ITMO UNIVERSITY

How to Win Coding Competitions: Secrets of Champions

Week 3: Sorting and Search Algorithms
Lecture 2: Insertion sort

Maxim Buzdalov
Saint Petersburg 2016

Idea of the algorithm:

- A sequence of one element is sorted. Let's grow it!
- Increase the sorted part, step by step, until everything is sorted
  - Take the element adjacent to the sorted part
  - Push it backwards, step by step, while it is greater than the predecessor

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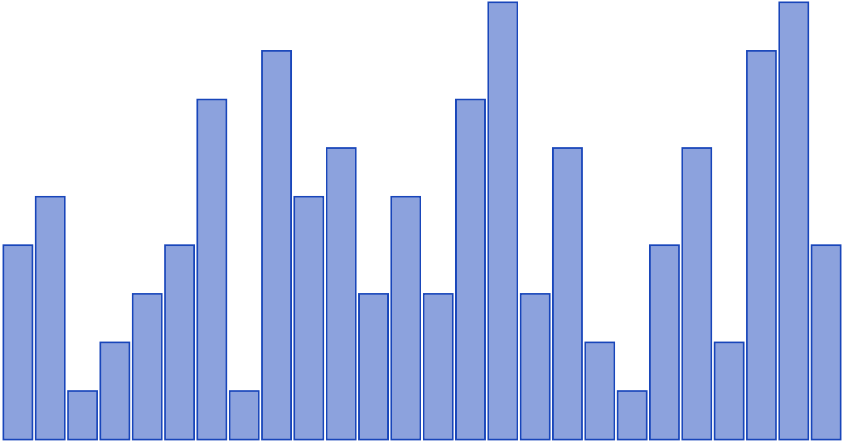
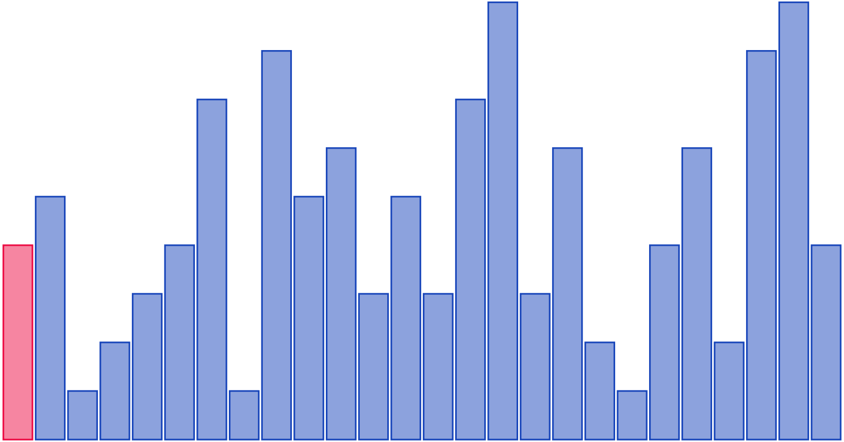
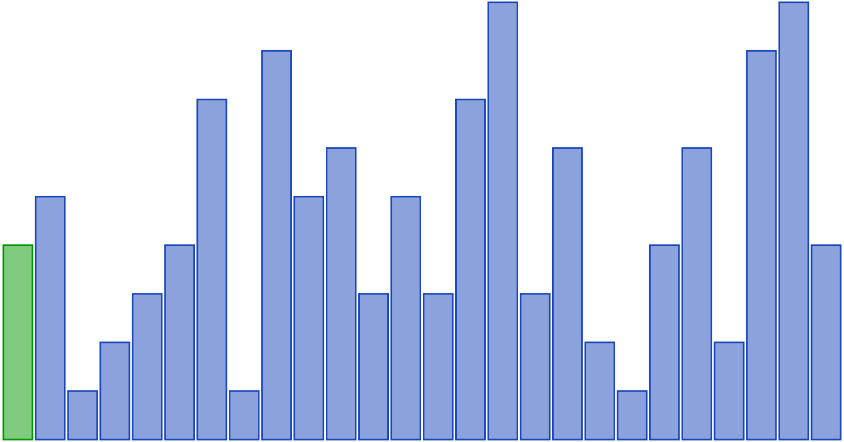
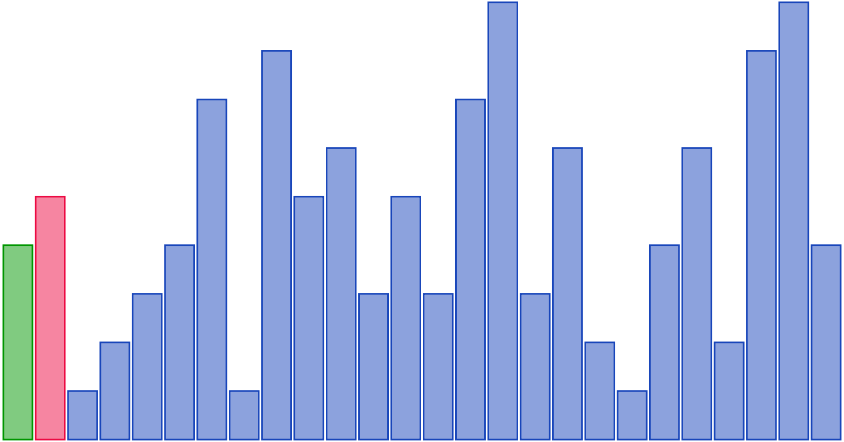
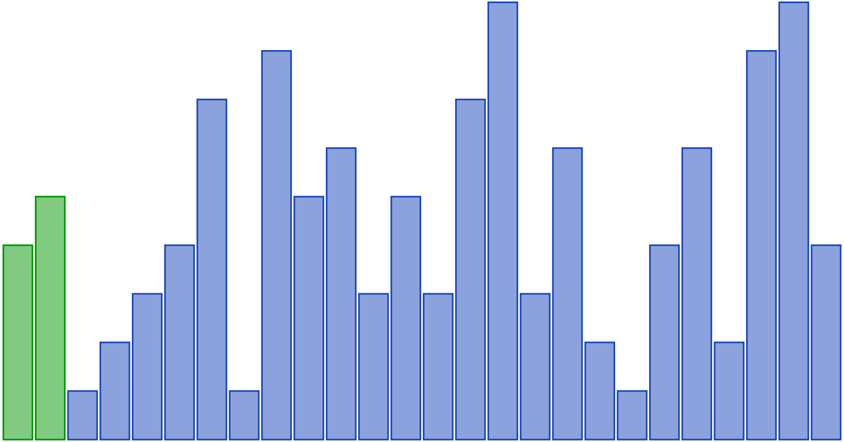
procedure $\textsc{InsertionSort}(A, \leq)$
  for i **from** 1 **to** $|A|$ **by** 1 **do**
    $k \leftarrow i$
    while $(k > 1)$ **and not** $(A[k-1] \leq A[k])$ **do**
      $A[k-1] \Leftrightarrow A[k]$
      $k \leftarrow k - 1$
    end while
  end for
end procedure

