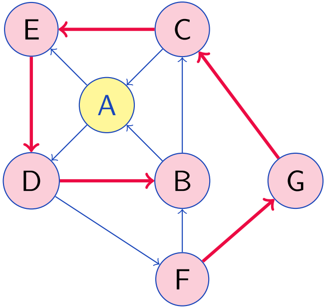

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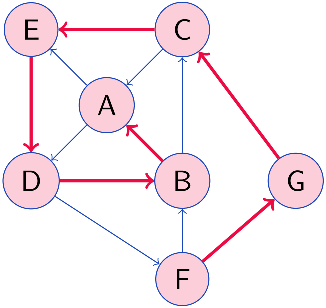

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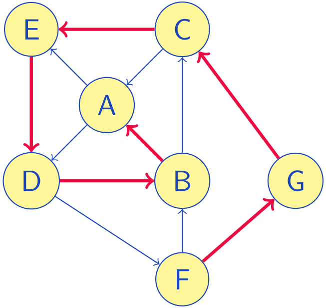

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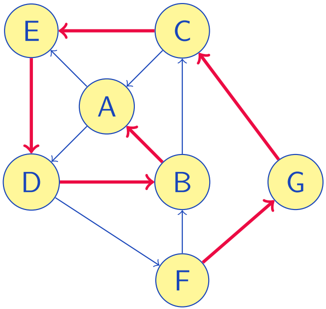

A Hamiltonian tour is a Hamiltonian path which starts and ends on the same vertex

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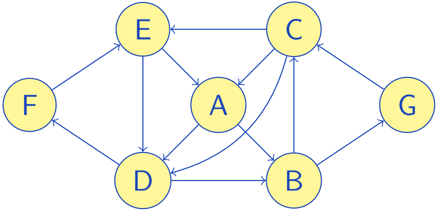

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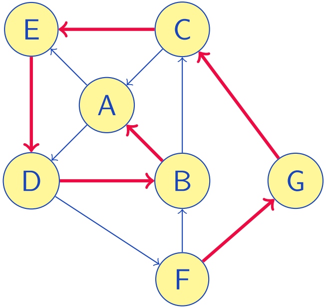

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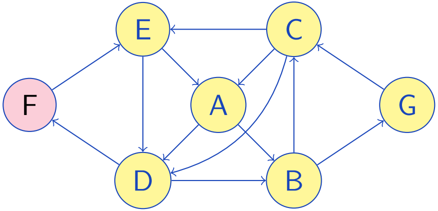

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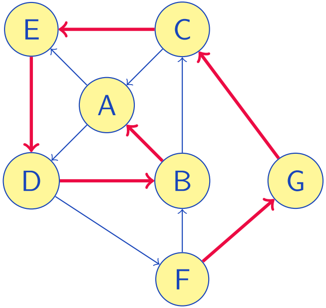

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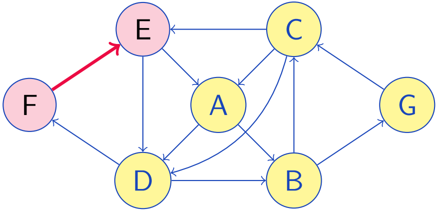

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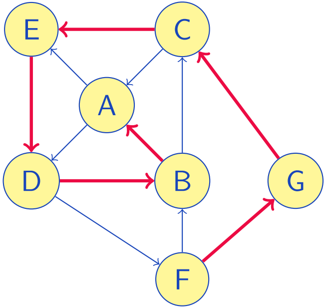

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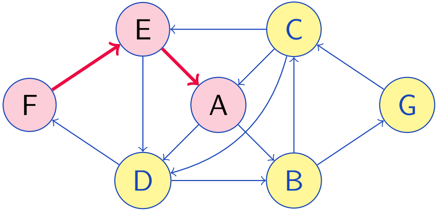

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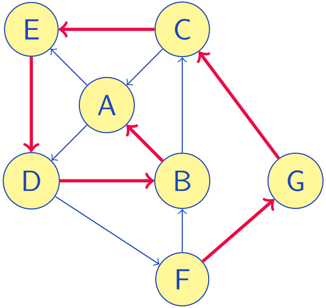