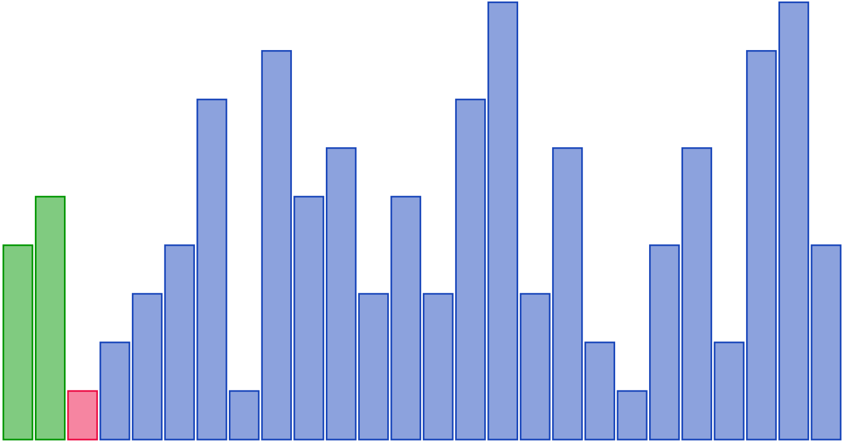


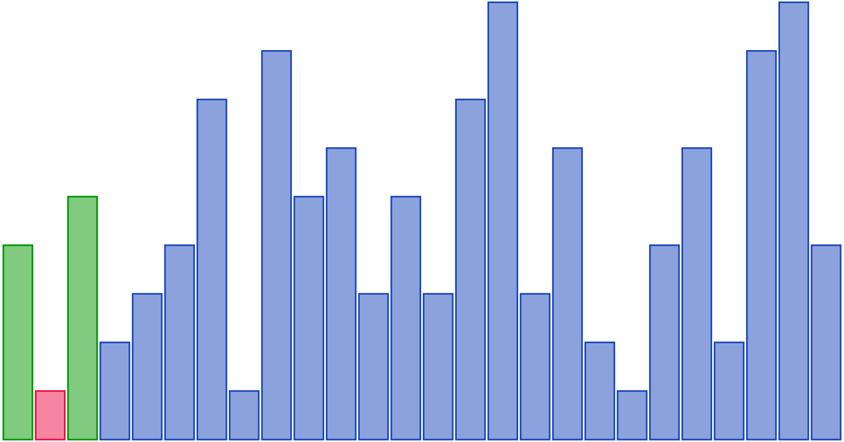


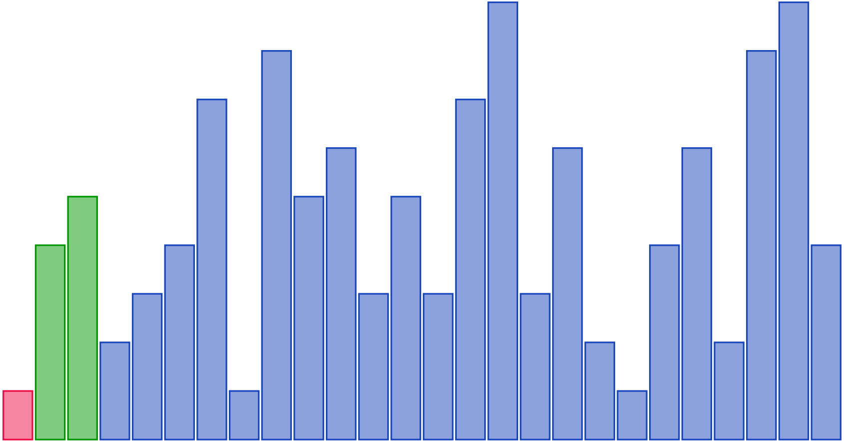


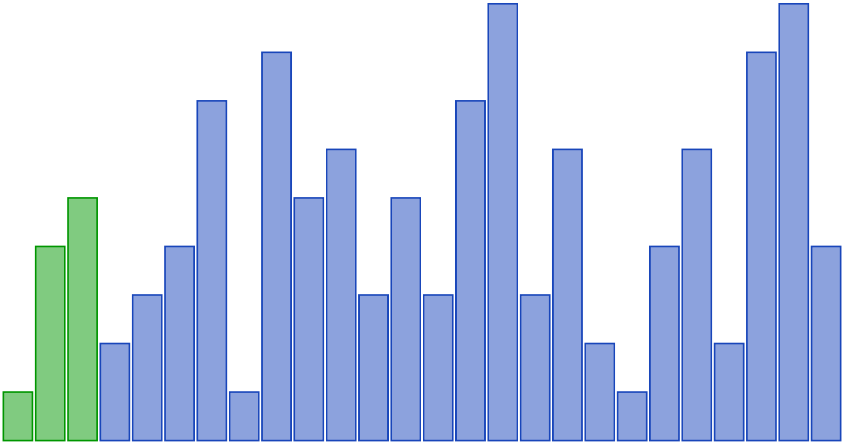


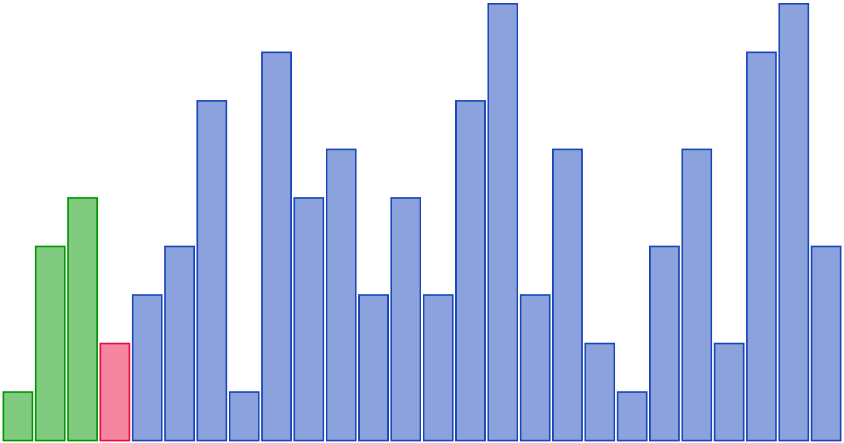


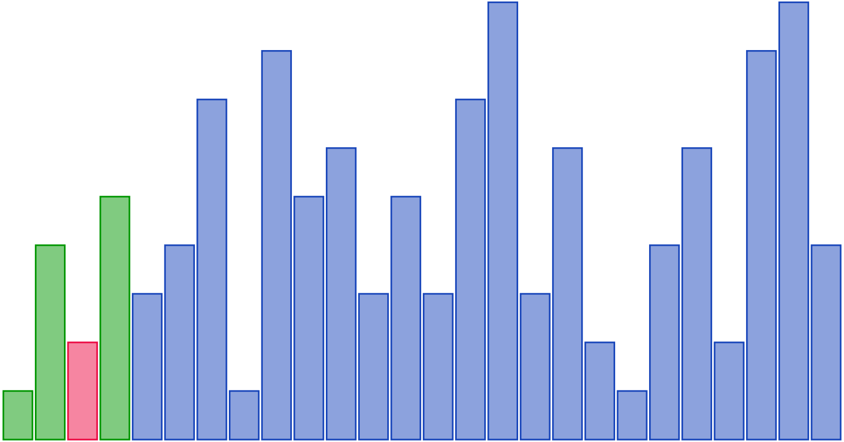


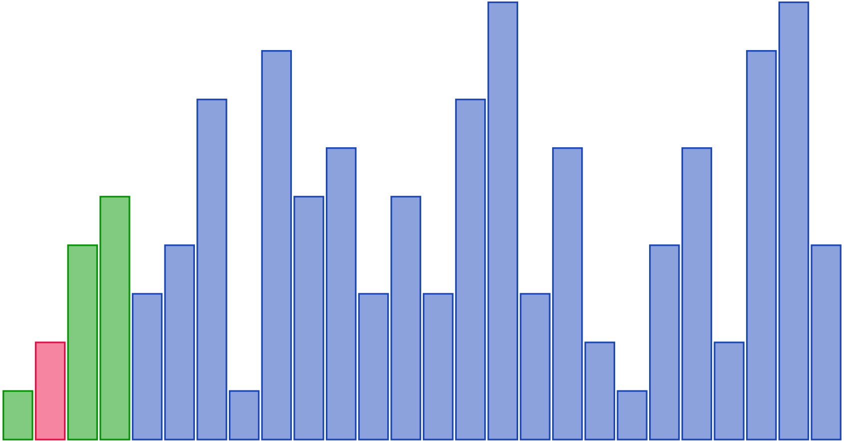


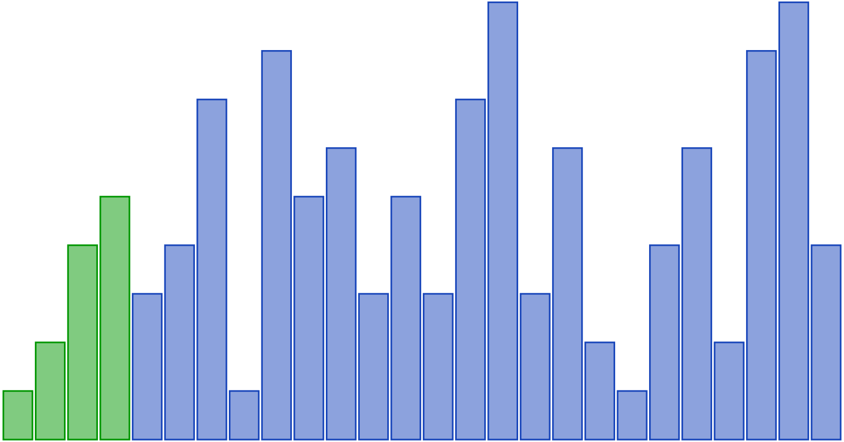


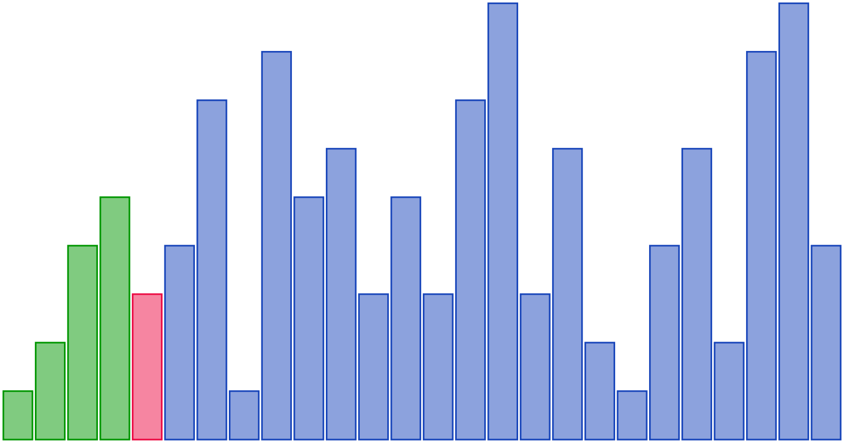


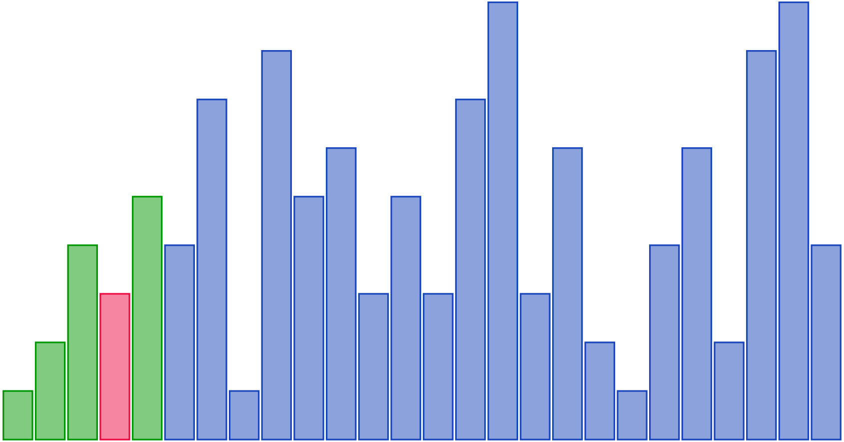


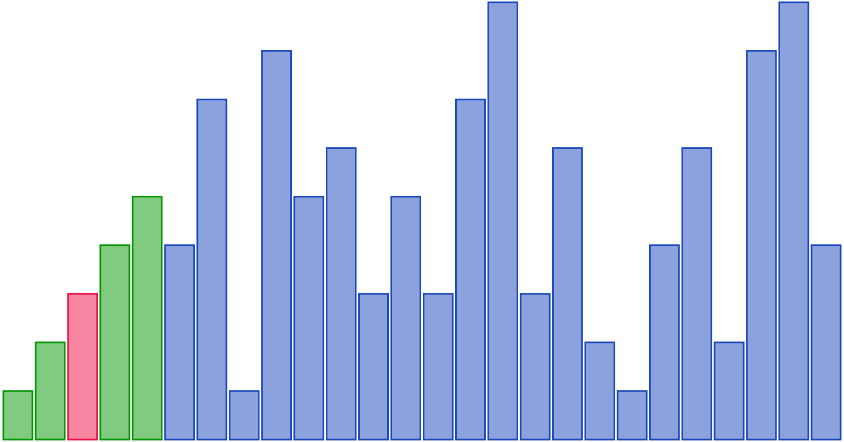


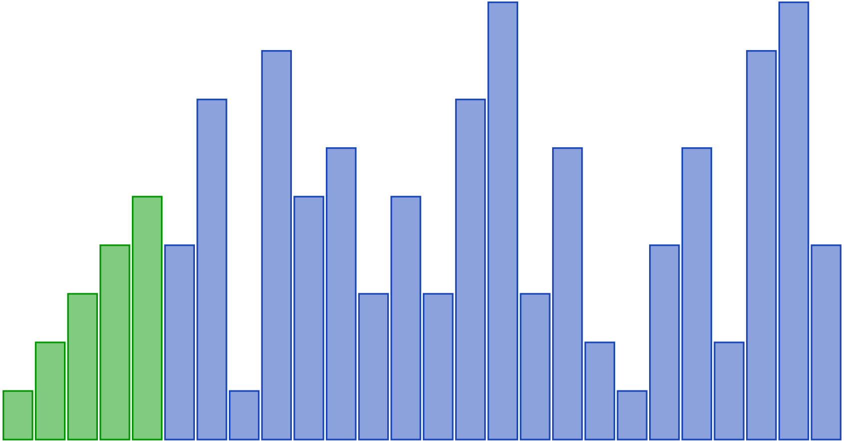


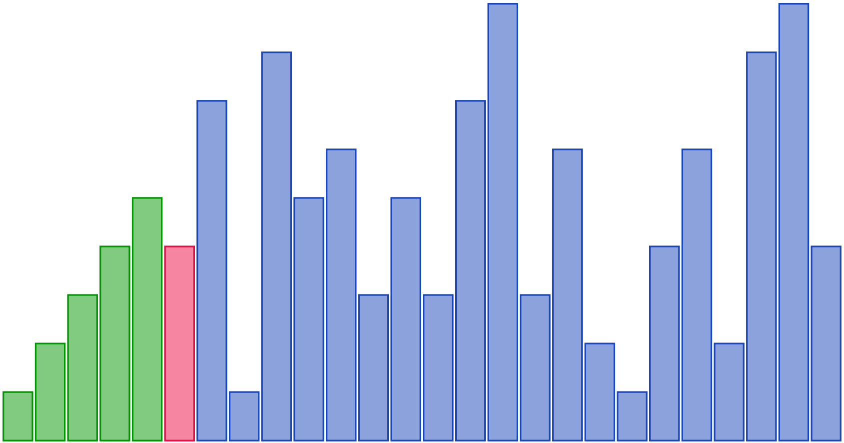


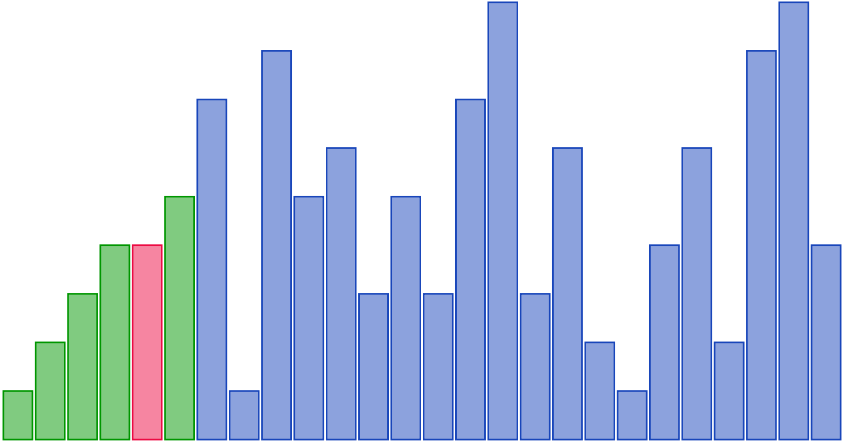


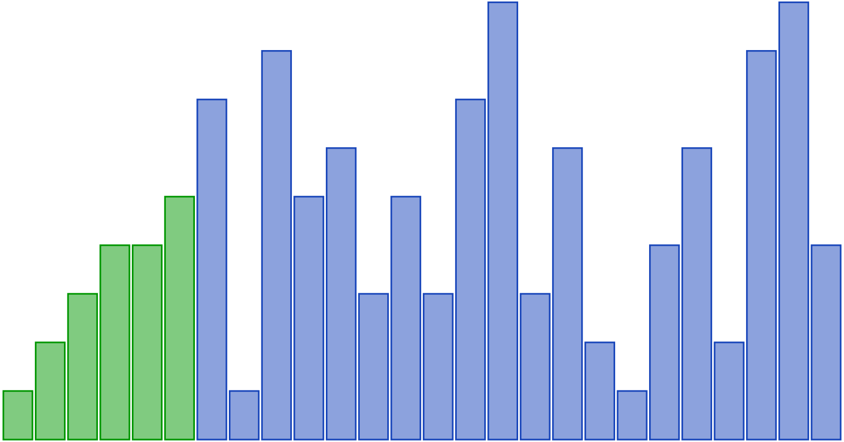


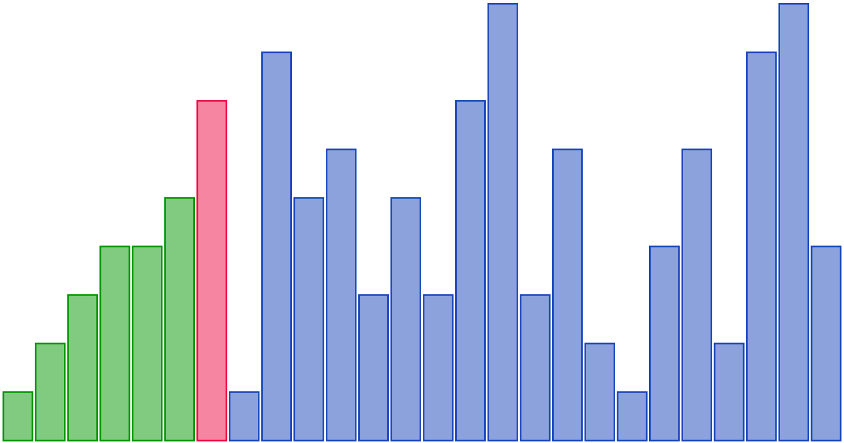


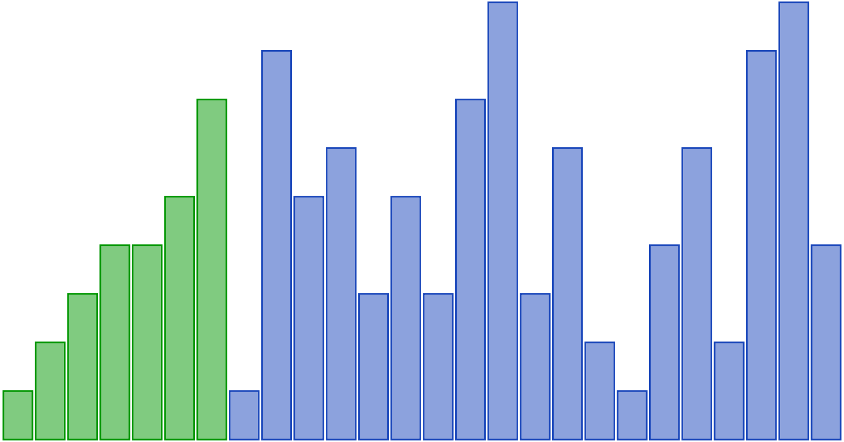


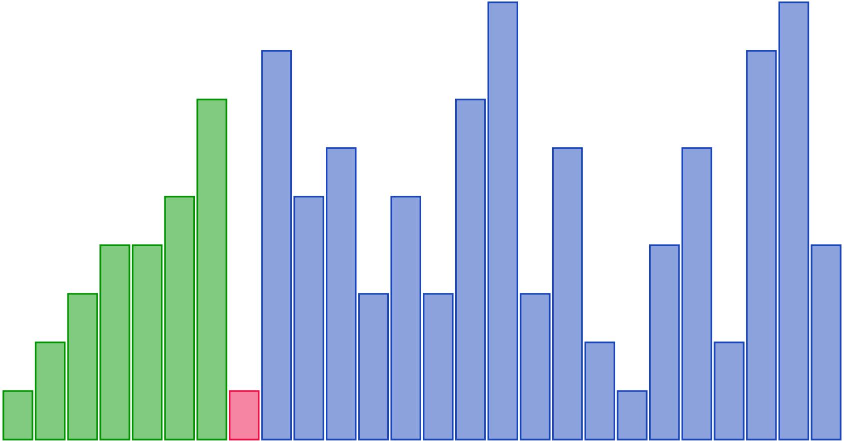


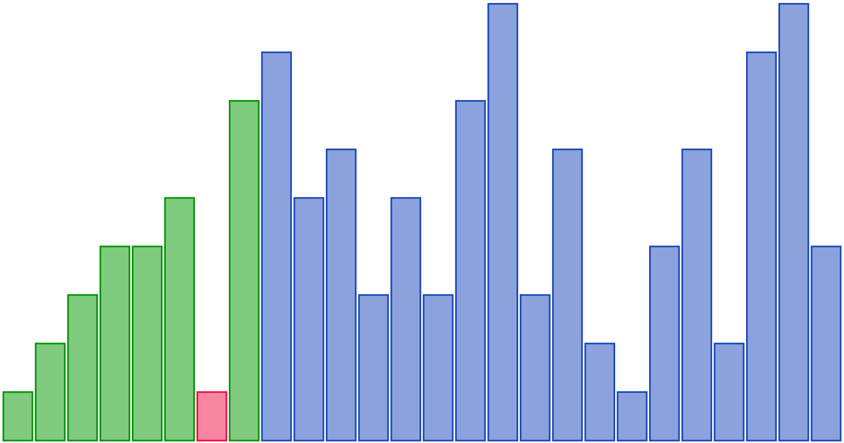


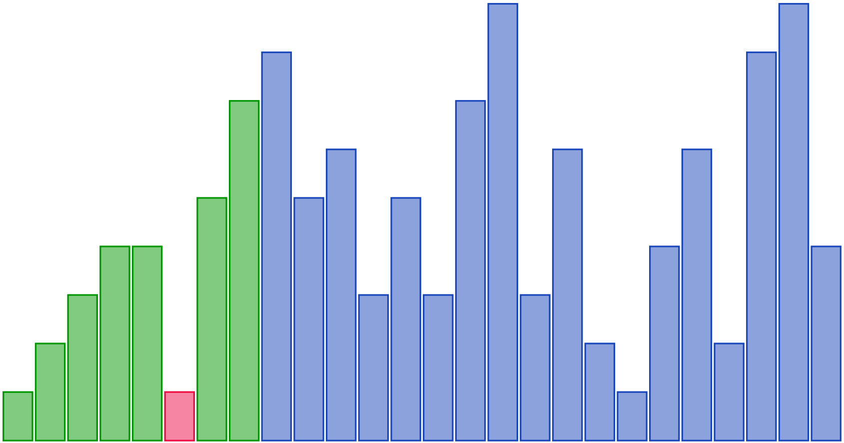


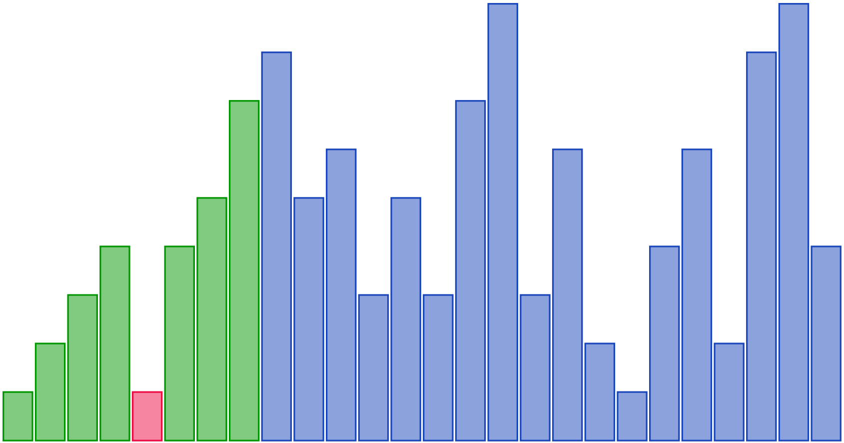


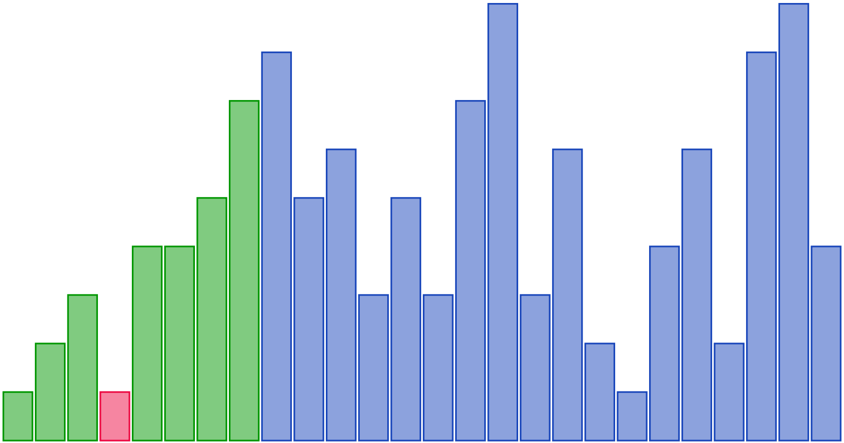


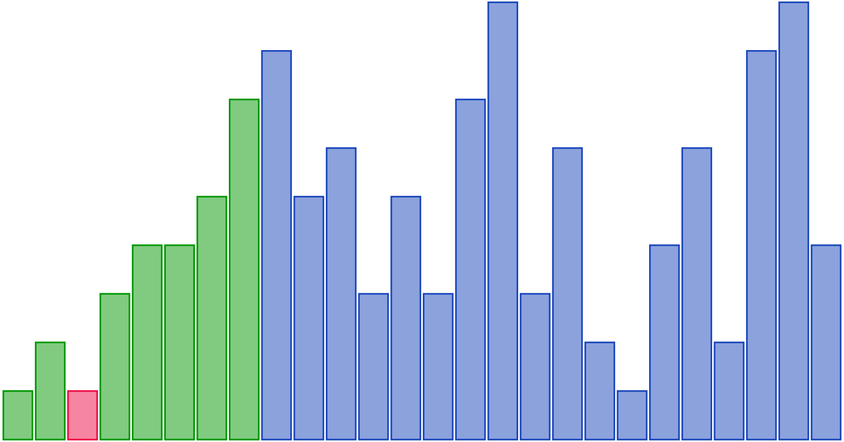


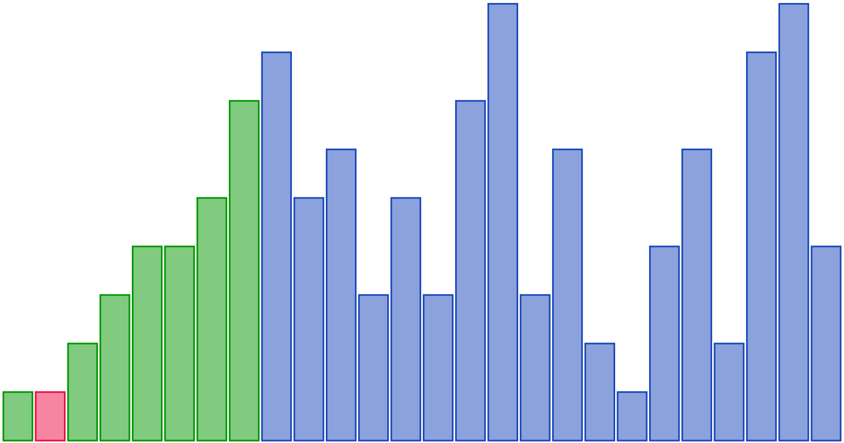


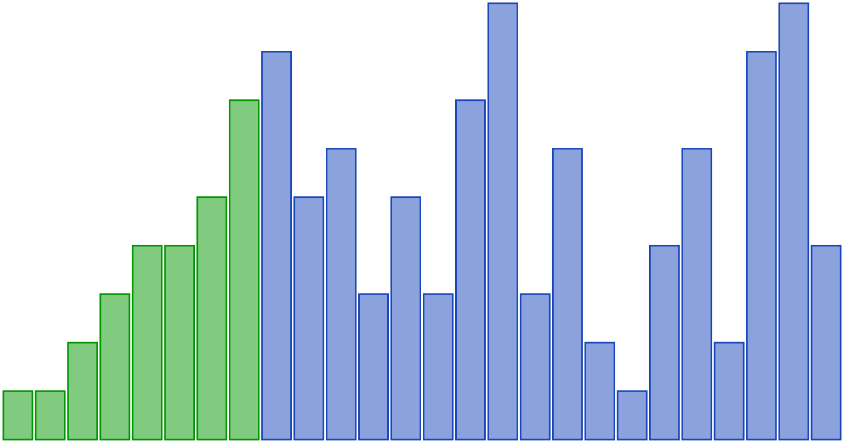


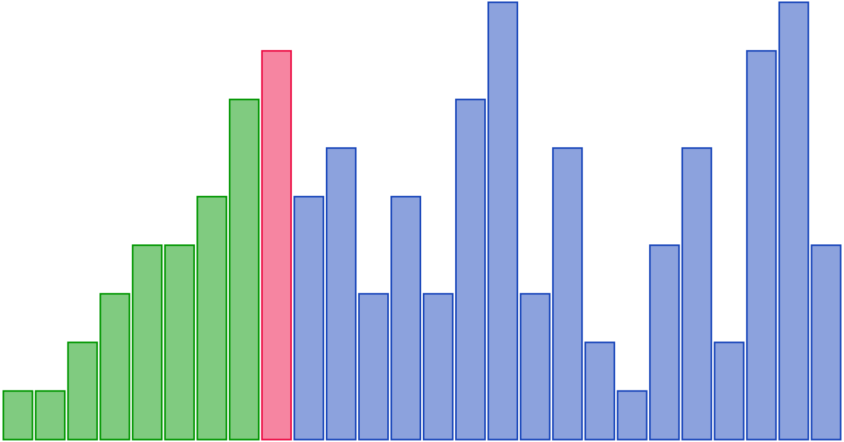


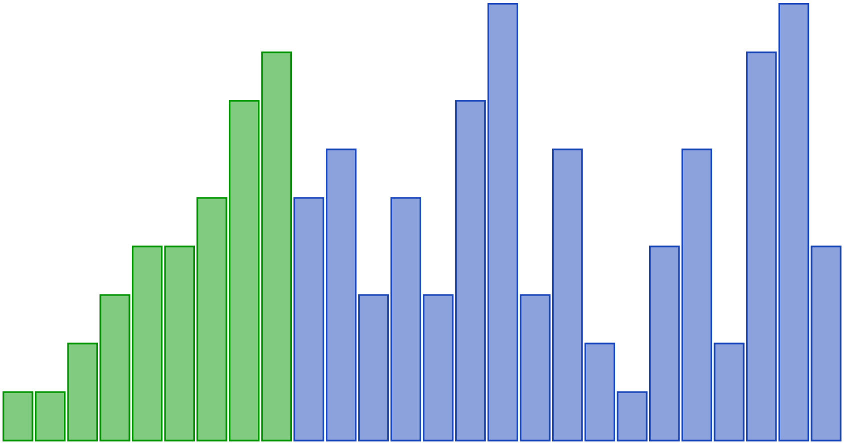


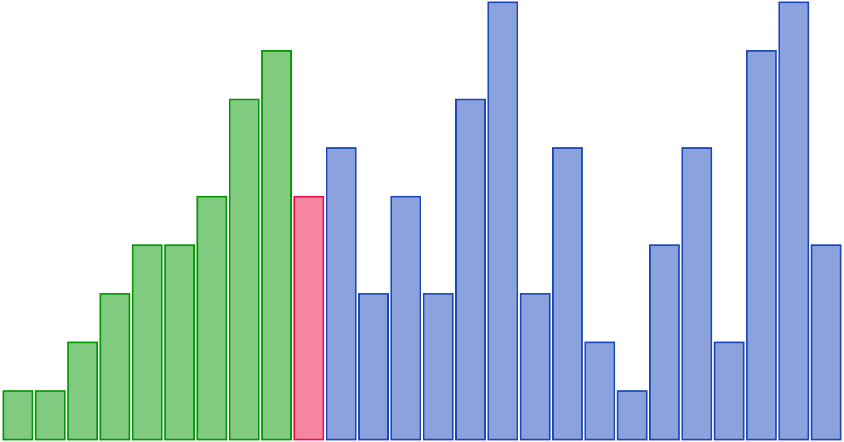


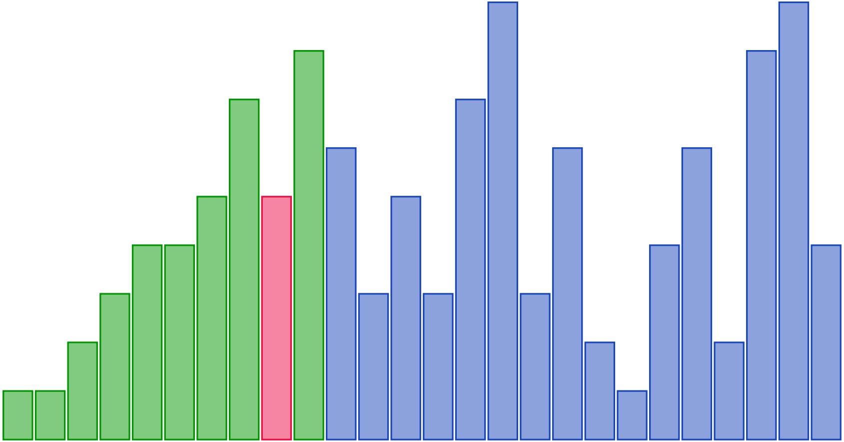


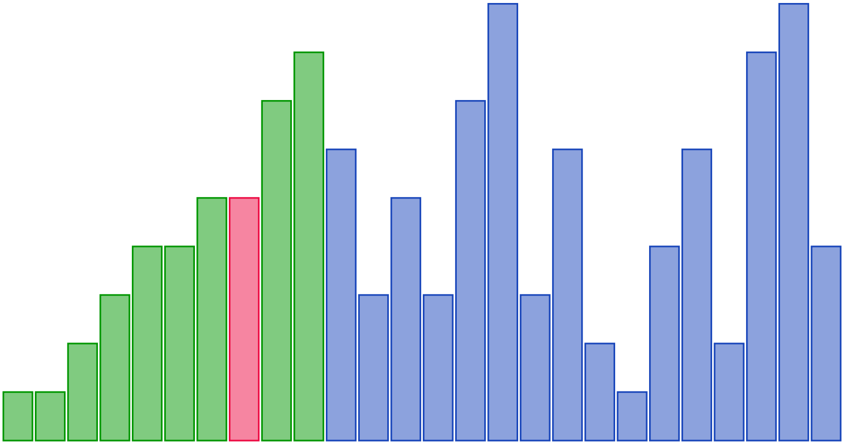


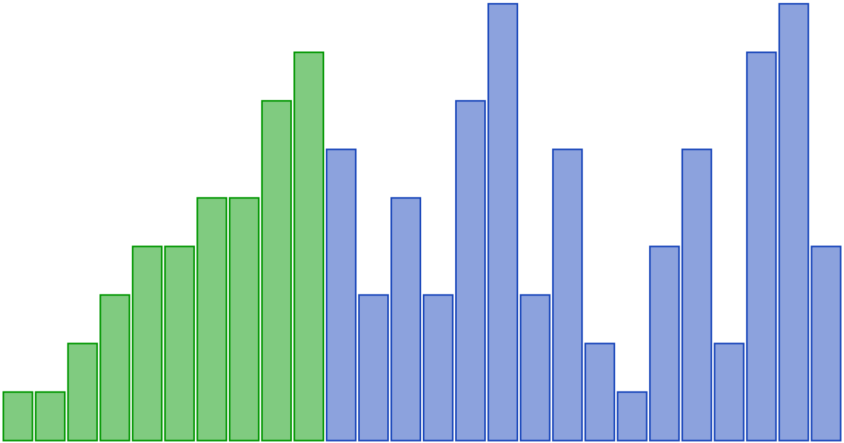


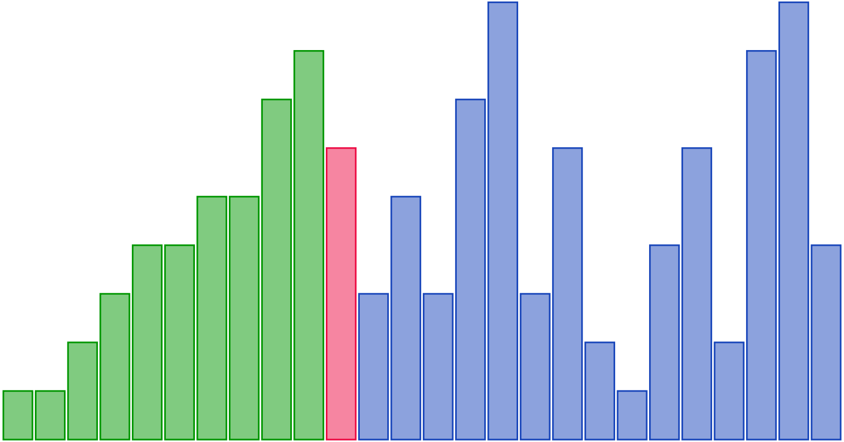


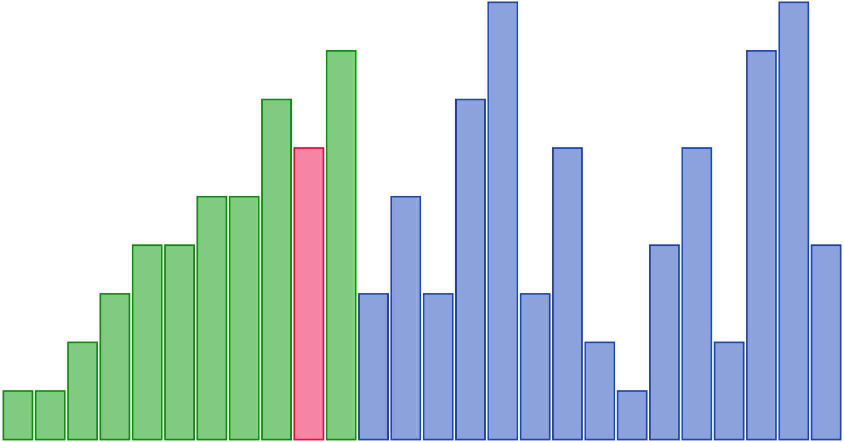


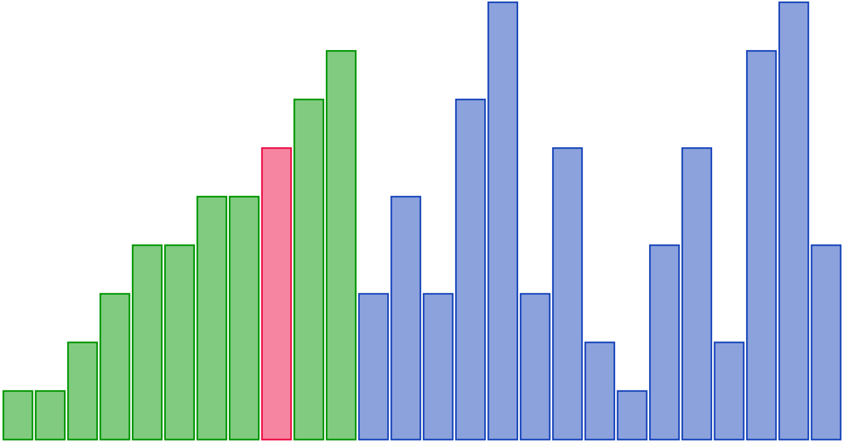


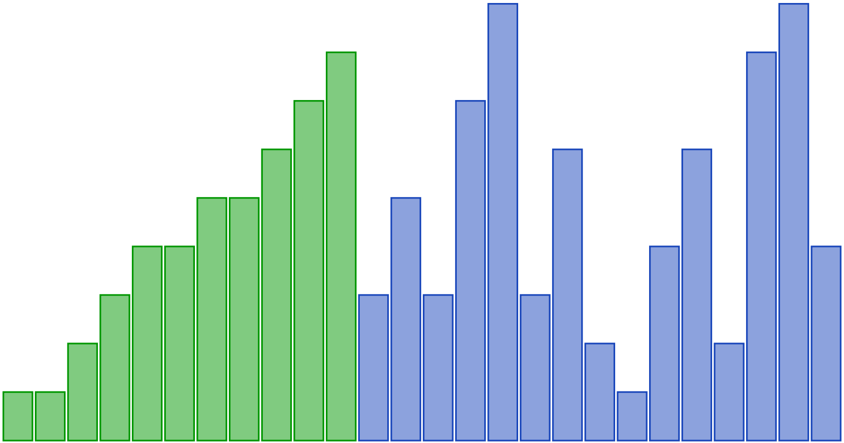


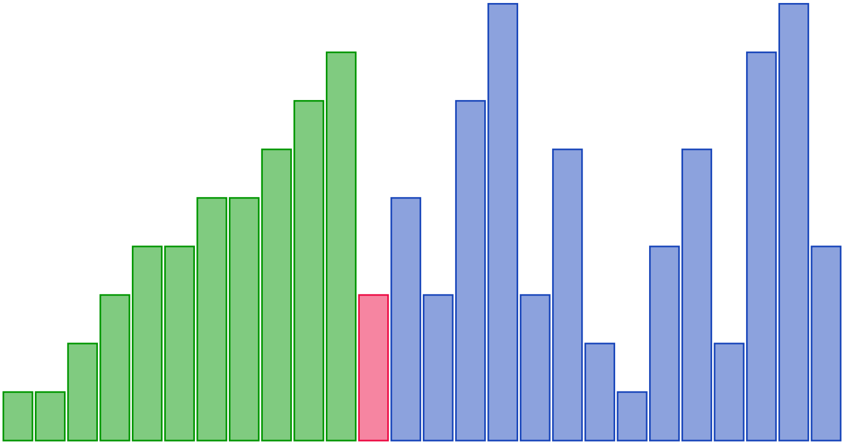


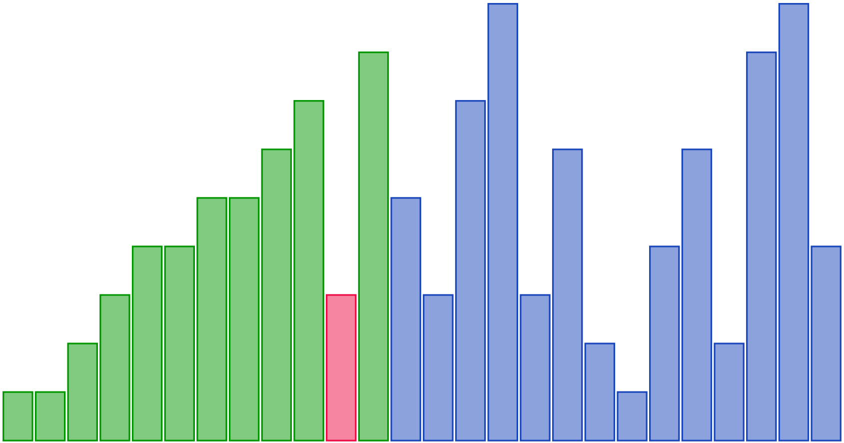


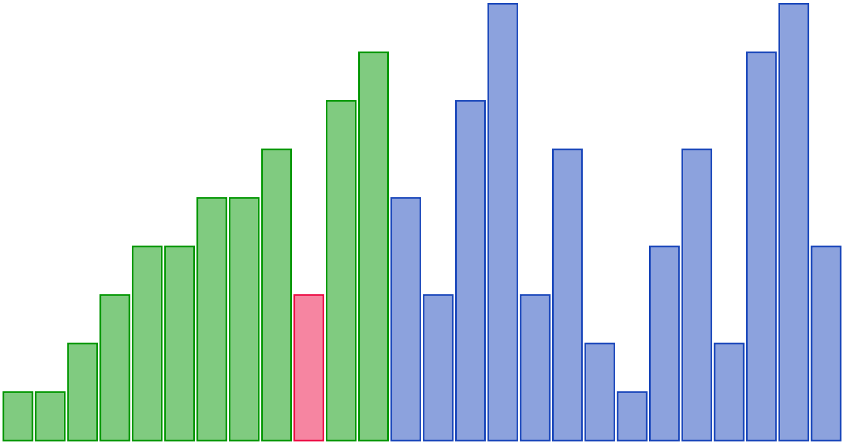


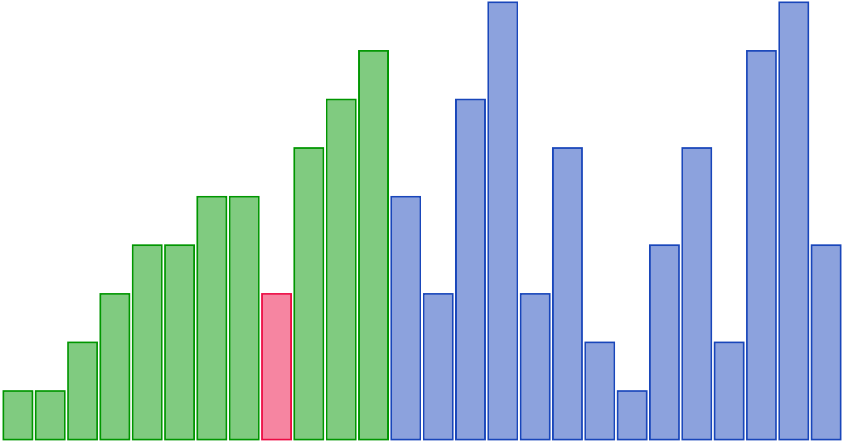


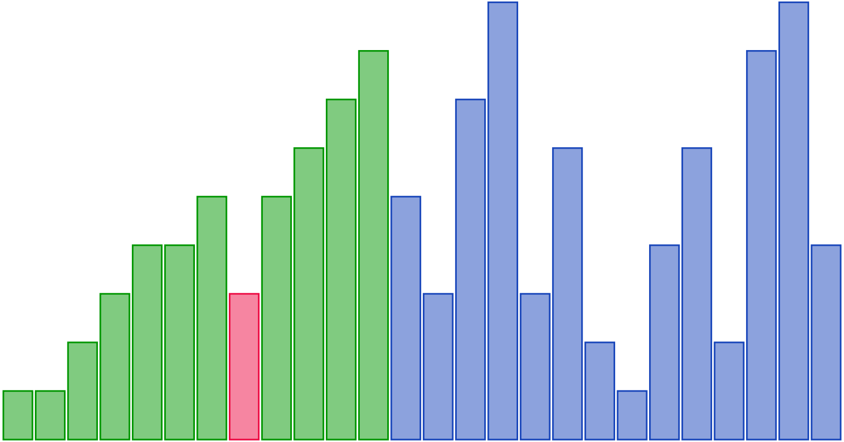


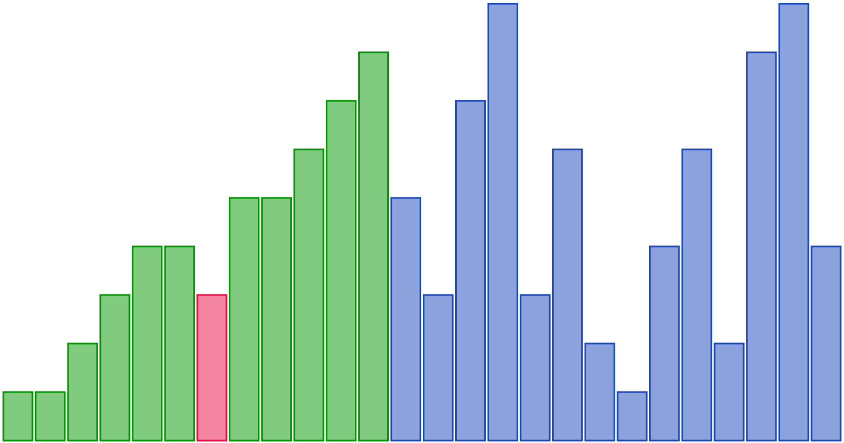


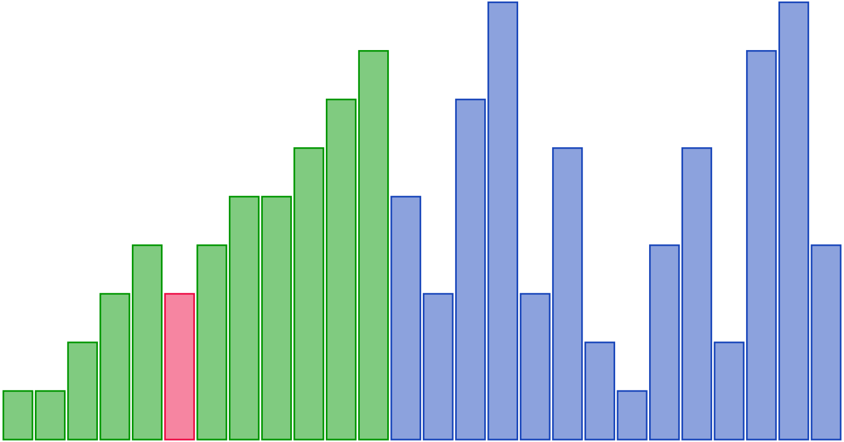


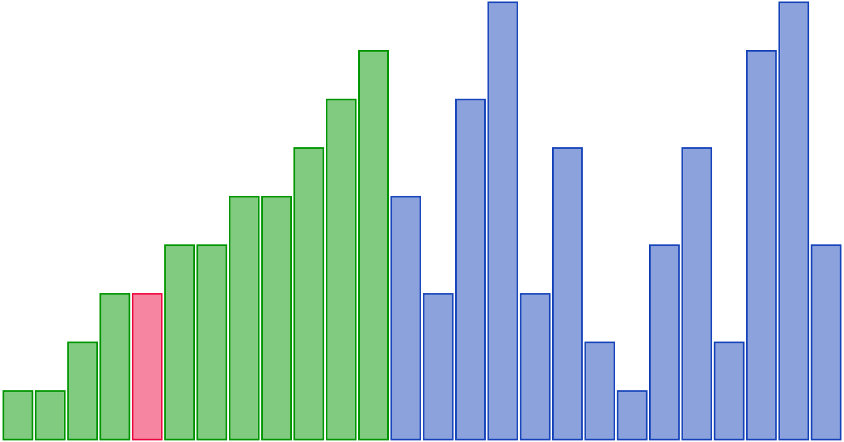


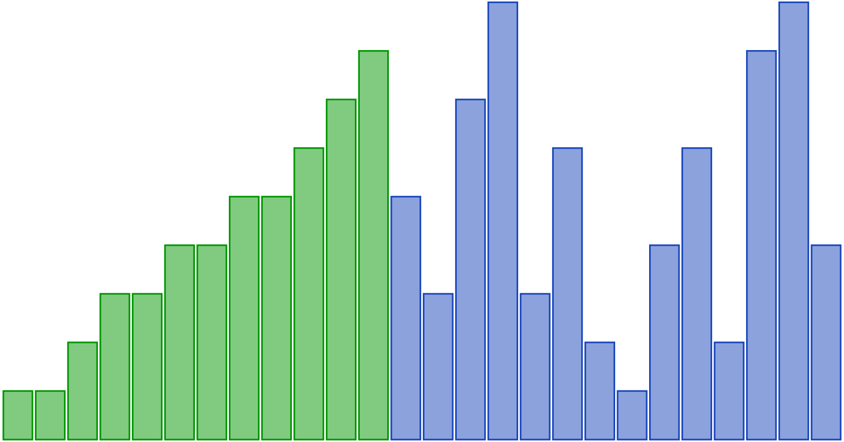


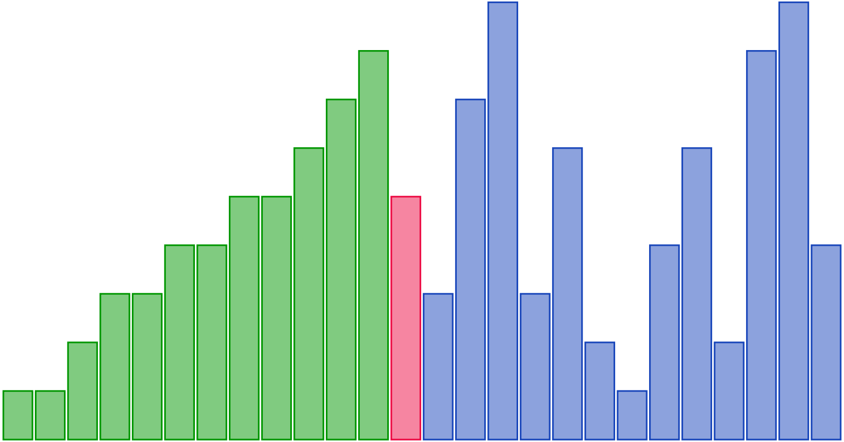


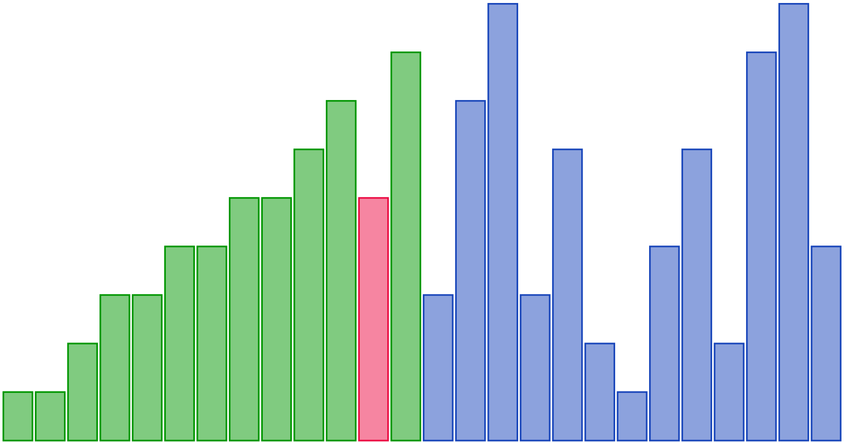


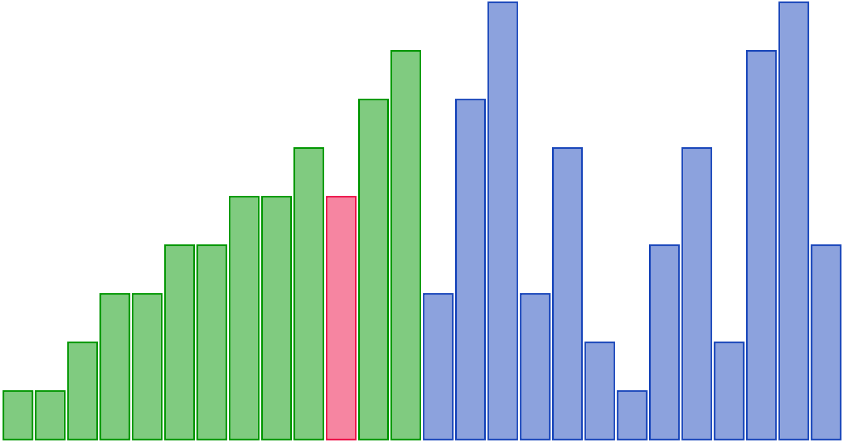


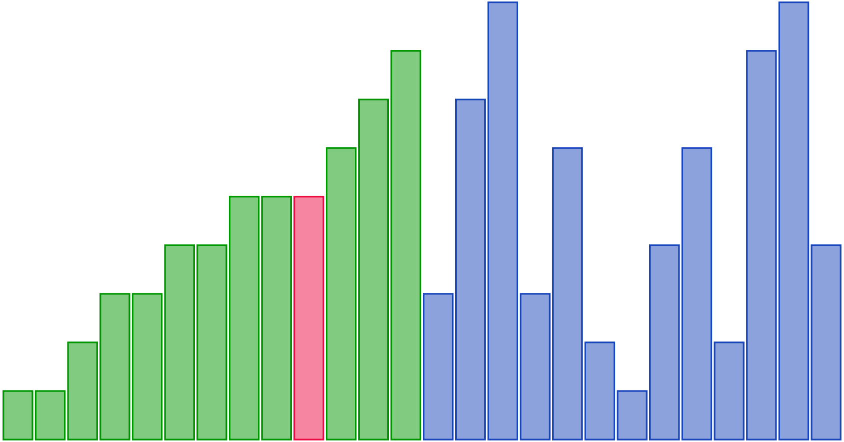


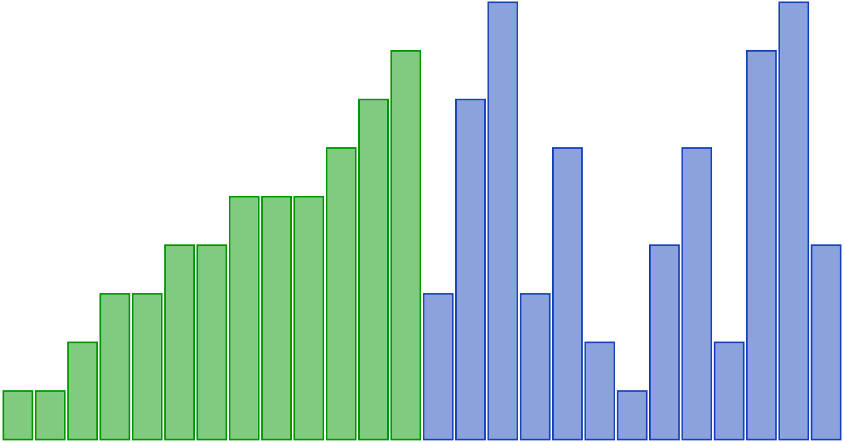


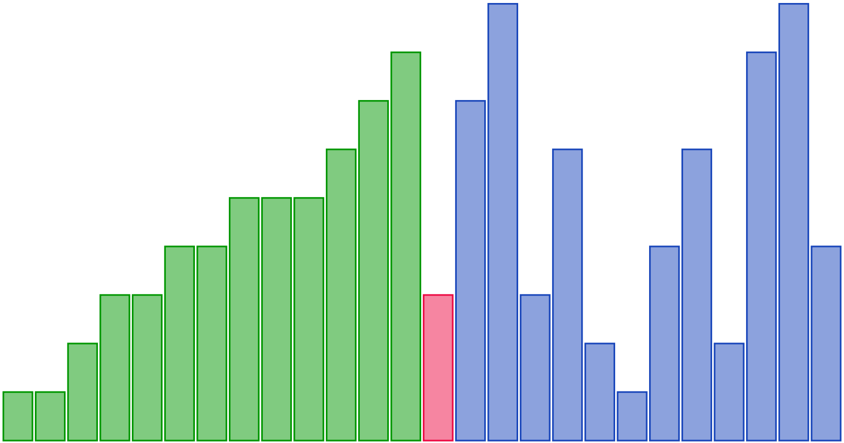


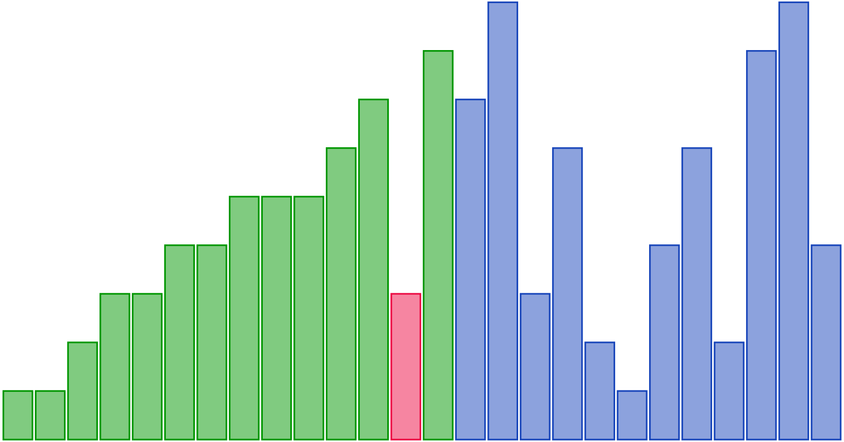