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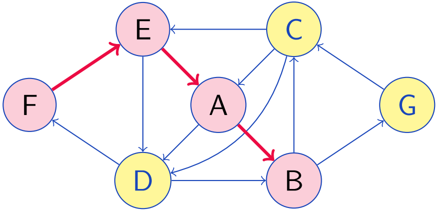

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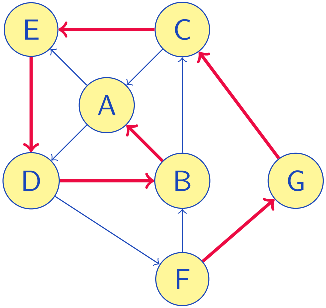

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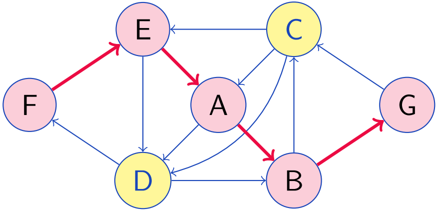

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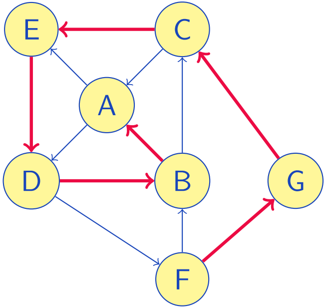

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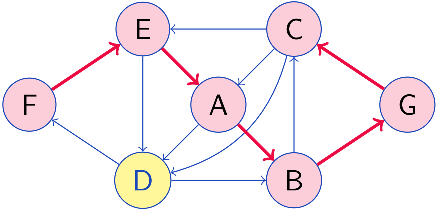

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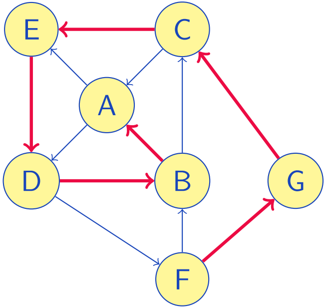

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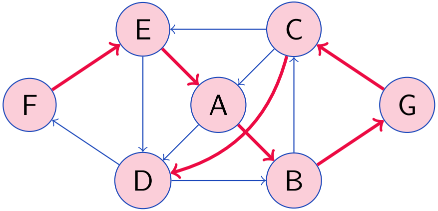

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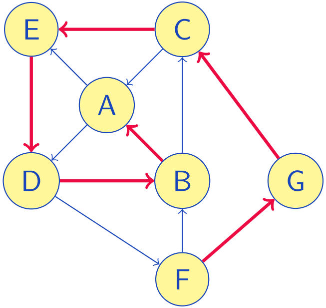

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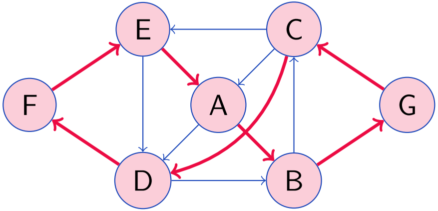

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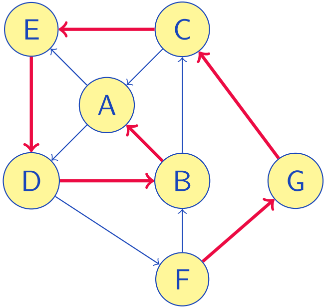

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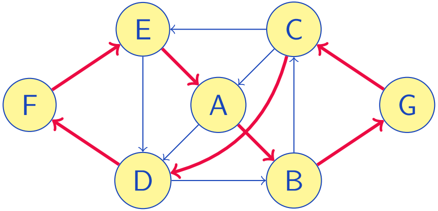