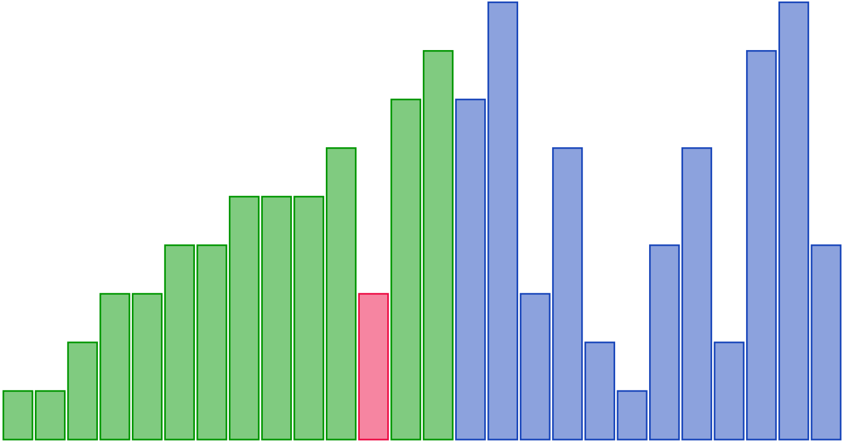


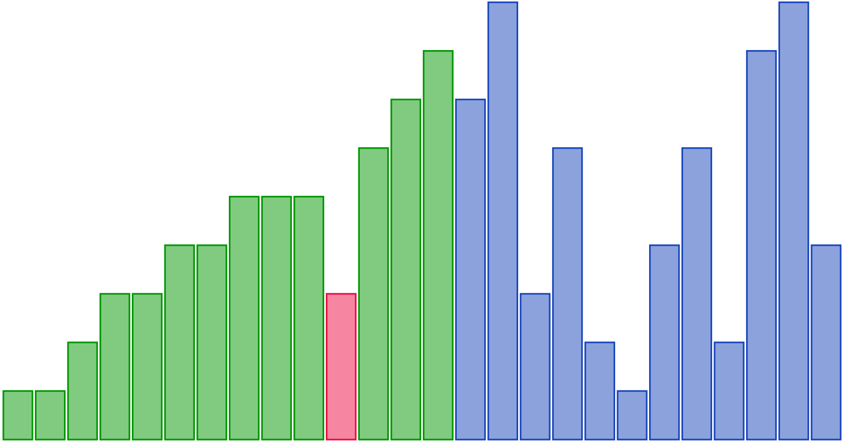


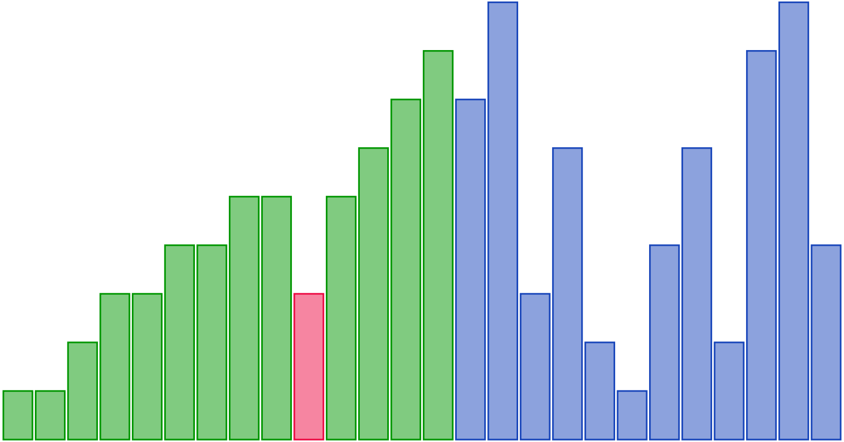


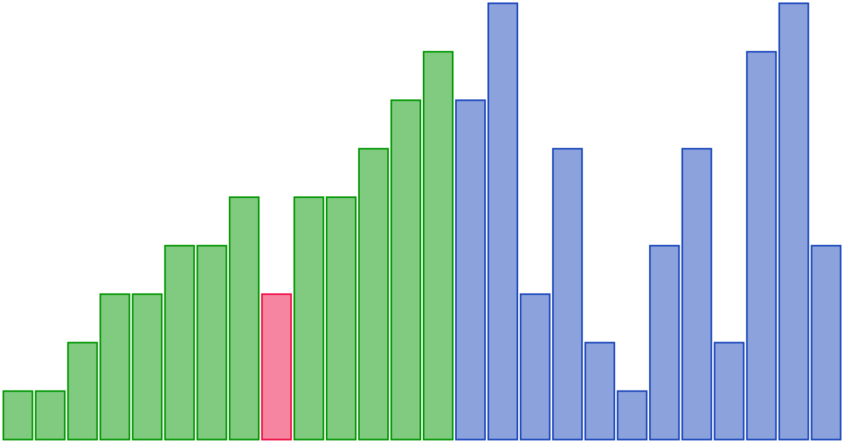


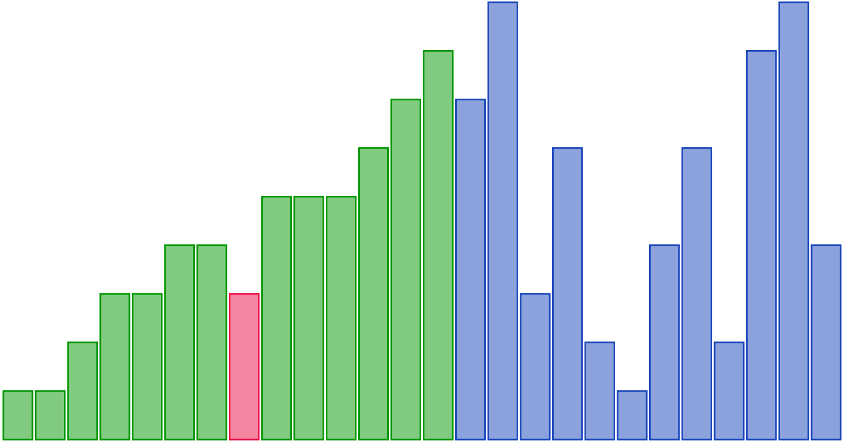


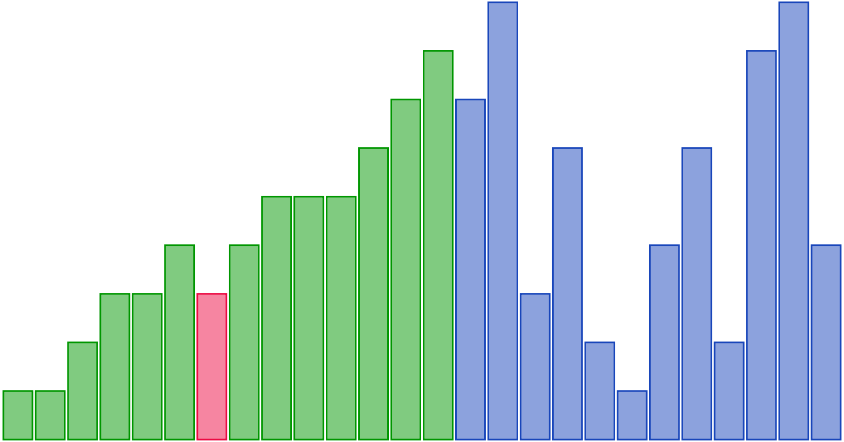


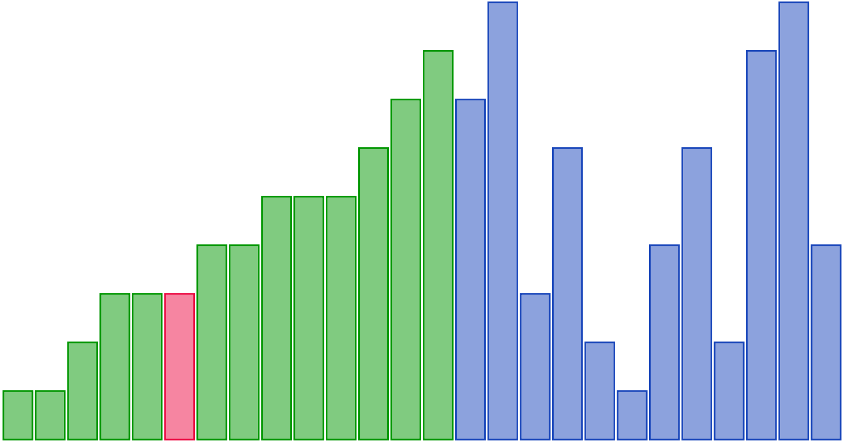


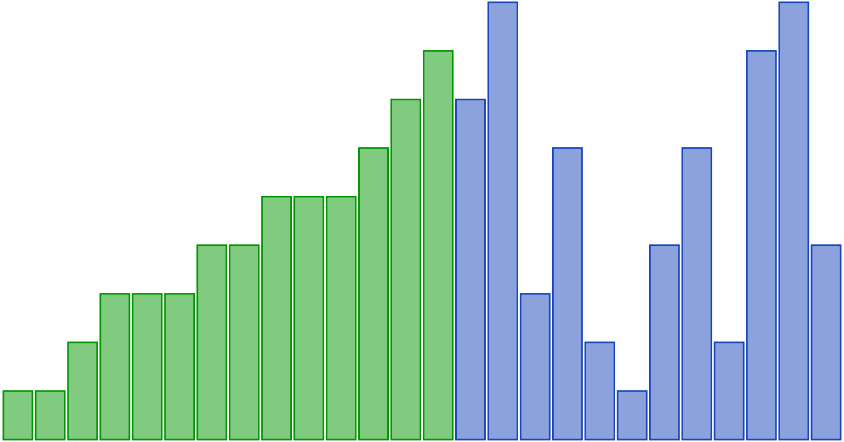


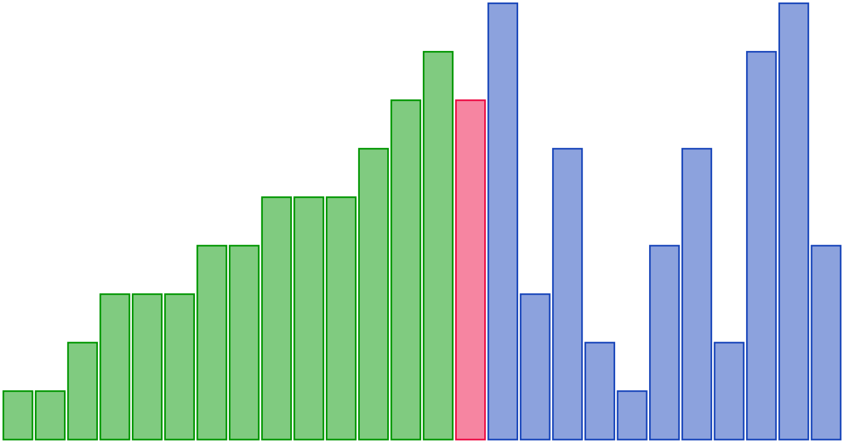


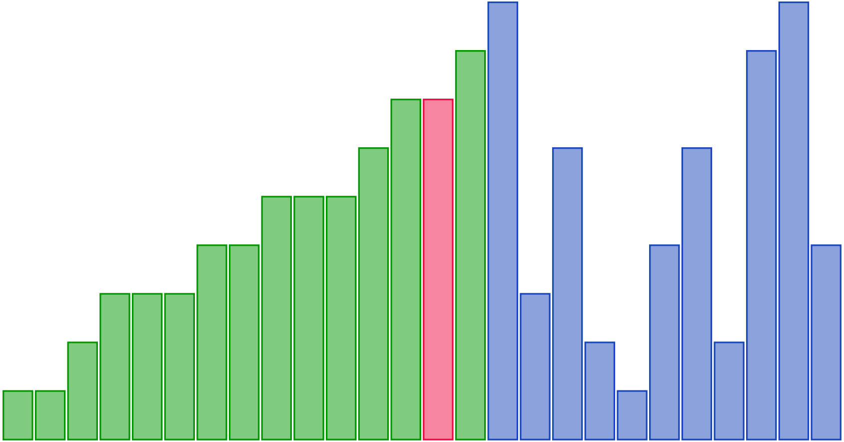


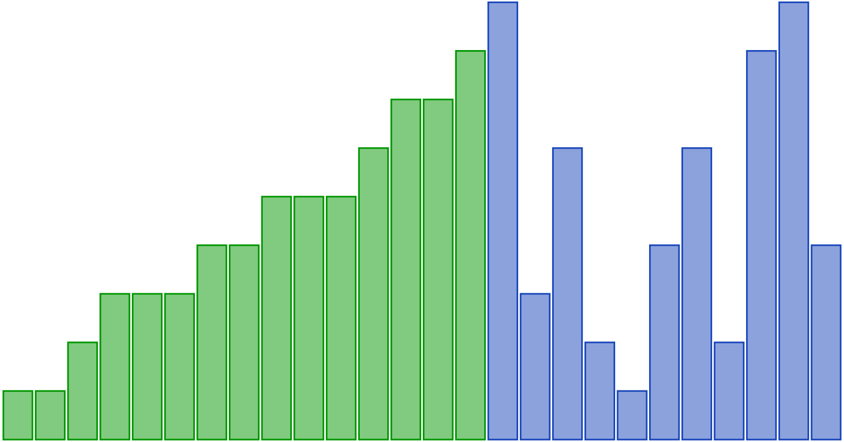


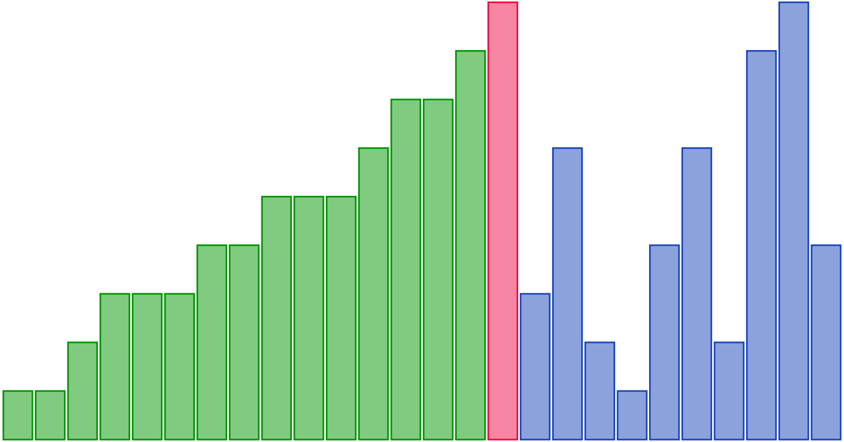


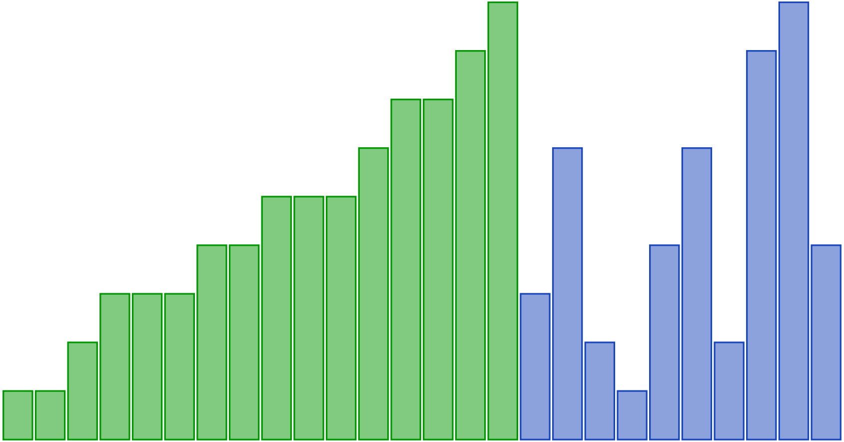


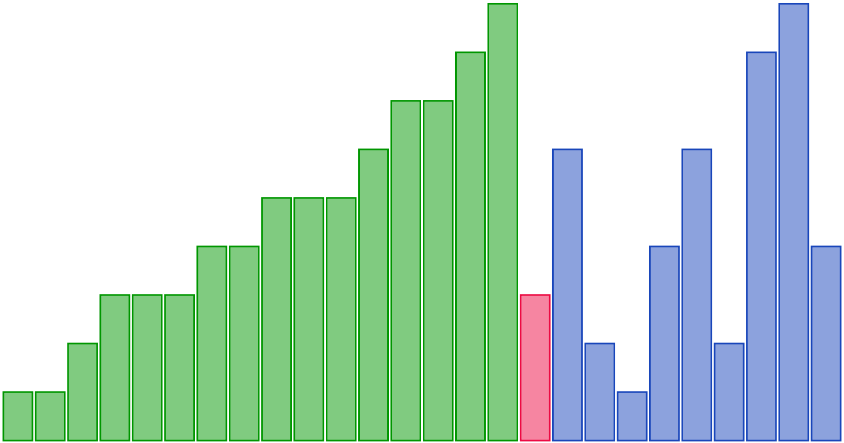


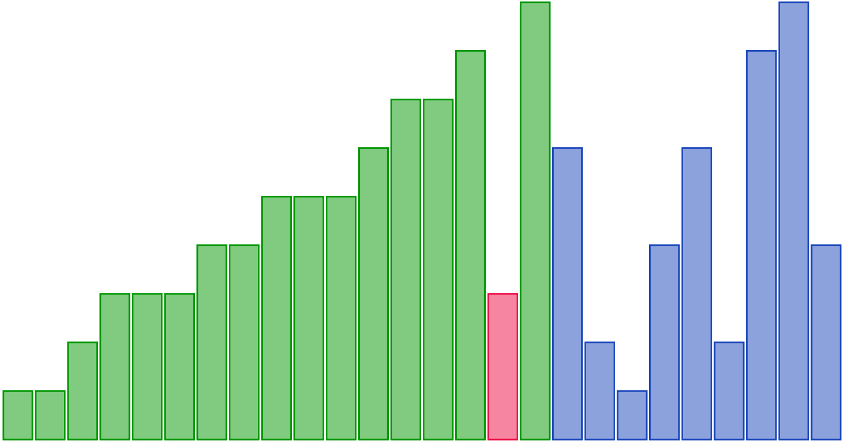


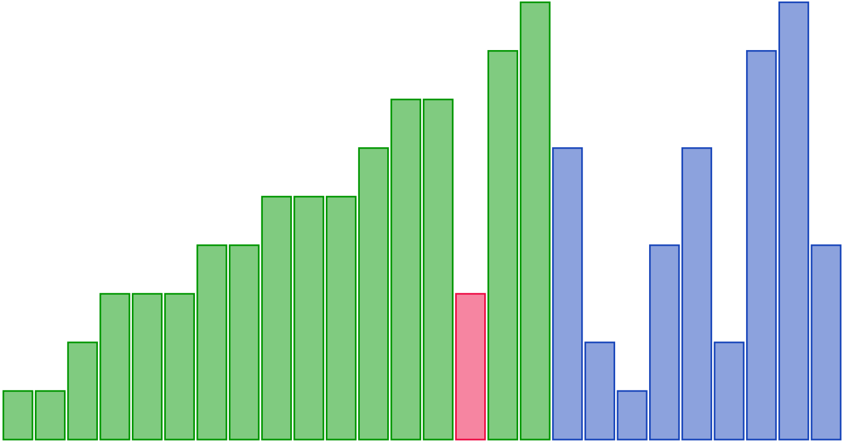


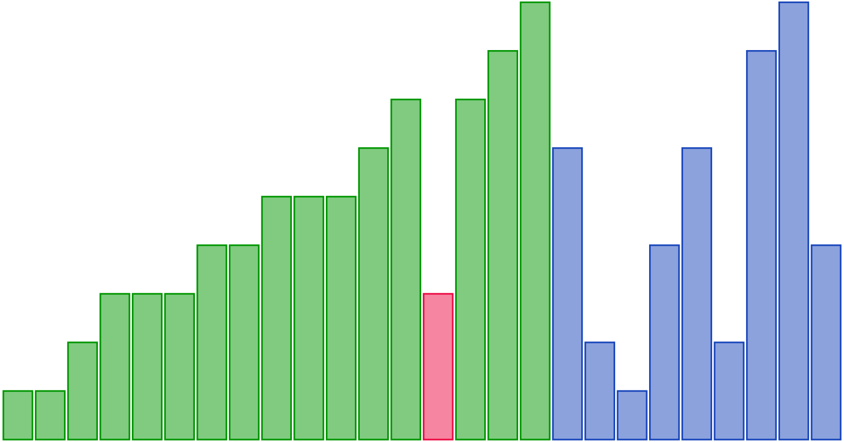


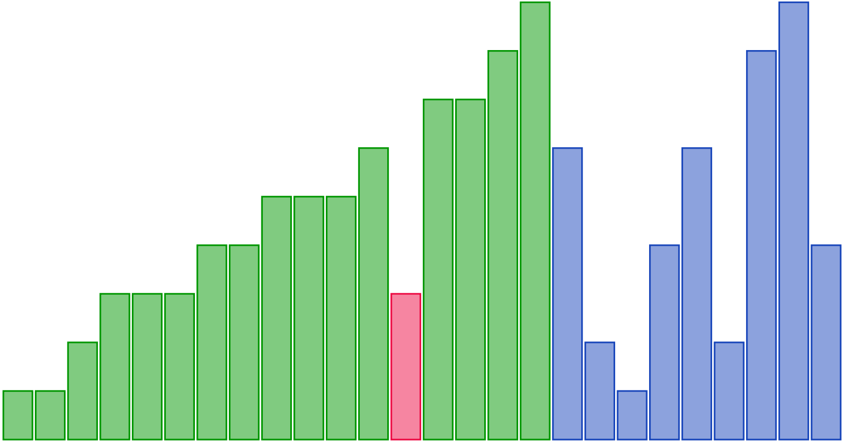


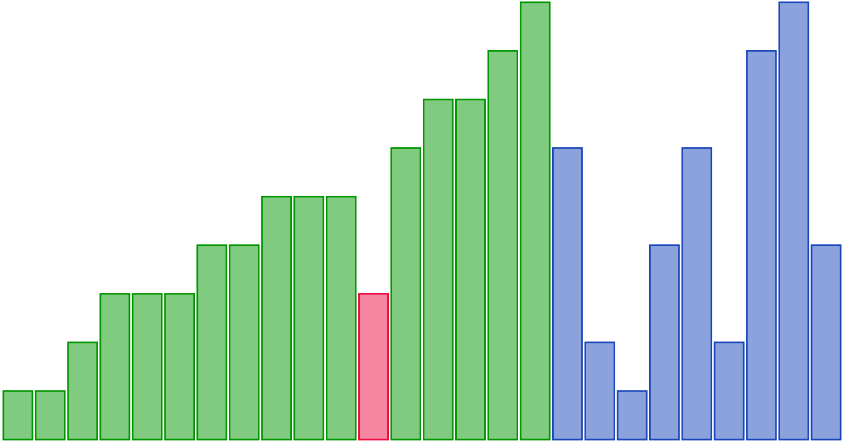


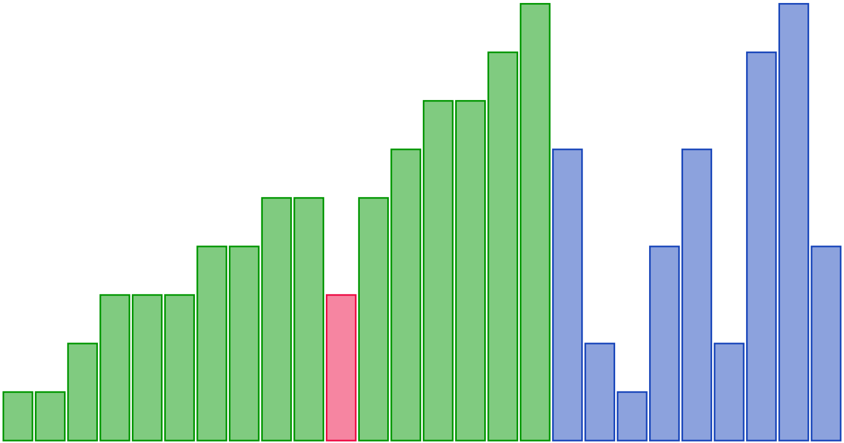


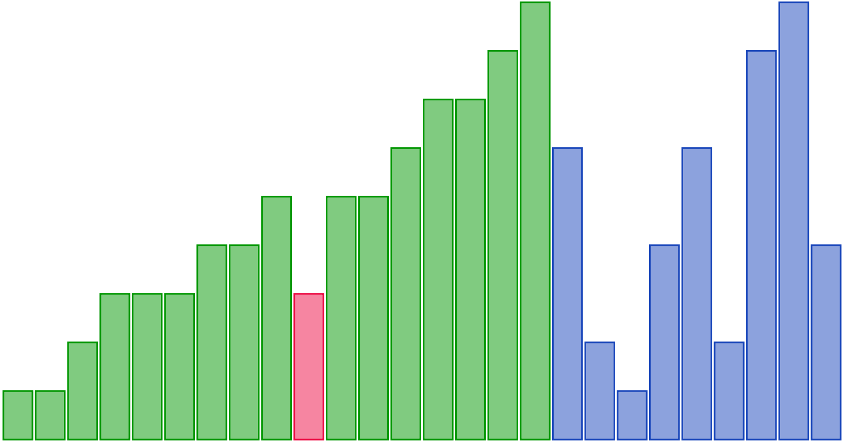


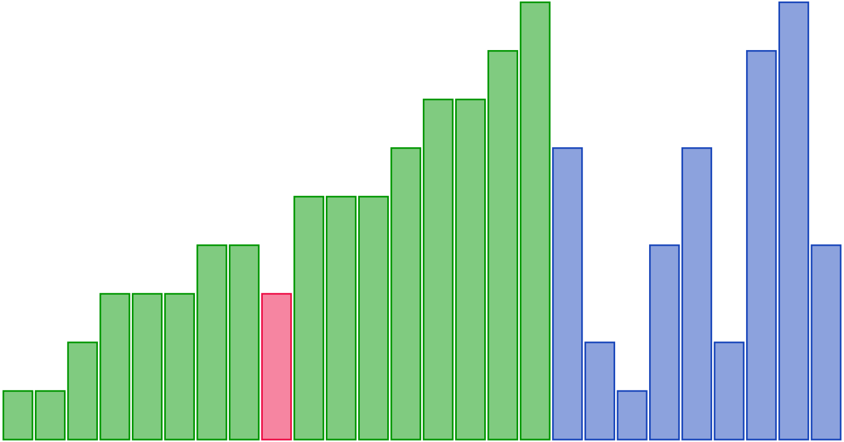


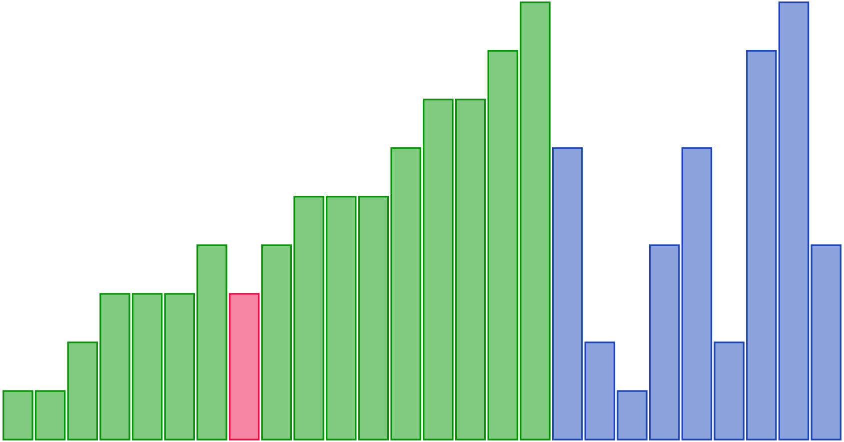


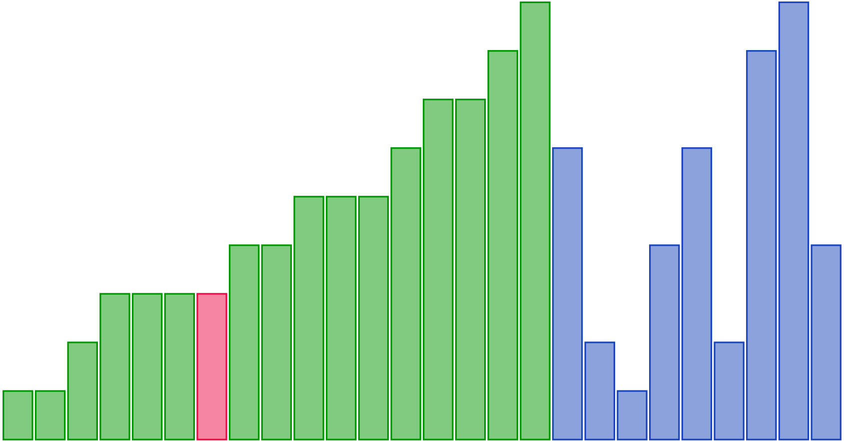


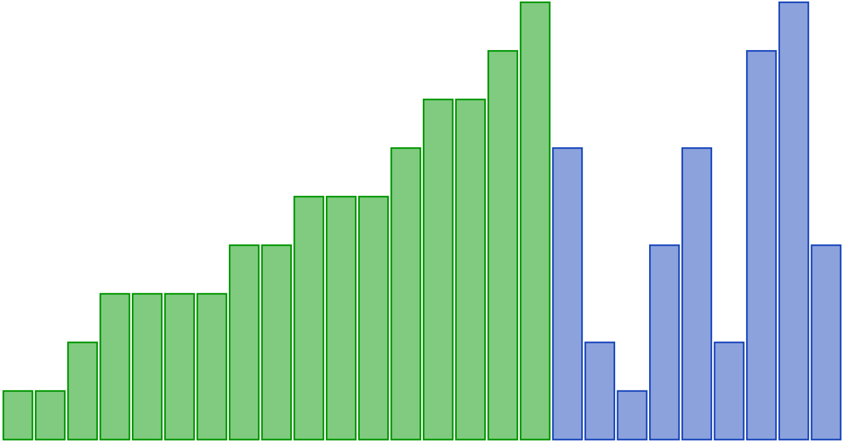


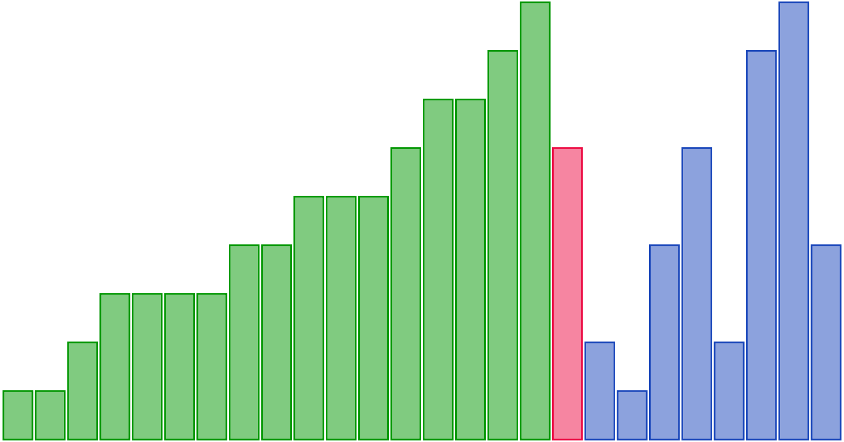


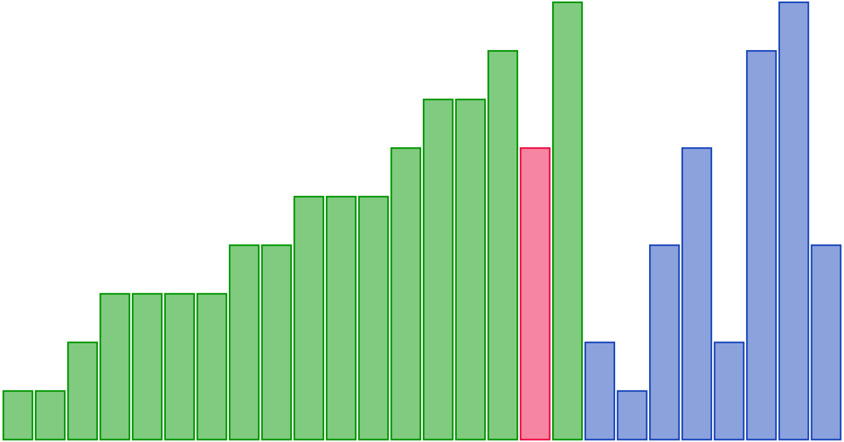


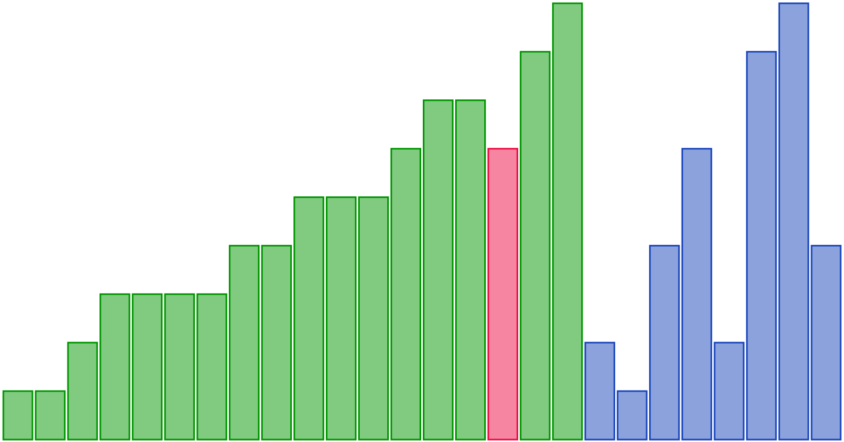


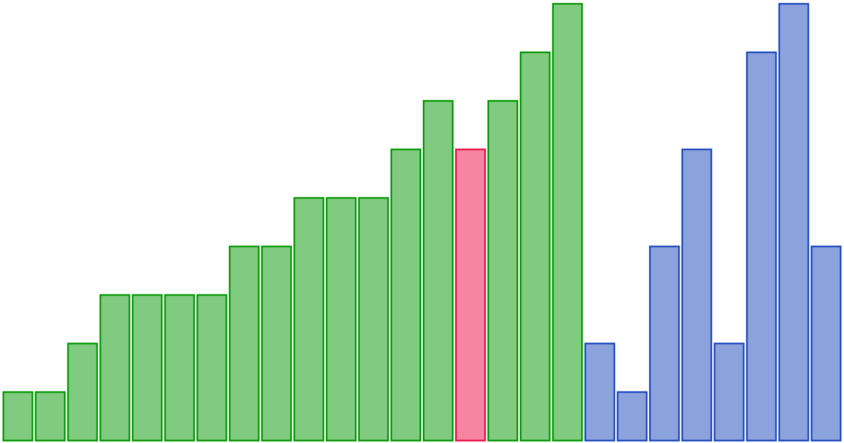


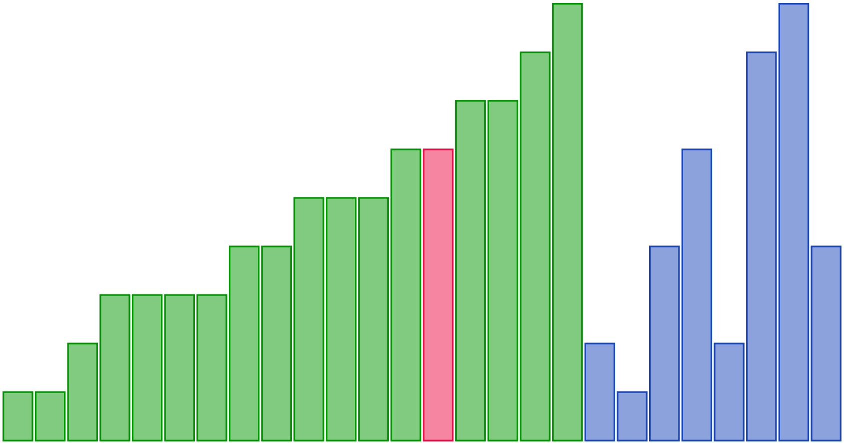


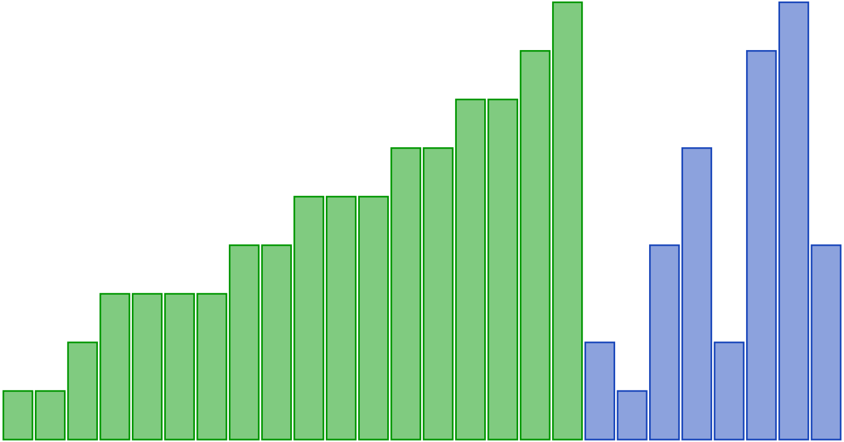


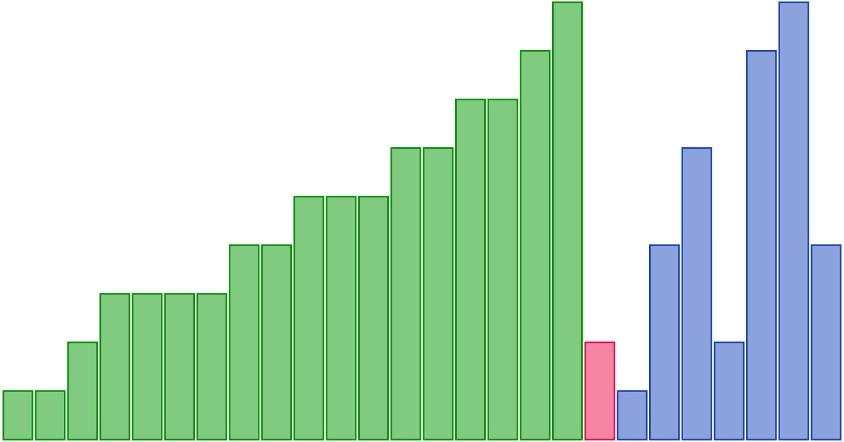


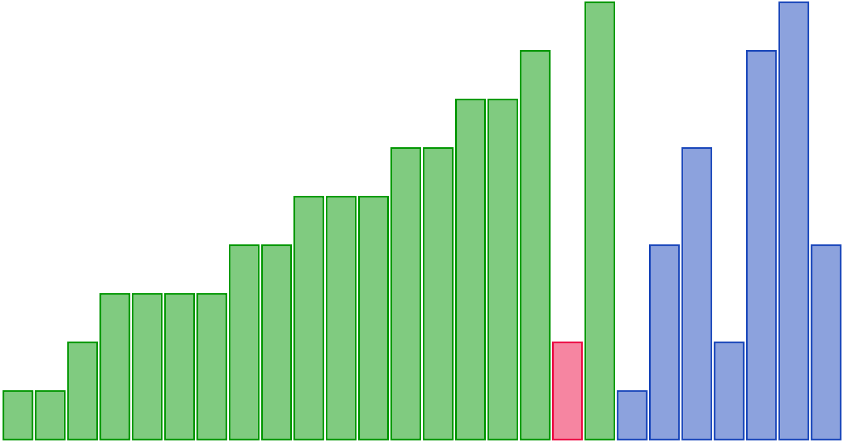


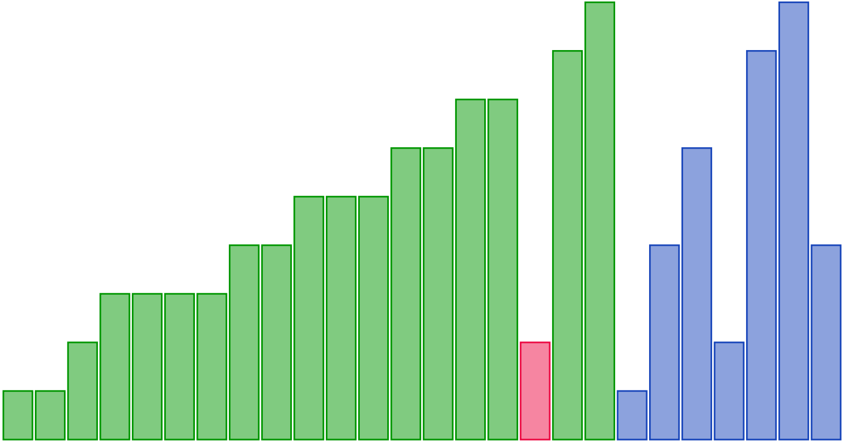


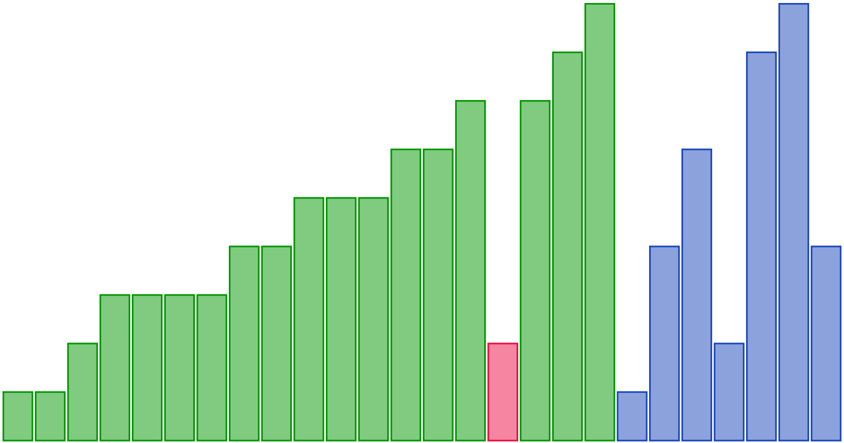


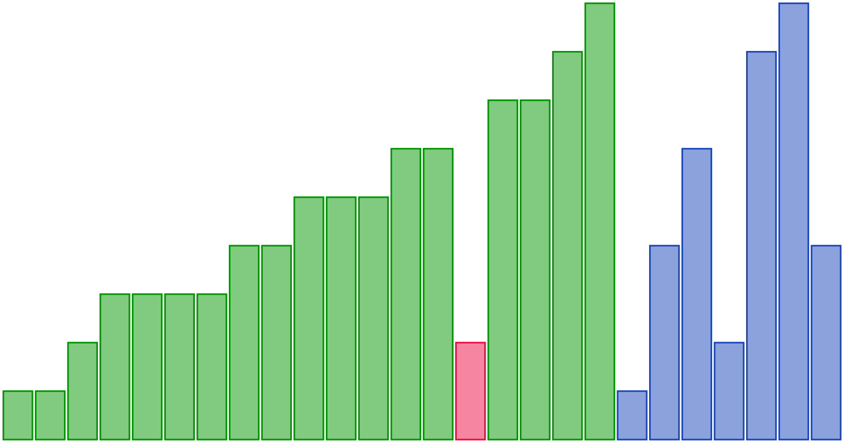


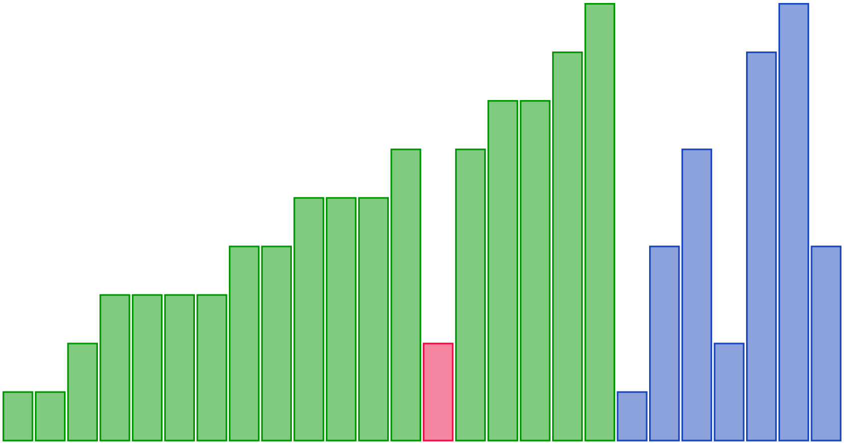


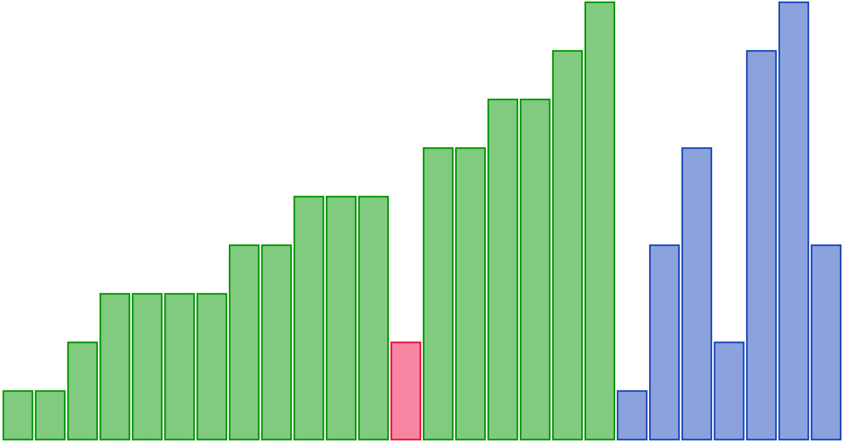


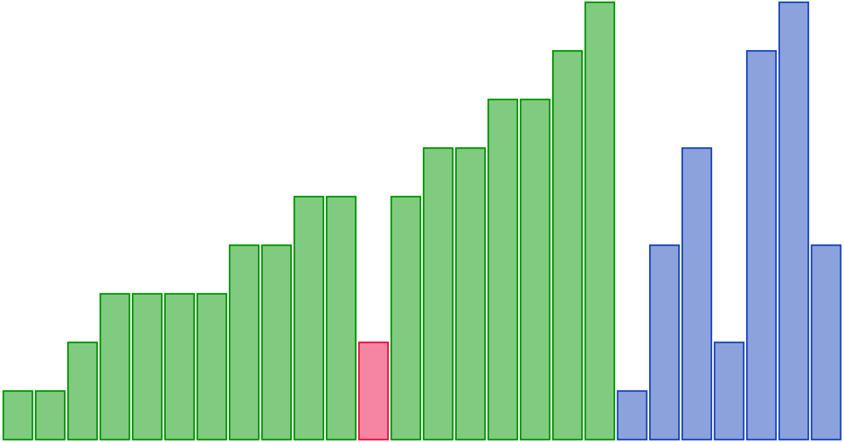


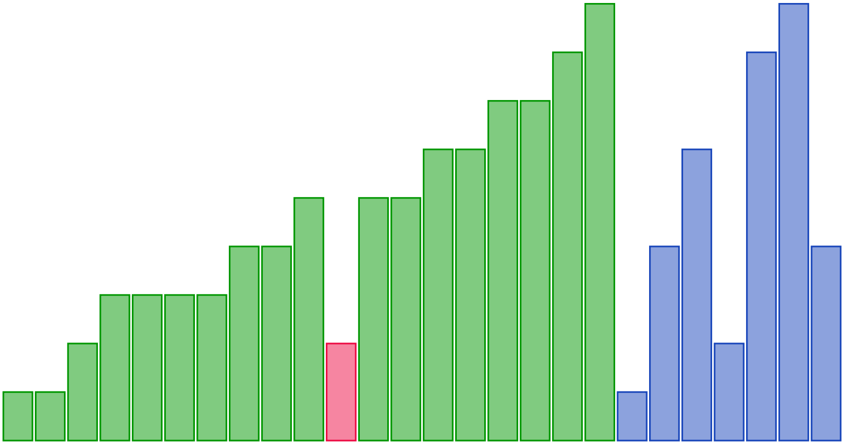


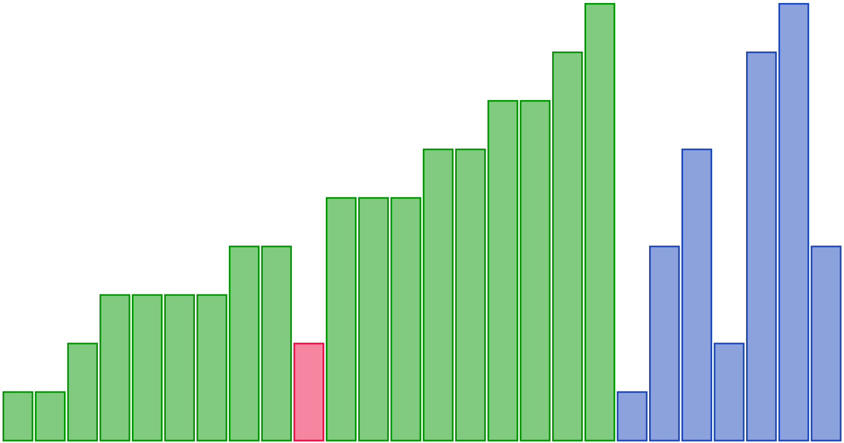


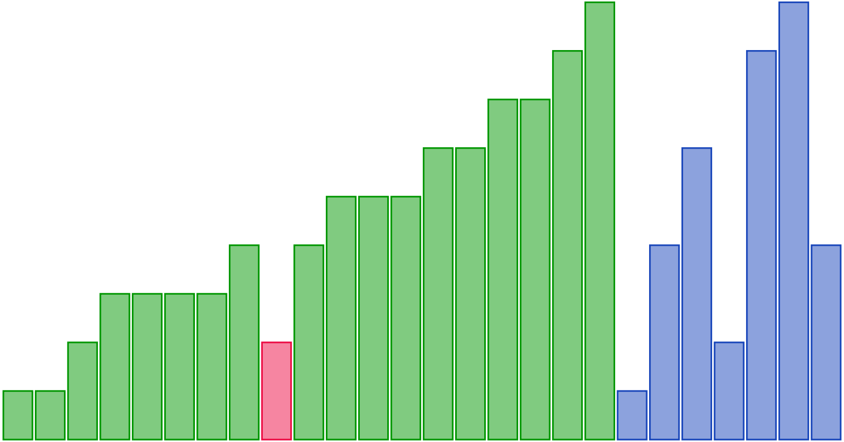


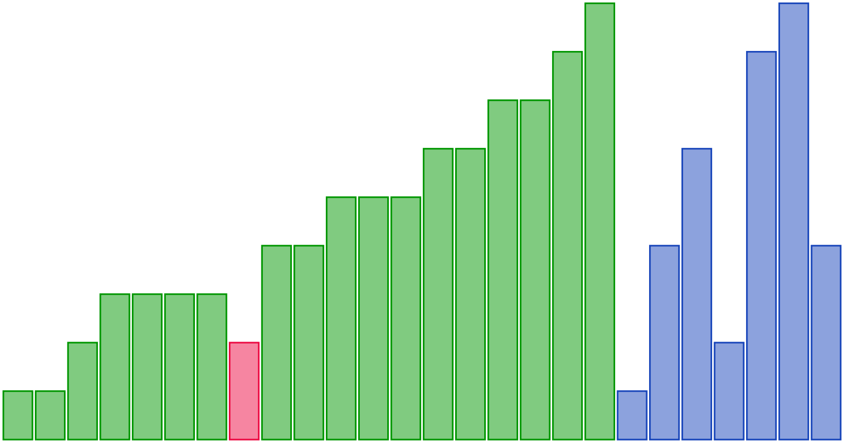


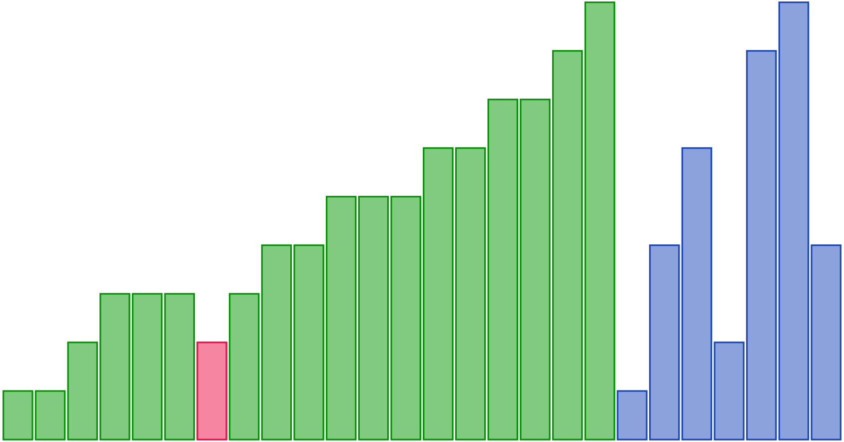


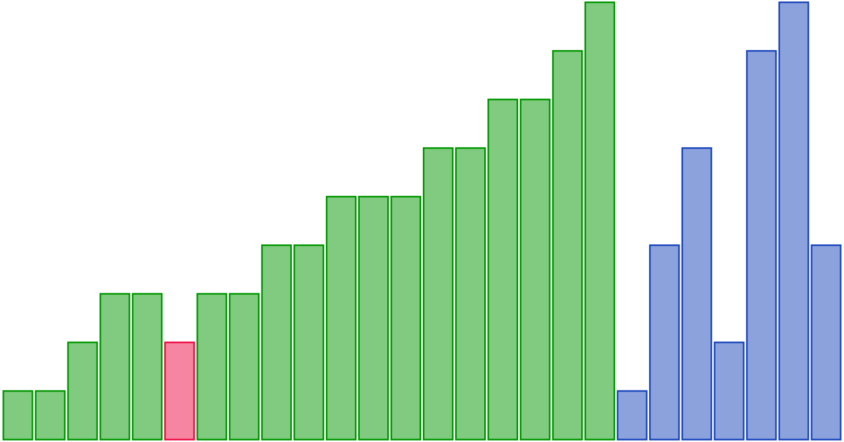


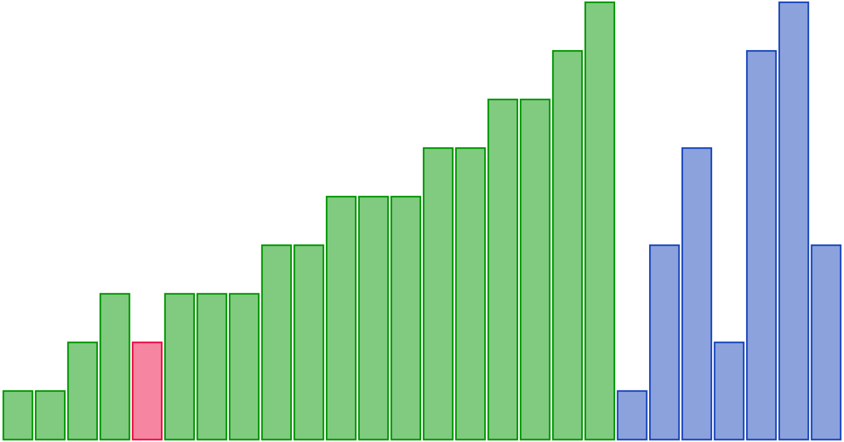


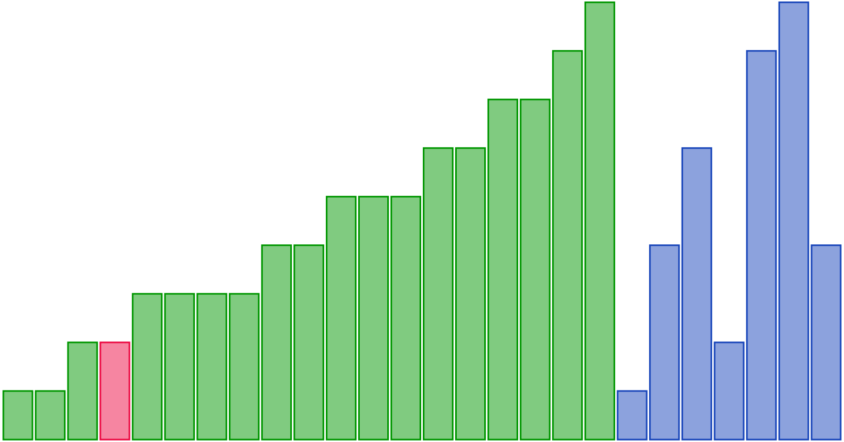


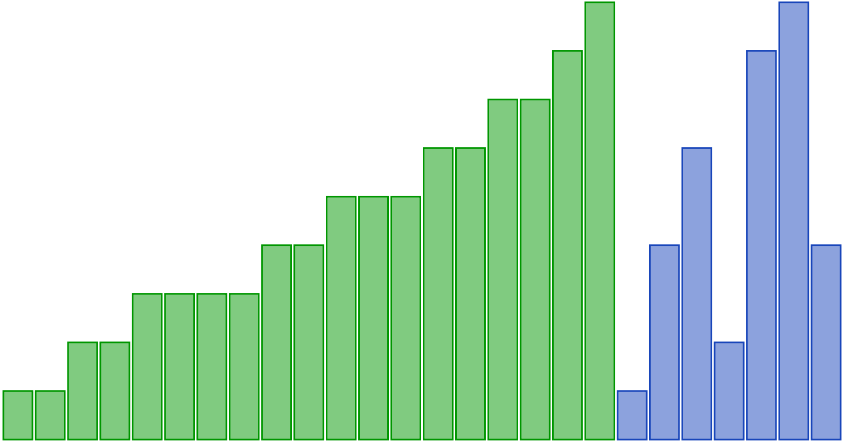


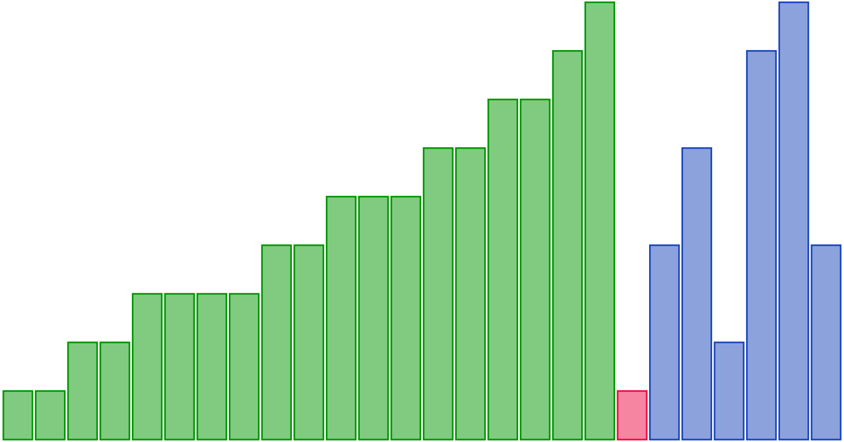


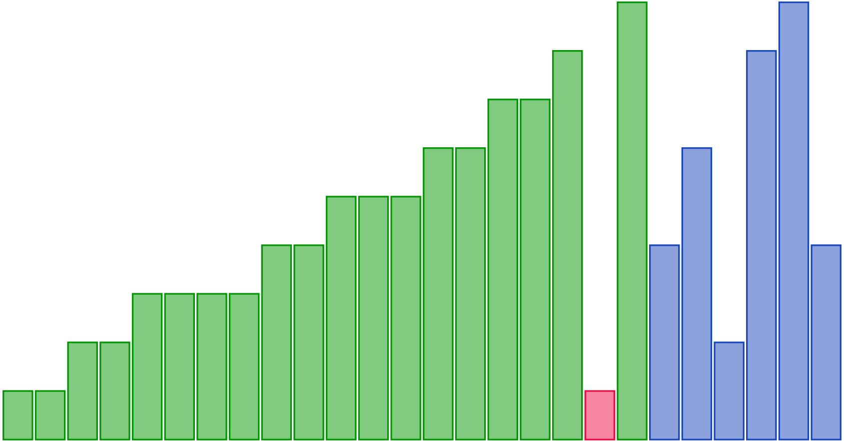