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- Vertex sets are stored as bitmasks
  - Numbers from 0 to $2^N - 1$
  - i-th bit is set if vertex number i is in the set
- We can solve Hamiltonian-related problems using the values of $d[S][v]$

procedure $\textsc{HamiltonianDP}(V, E)$
    $d[S][v]$: if a path exists which starts at 1, ends at v and visits vertices from S
    $d[\{1\}][1] \leftarrow \textsc{true}$
    for $S \in 2^V$ in non-decreasing order of $|S|$ where $|S| \geq 2$ **do**
        for $v \in S \setminus \{1\}$ **do**
            $d[S][v] \leftarrow \textsc{false}$
            $S' \leftarrow S \setminus \{v\}$
            for $u \in S'$ **do**
                if $(u, v) \in E$ **then** $d[S][v] \leftarrow d[S][v]$ **or** $d[S'][u]$ **end if**
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procedure HamiltonianDP(V, E)
    d[S][v]: if a path exists which starts at 1, ends at v and visits vertices from S
    d[{1}][1] ← TRUE                        ▷ A path consisting of vertex 1 exists
    for S ∈ 2^V in non-decreasing order of |S| where |S| ≥ 2 do
        for v ∈ S \ {1} do
            d[S][v] ← FALSE
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        for v ∈ S \ {1} do                                   ▷ Check all possible endpoints
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boolean[][] hamiltonianDP(boolean[][] graph) {
    int n = graph.length;
    boolean[][] d = new boolean[(1 << (n - 1))][n];          // save one bit, reduce memory 2x times
    d[0][0] = 1;                                               // count vertices from 0
    for (int mask = 1; mask < d.length; ++mask) {             // locally ordered by size
        for (int v = 1; v < n; ++v) {
            if ((mask & (1 << (v - 1))) != 0) {               // mask contains v
                int prev = mask ^ (1 << (v - 1));             // previous mask
                boolean curr = d[prev][0] && graph[v][0];     // consider 0 separately
                for (int u = 1; u < n; ++u) {                 // check previous vertices
                    if (graph[v][u]) {                         // if graph has the (v,u) edge ...
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An efficient implementation

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            }
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    return d;
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```
int[] hamiltonianDP(int[] graph) {                    // running time looks more like 'O(2^n * n)'
    int n = graph.length;                             //           and memory more like 'O(2^n)'
    int[] d = new int[(1 << (n - 1))];
    d[0] = 1;                                         // count vertices from 0
    for (int mask = 1; mask < d.length; ++mask) {     // locally ordered by size
        for (int v = 1; v < n; ++v) {
            if ((mask & (1 << (v - 1))) != 0) {       // mask contains v
                int prev = mask ^ (1 << (v - 1));     // previous mask
                if ((d[prev] & graph[v]) != 0) {      // if u exists with path 1-u and with edge (u,v)
                    d[mask] |= 1 << v;                // saying the path 1-v also exists
                }
            }
        }
    }
    return d;
}
```

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- Does a Hamiltonian path between a and b exist?
  - Evaluate $d[S][v]$ for all S and v
  - If there exists $S' \subseteq 2^{V \setminus \{1,a,b\}}$, such that:
    - $d[S' \cup \{1, a\}][a] = \text{TRUE}$
    - $d[V \setminus S' \setminus \{a\}][b] = \text{TRUE}$

    then a Hamiltonian path between a and b exists, otherwise not
  - Simple special cases if $a = 1$ or $b = 1$
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- Count Hamiltonian paths/tours
  - $d[S][v]$ stores the number of paths from 1 to v using vertices from S
- Shortest Hamiltonian path/tour (Traveling Salesperson Problem)
  - $d[S][v]$ stores the shortest length of a path from 1 to v using vertices from S

Special case: Every tournament has a Hamiltonian path.

Special case: Every tournament has a Hamiltonian path. Proof:

- Start building this path from an arbitary vertex, say, v_1
- Assume a path $v_1 \dots v_k$ is built. Add a new vertex v:
  - If there is an edge $(v, v_1) \in E$, prepend v
  - Otherwise, if there is an edge $(v_k, v) \in E$, append v
  - Otherwise: find i such that $(v_i, v) \in E$ and $(v, v_{i+1}) \in E$ – it will exist because the graph is a tournament – then insert v between v_i and v_{i+1}