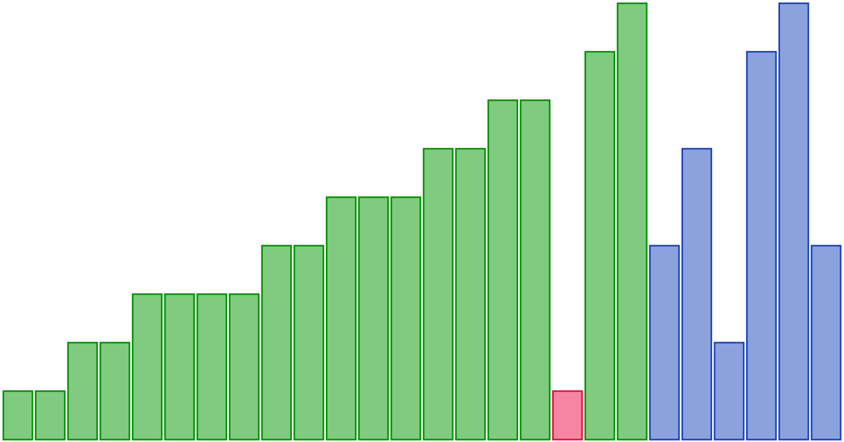


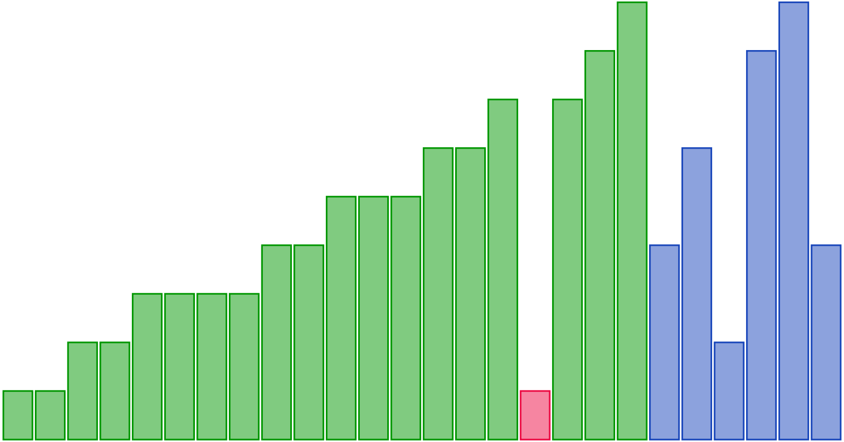


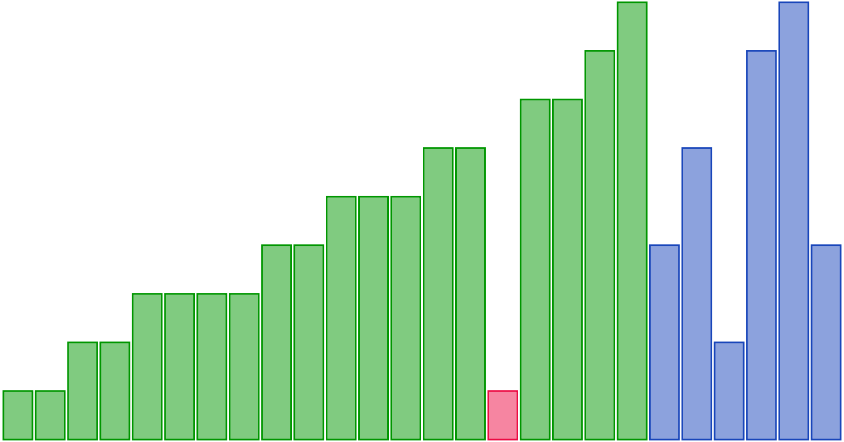


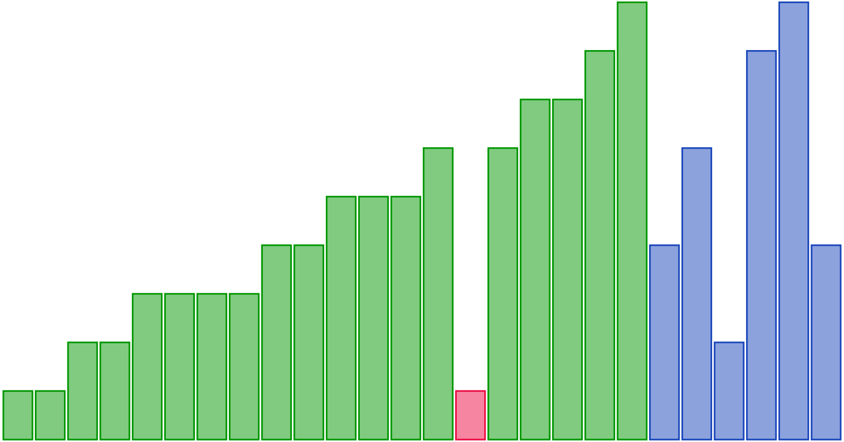


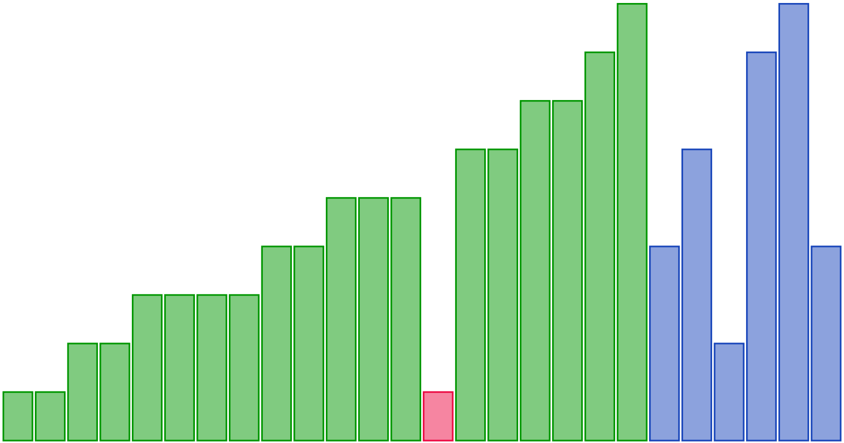


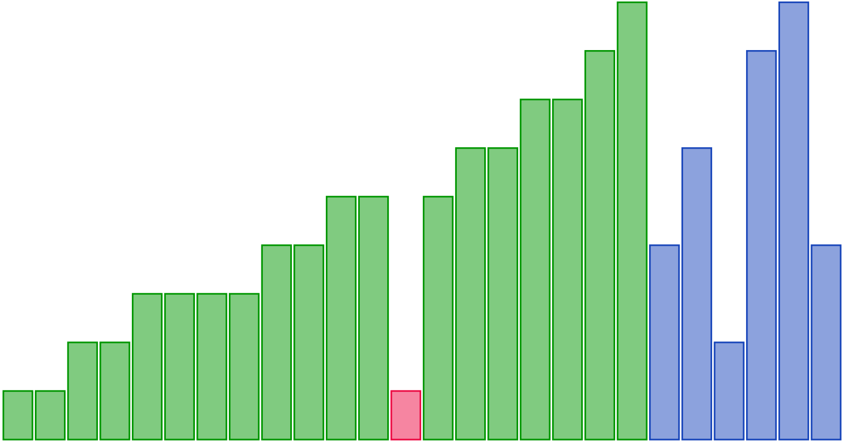


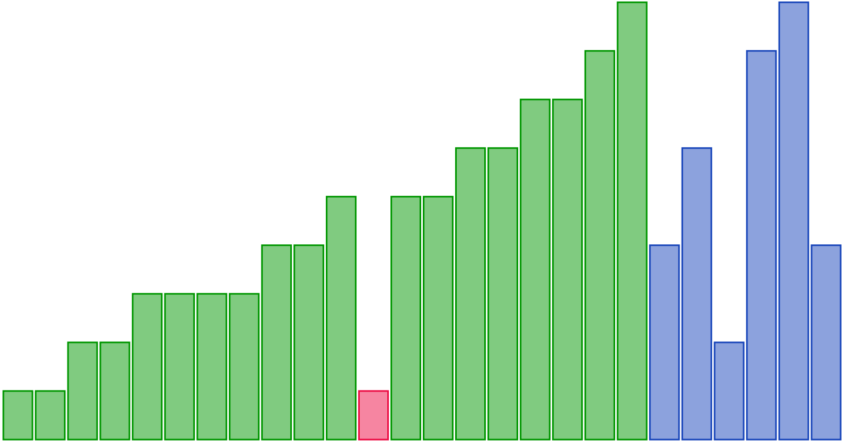


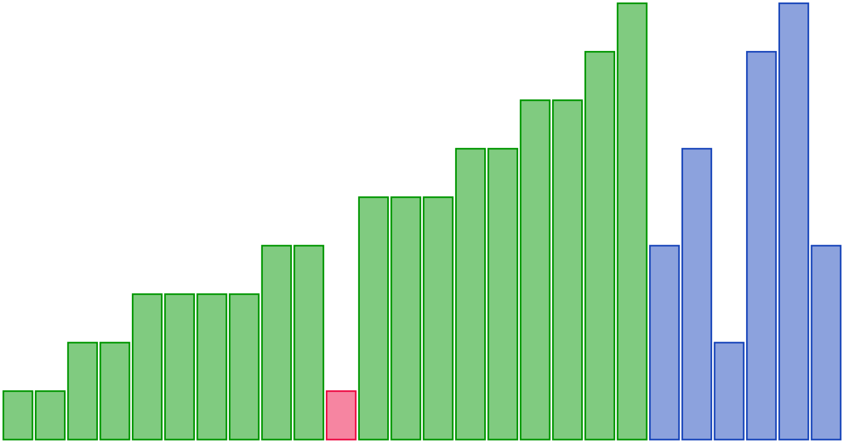


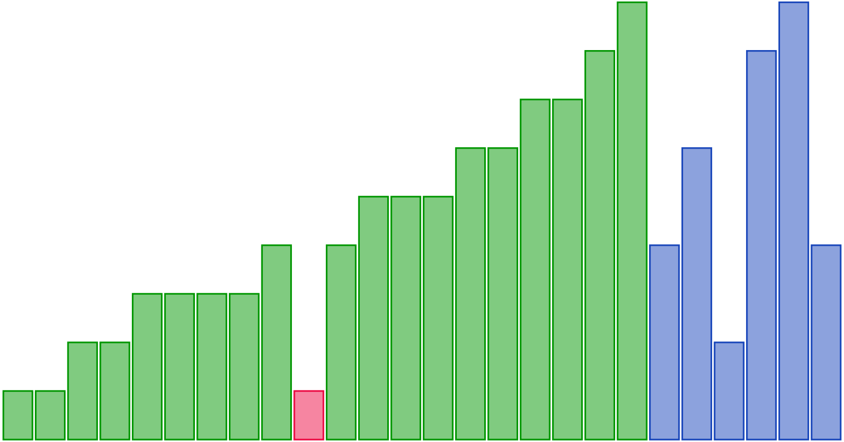


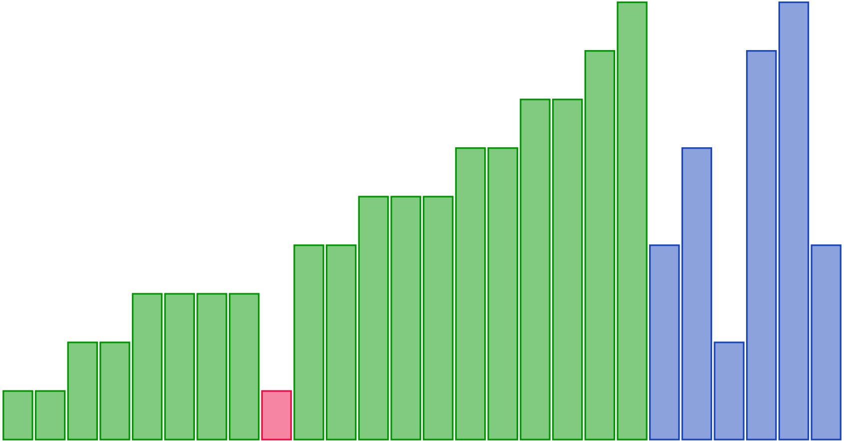


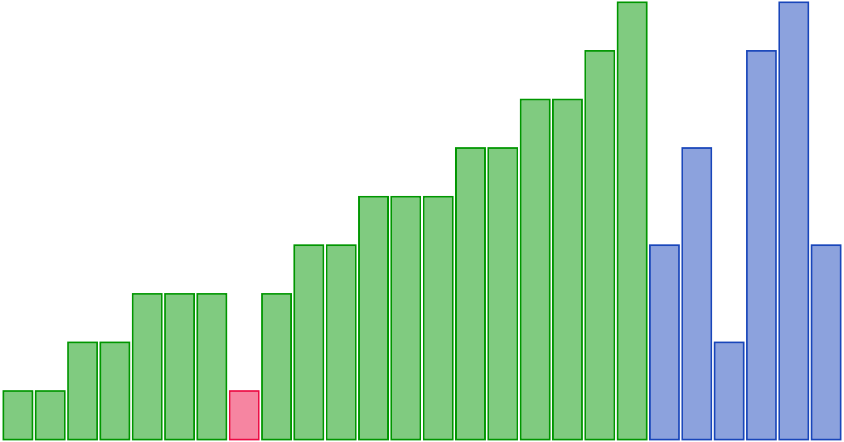


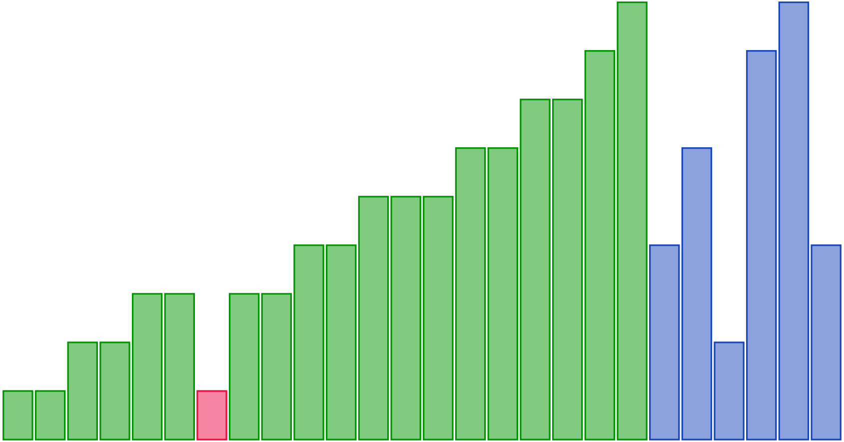


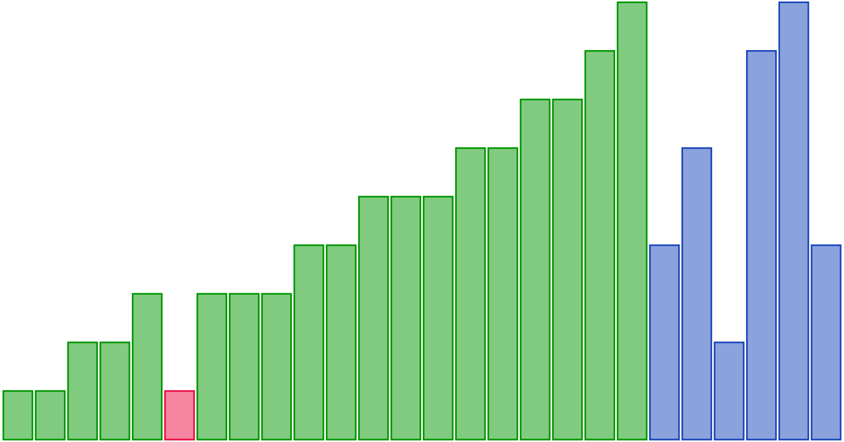


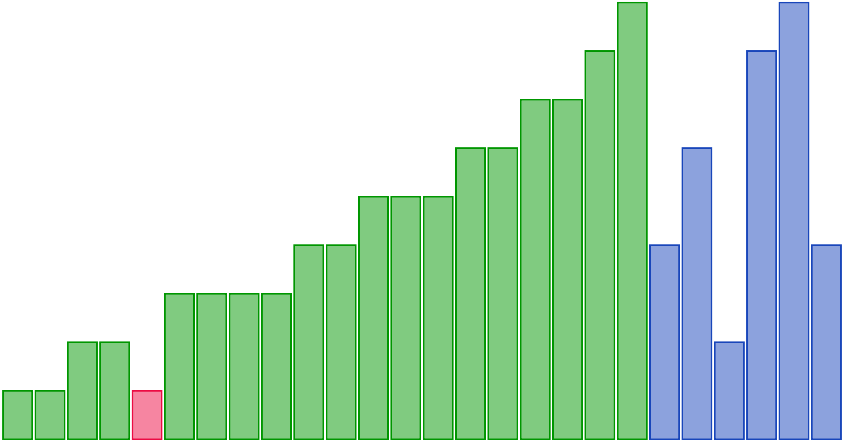


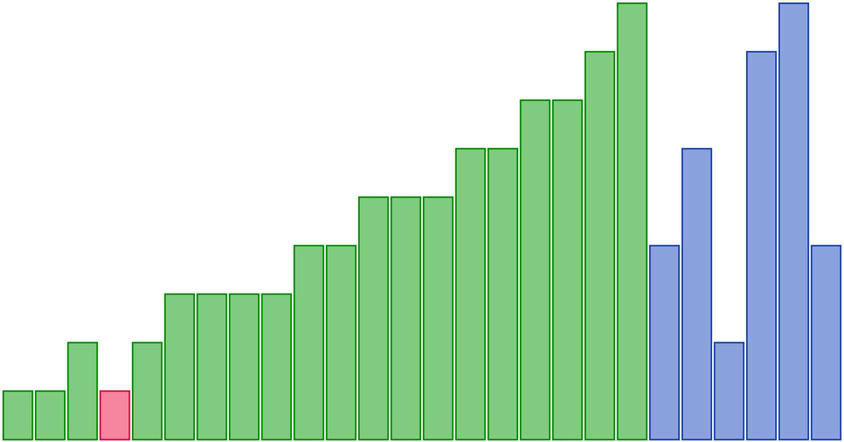


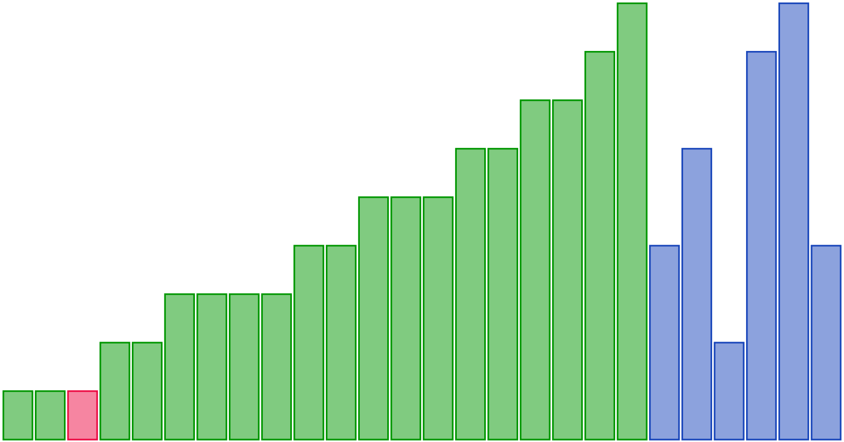


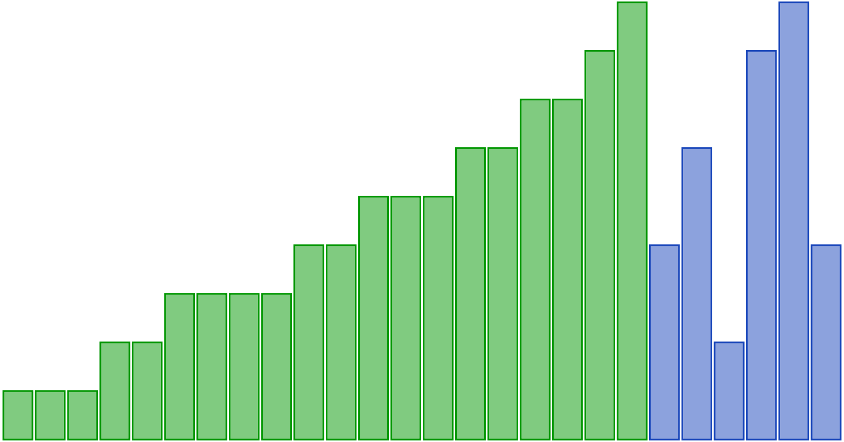


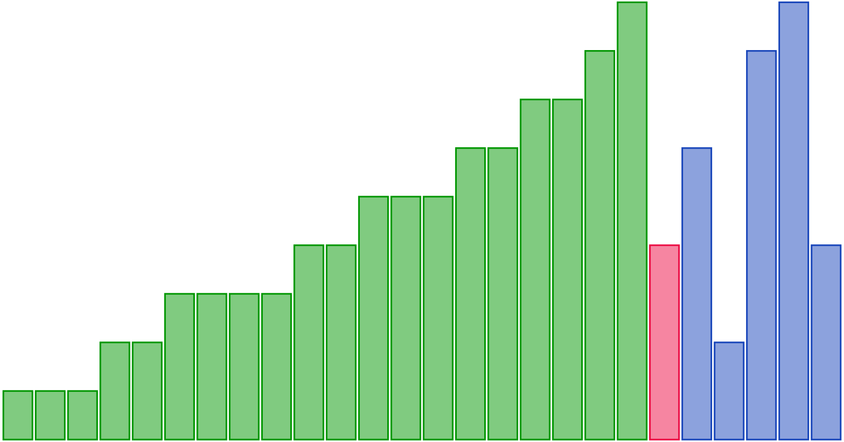


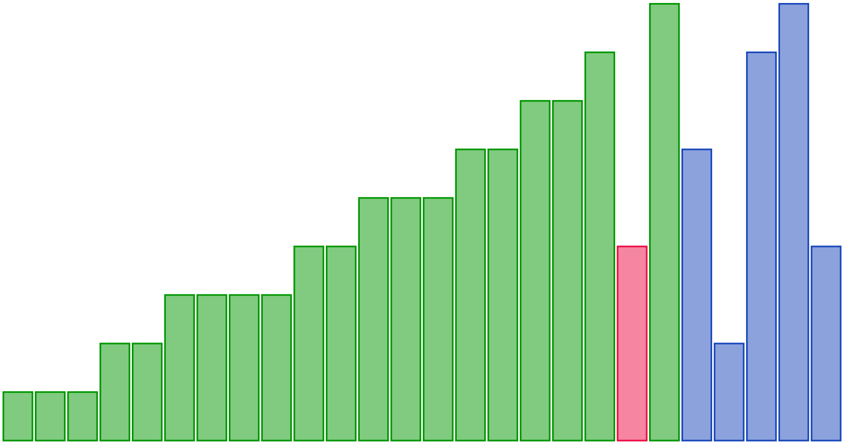


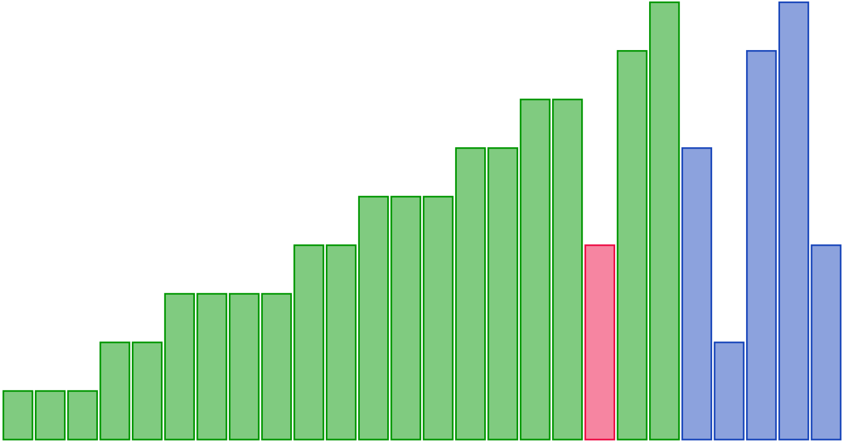


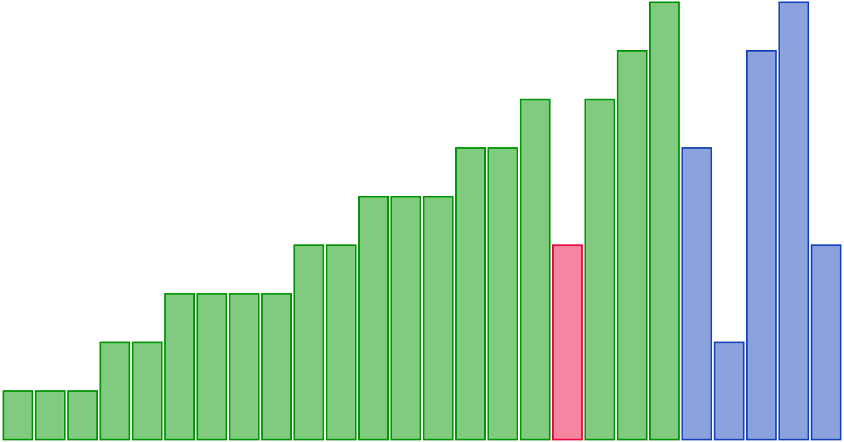


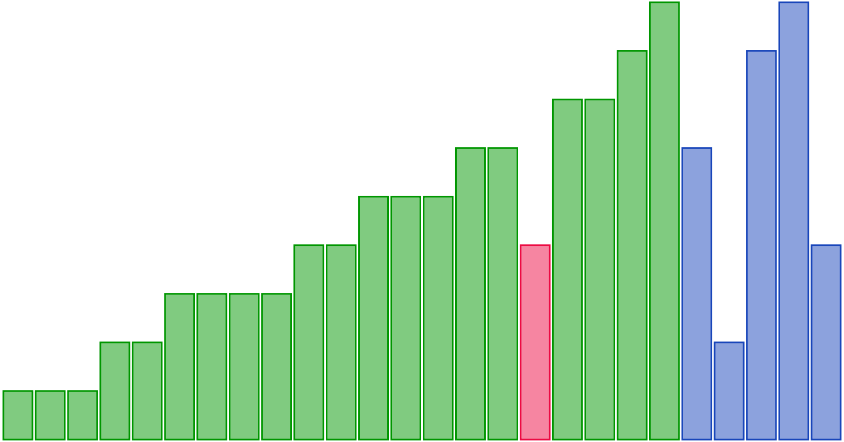


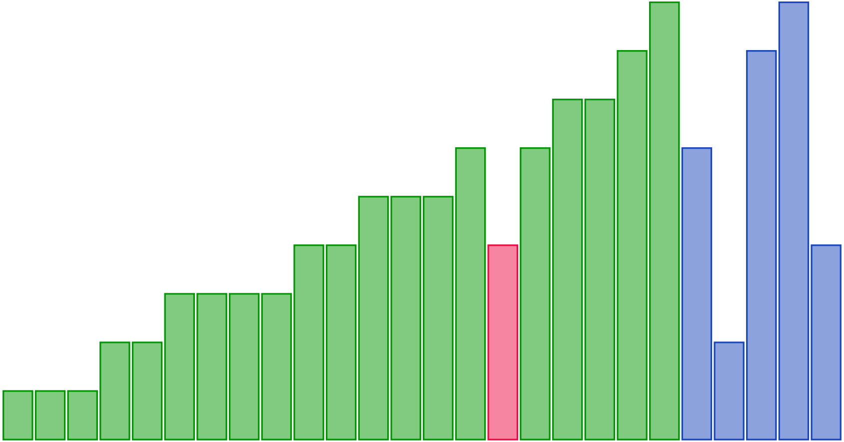












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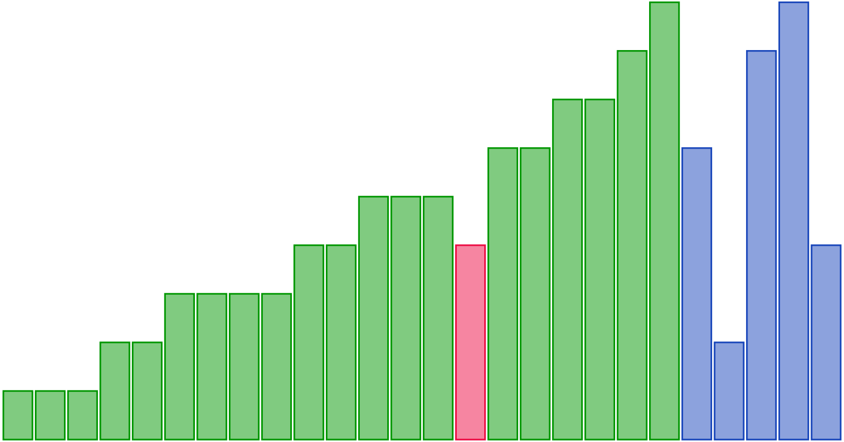

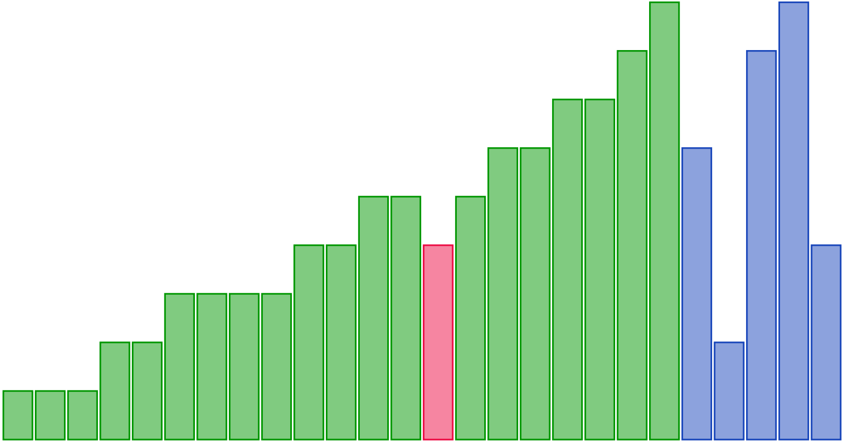

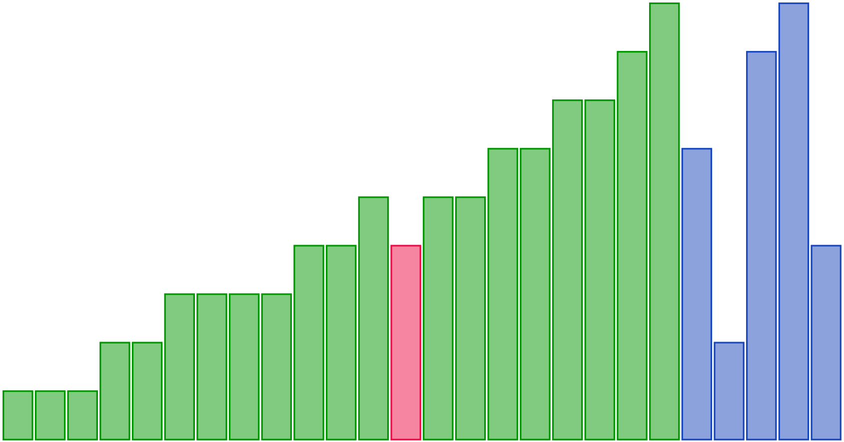

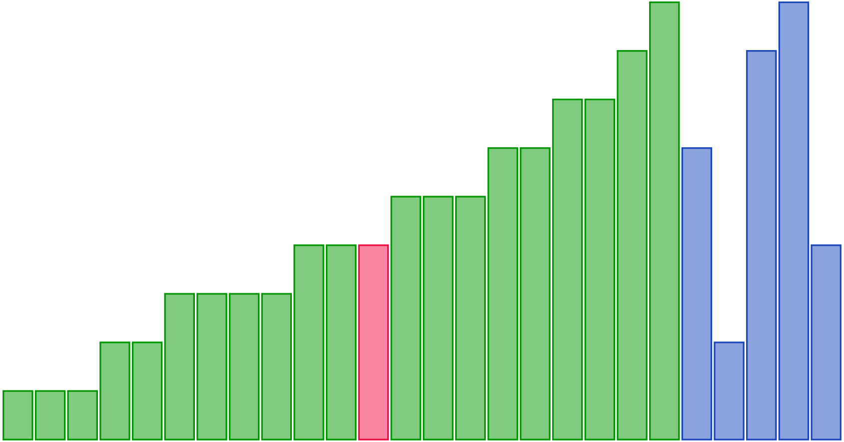

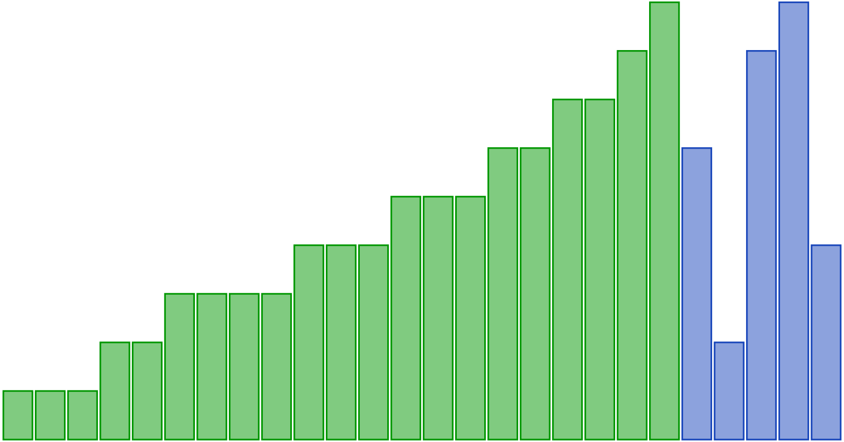

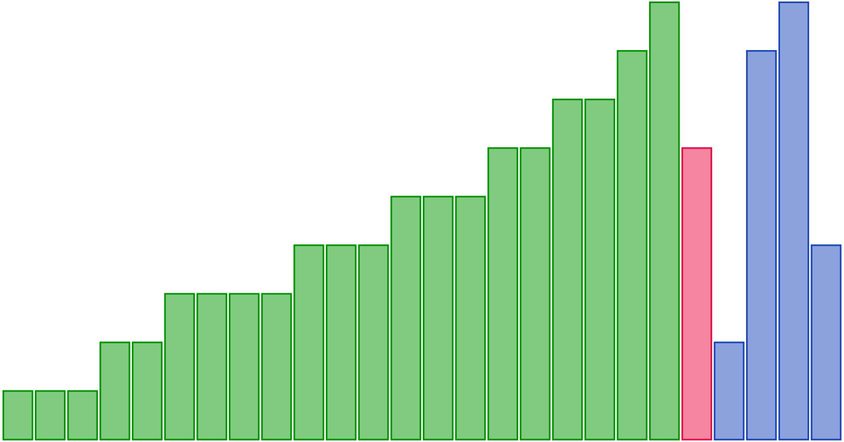

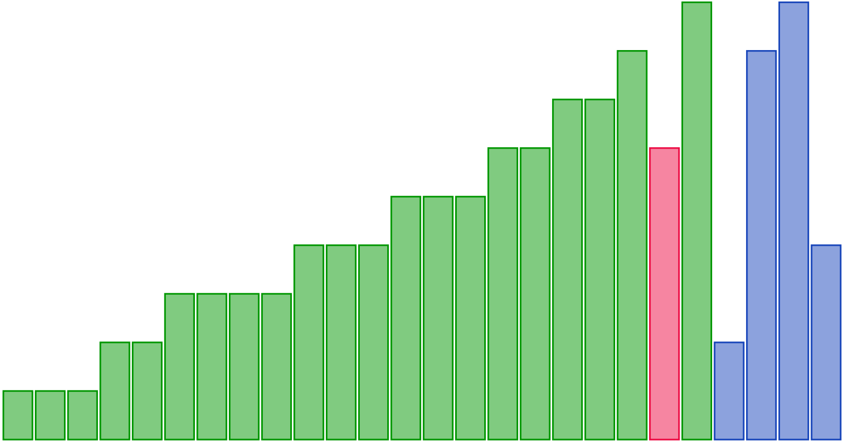

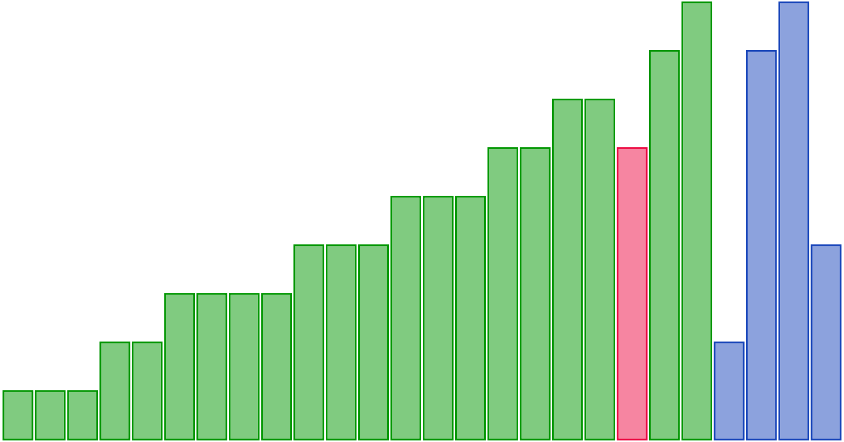

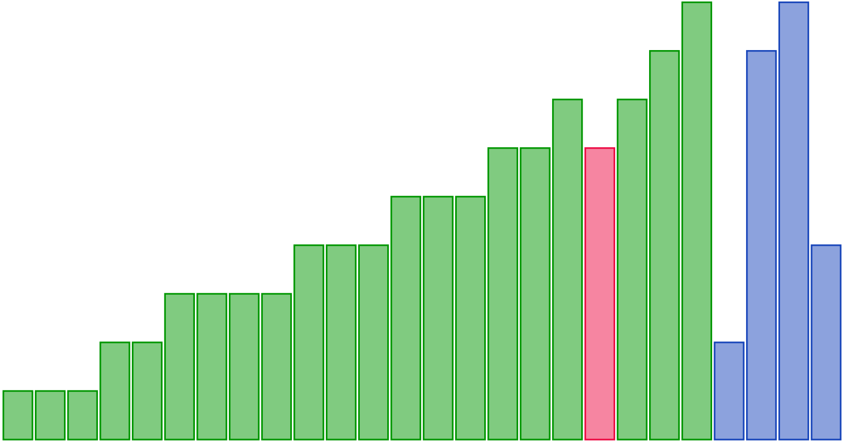

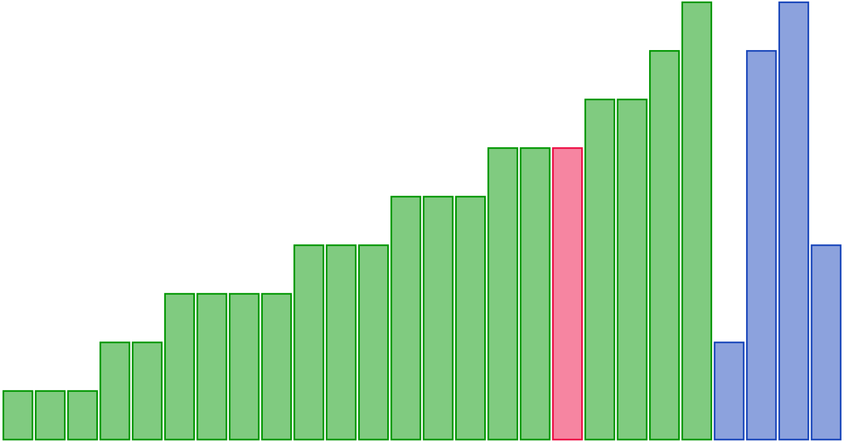

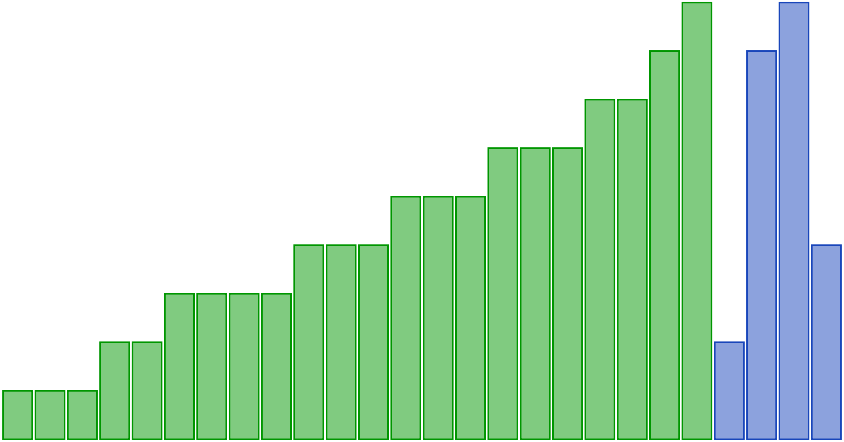

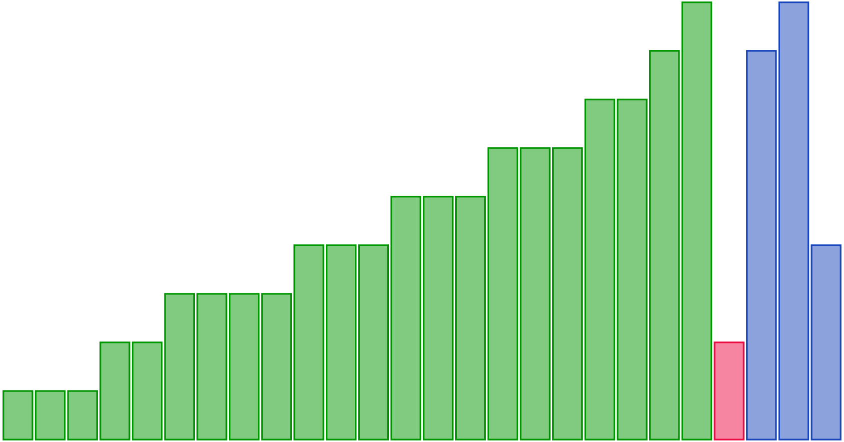

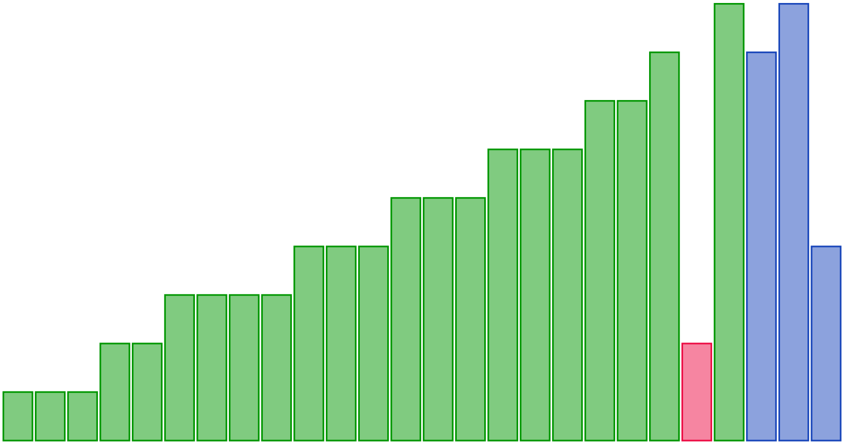

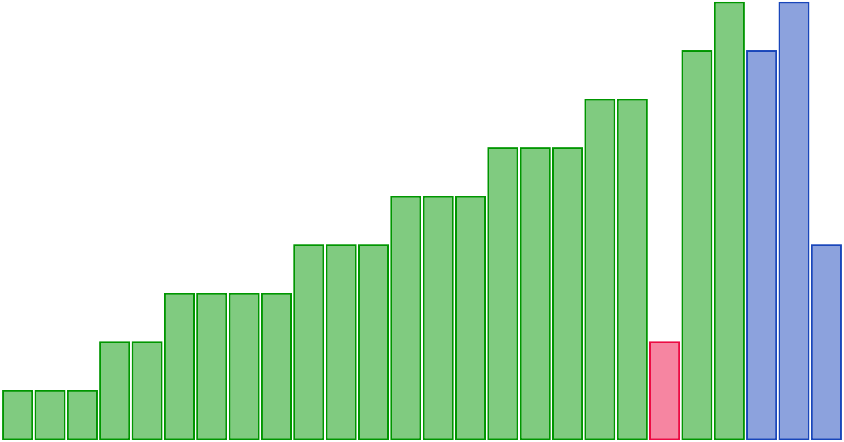

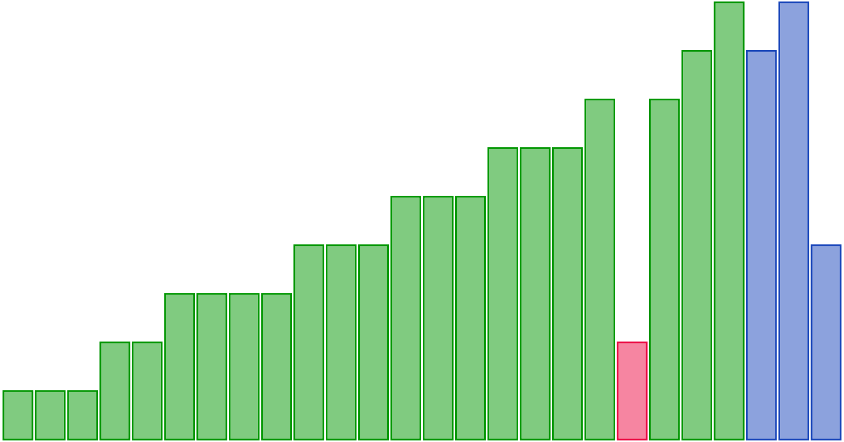

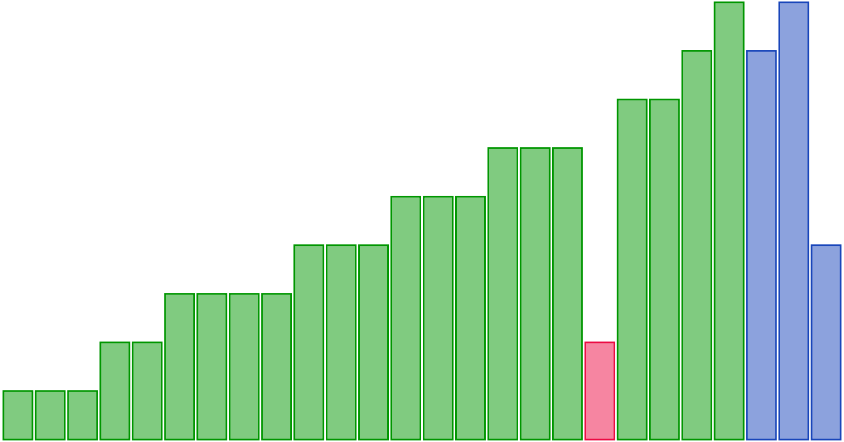

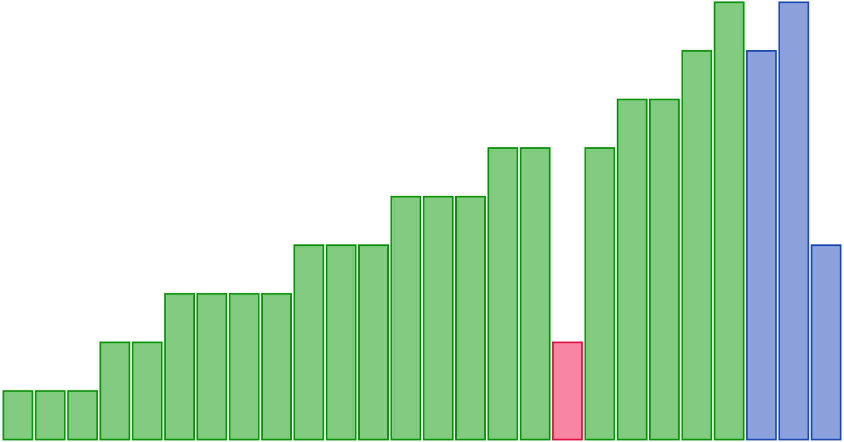

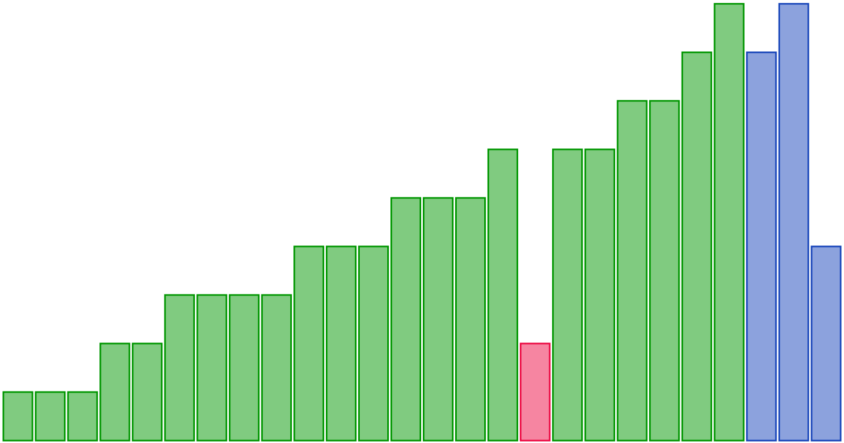

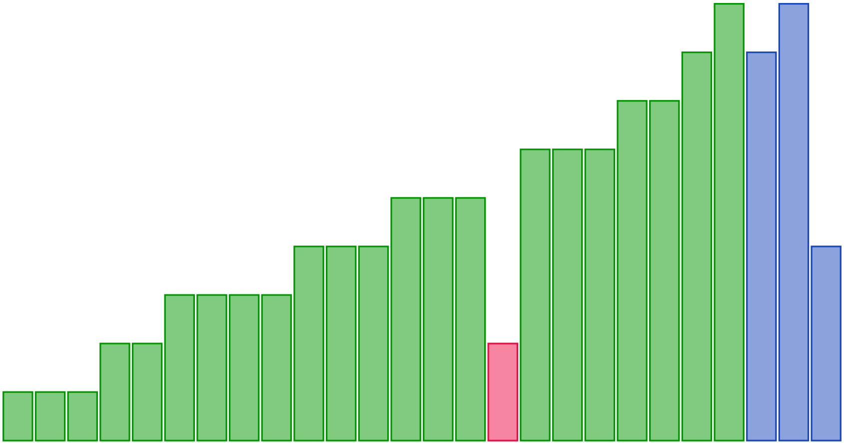

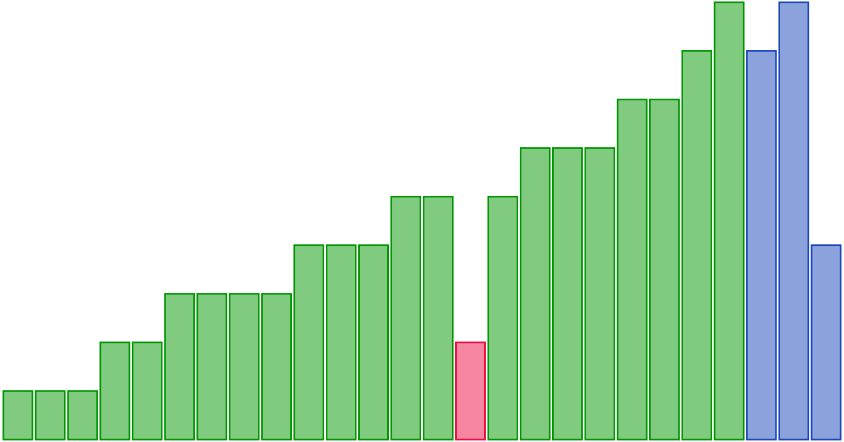

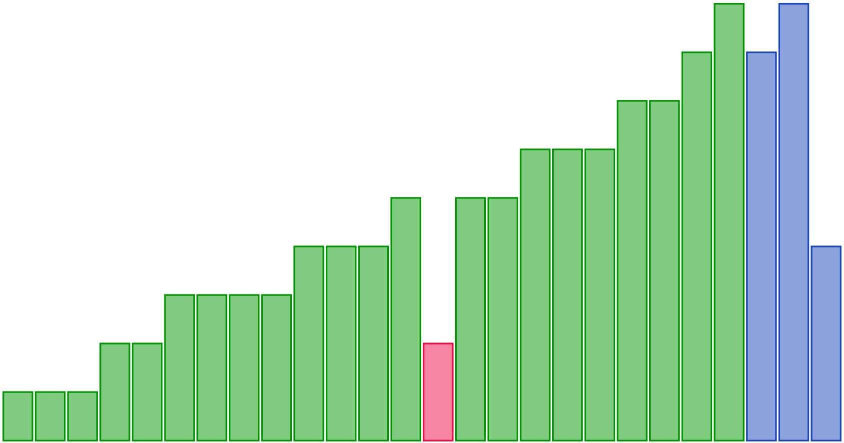

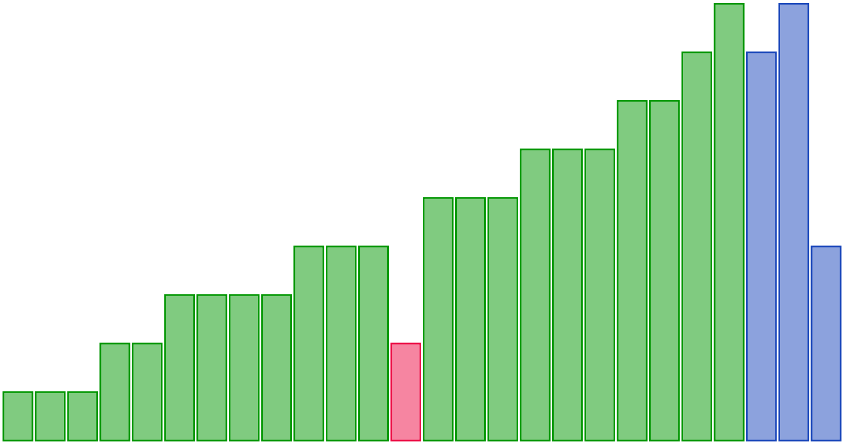

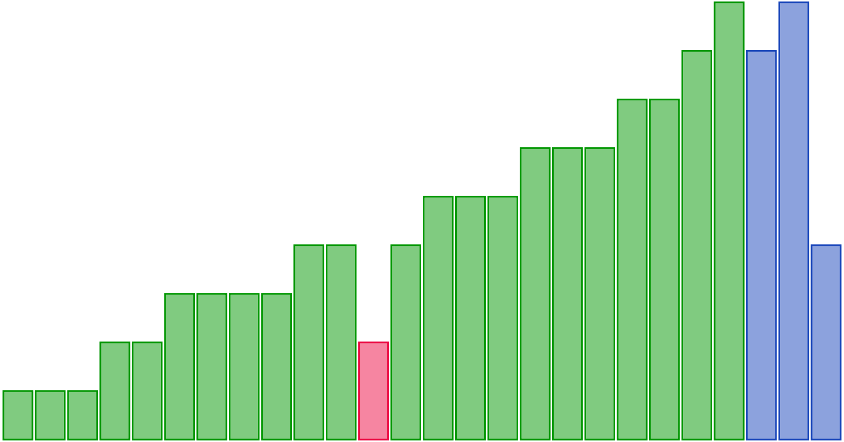

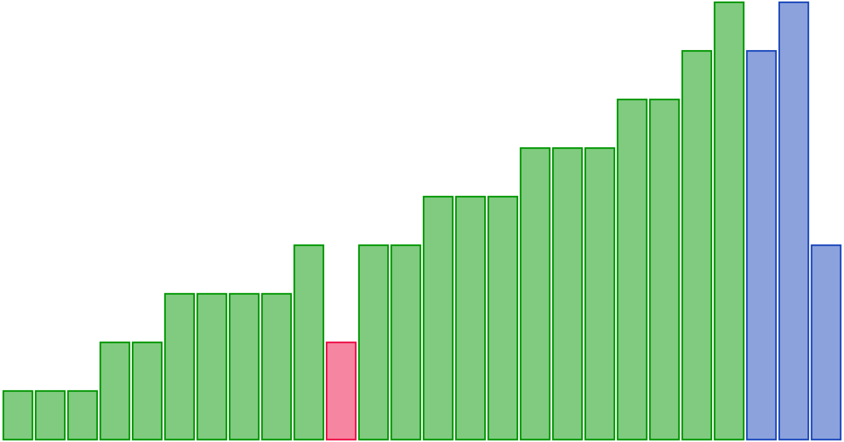

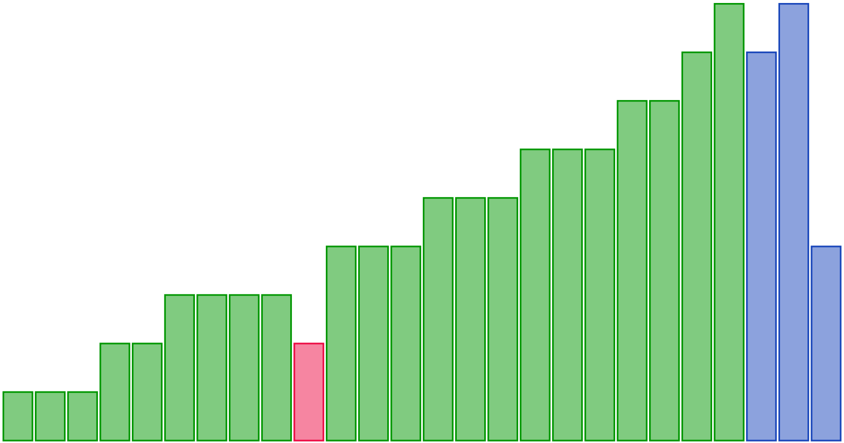

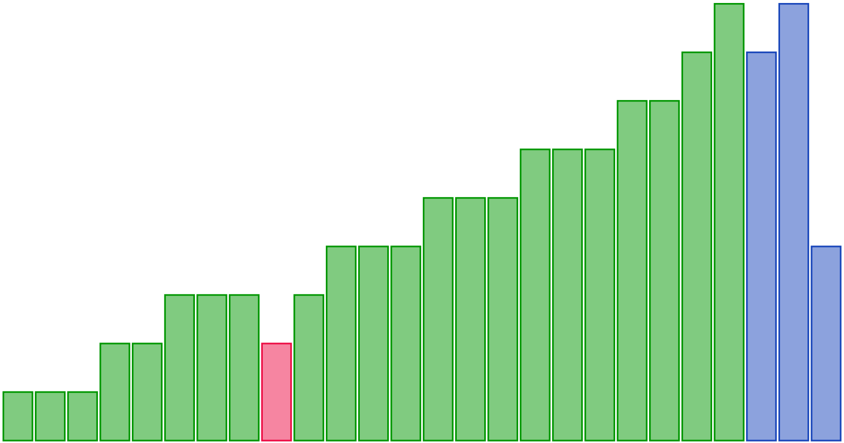

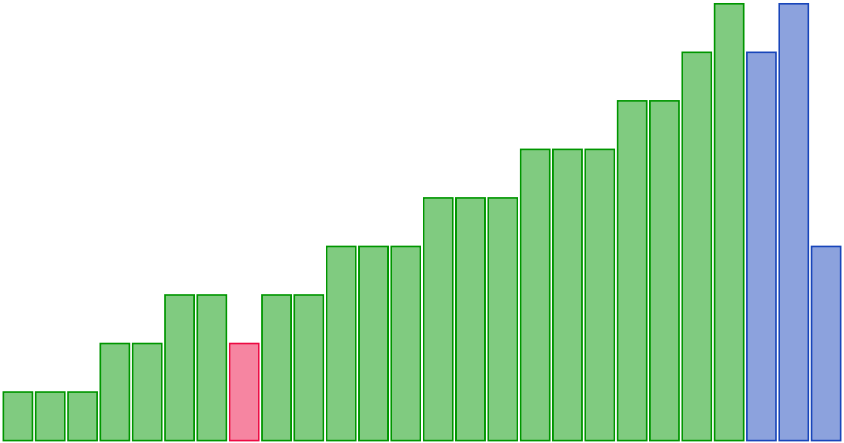

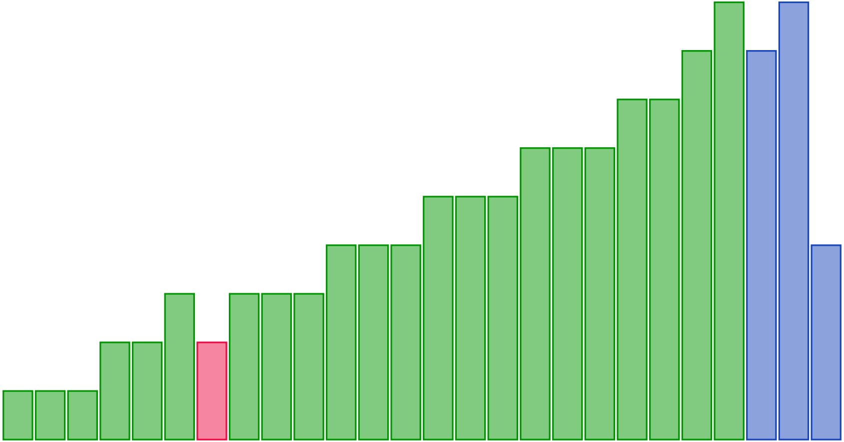

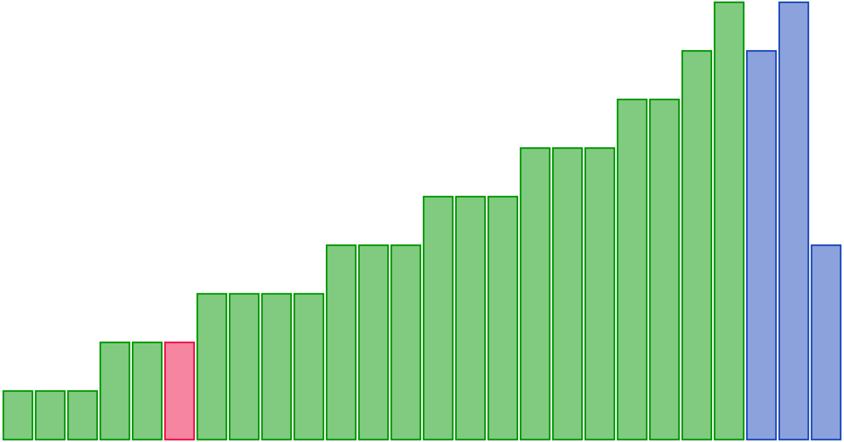

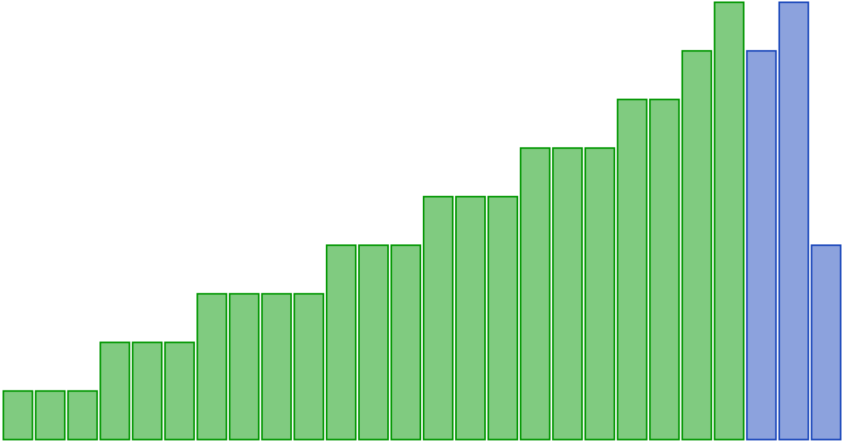

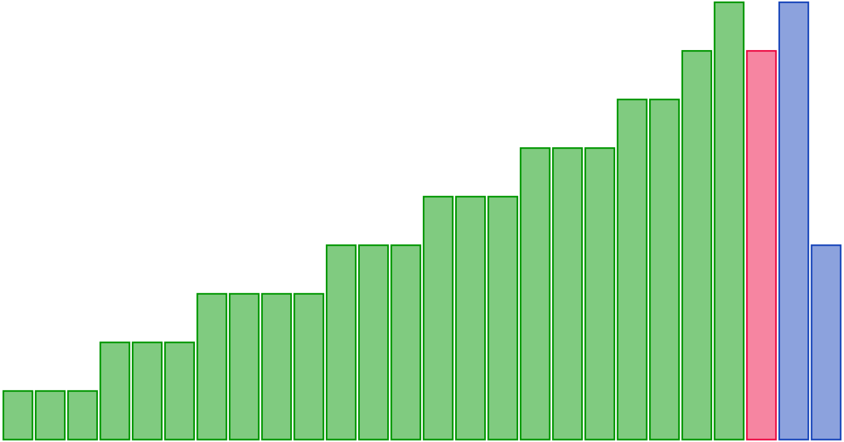

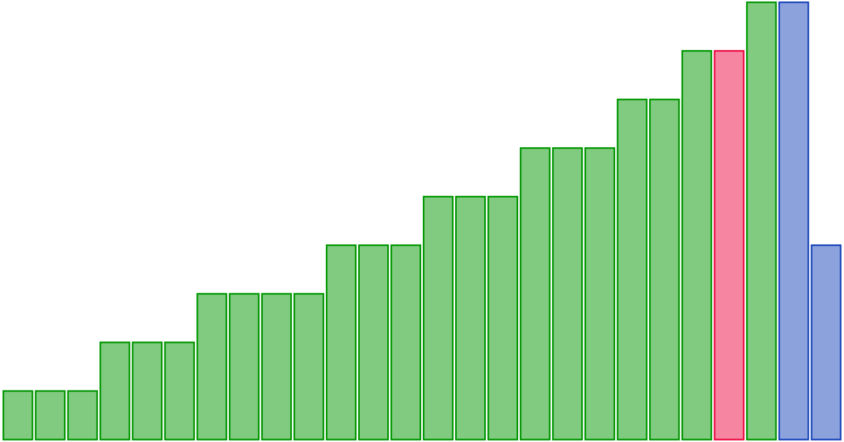

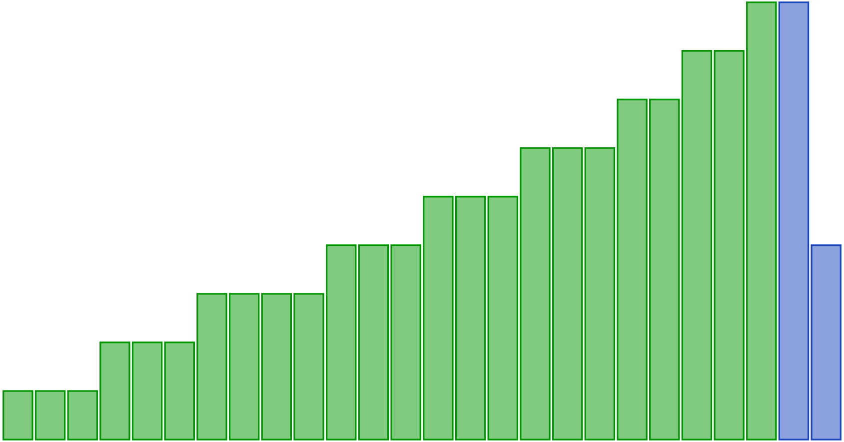

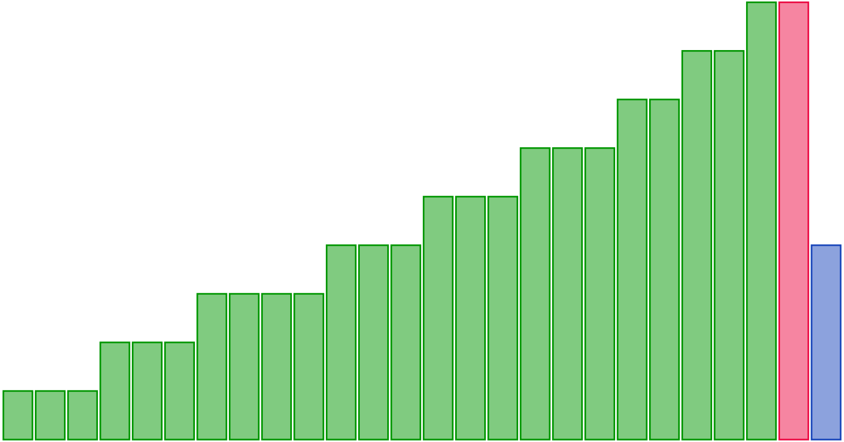

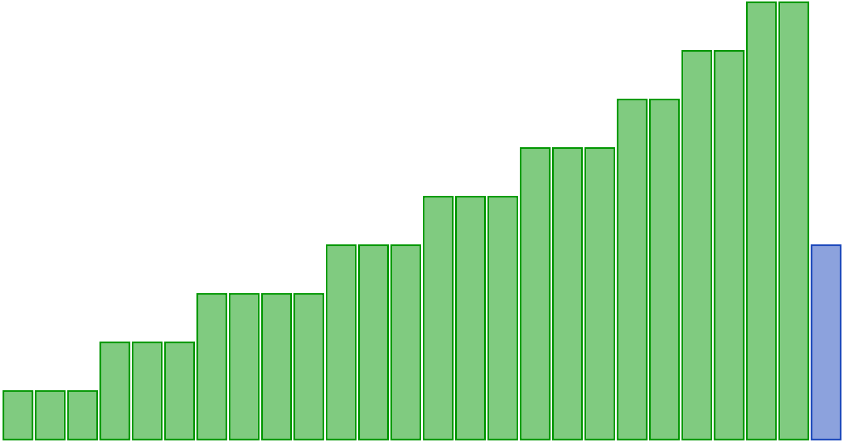

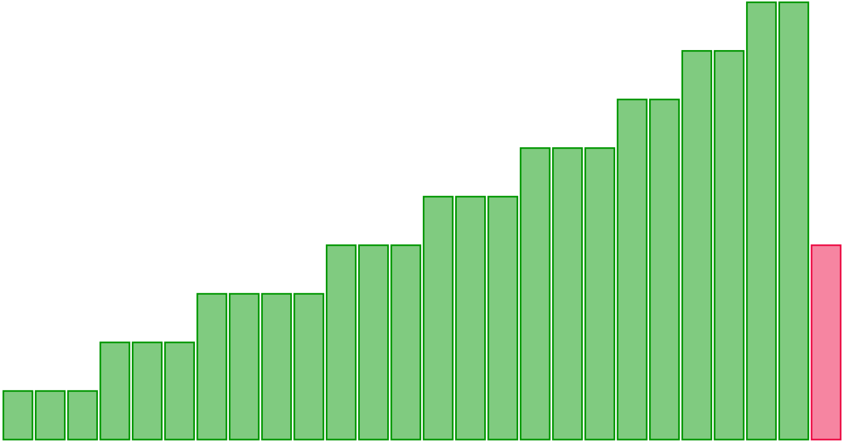

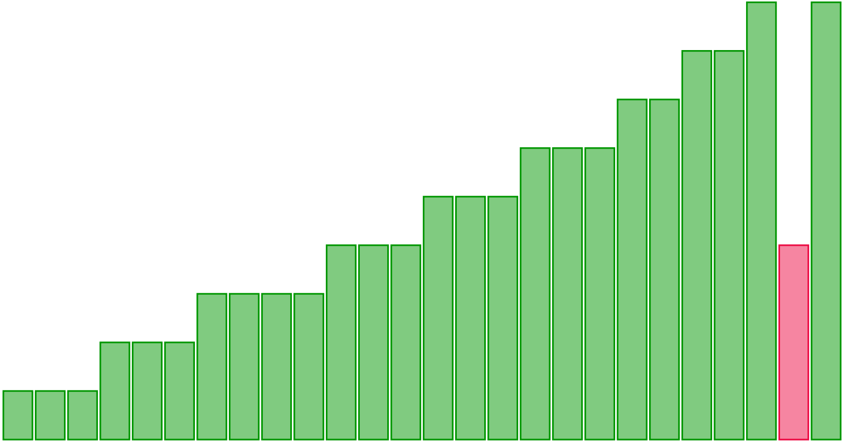

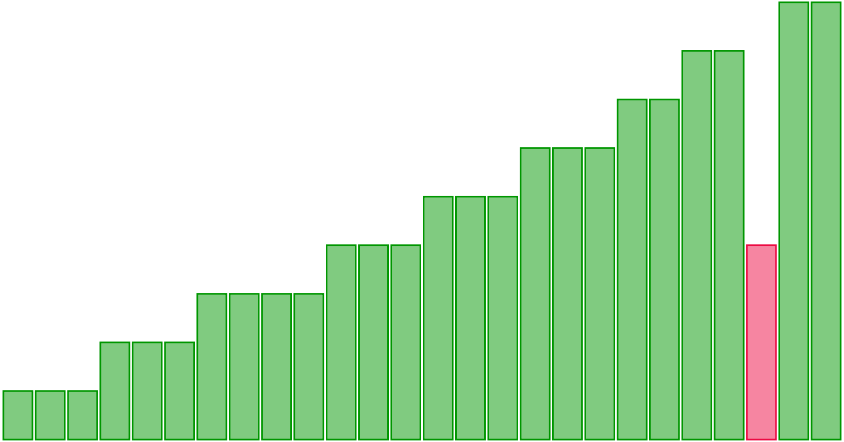

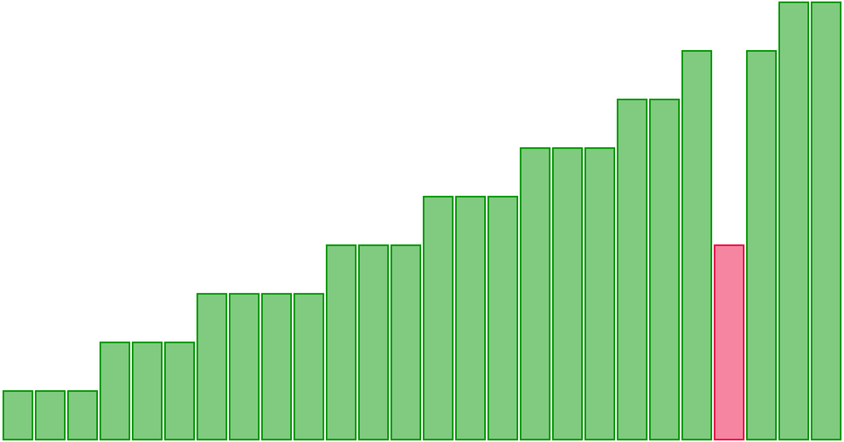

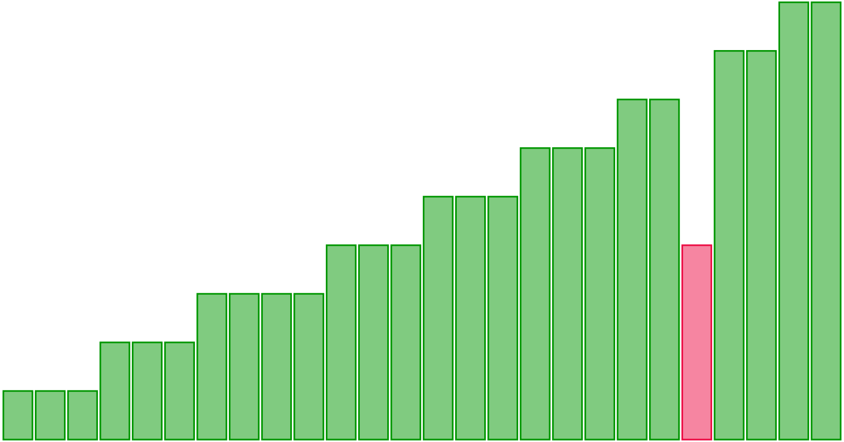

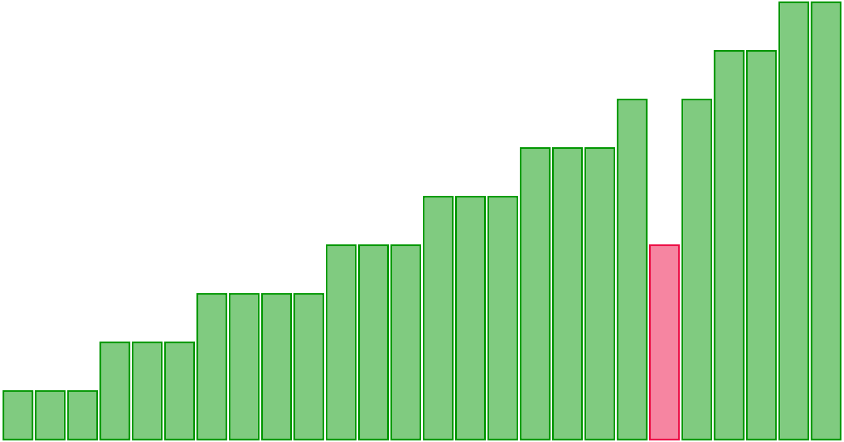

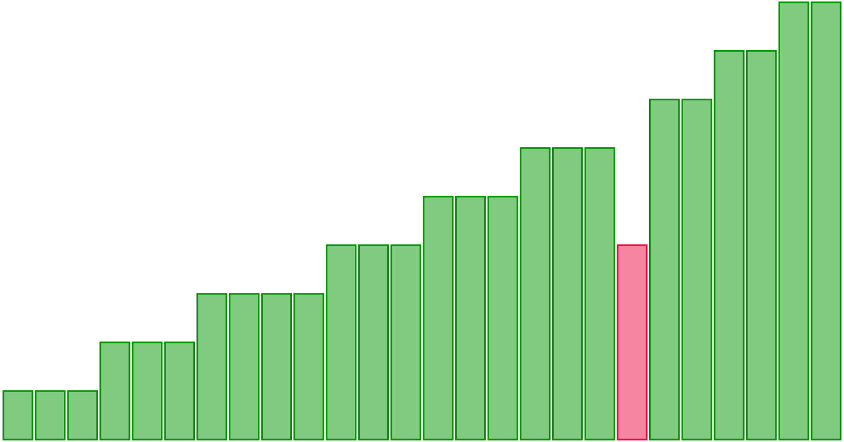

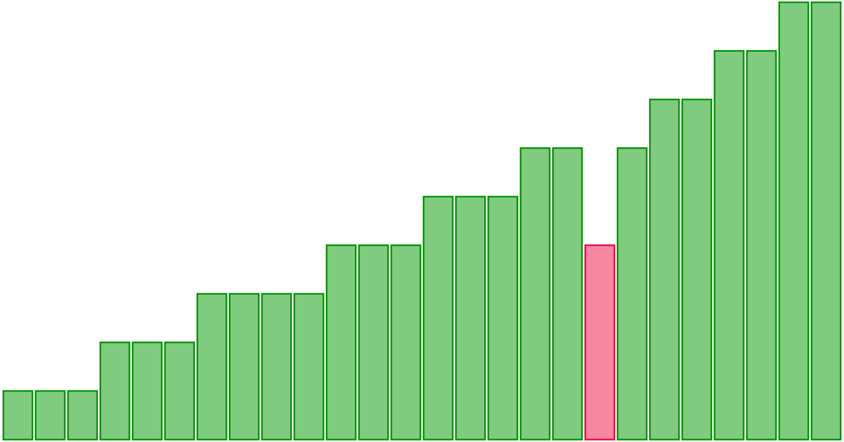

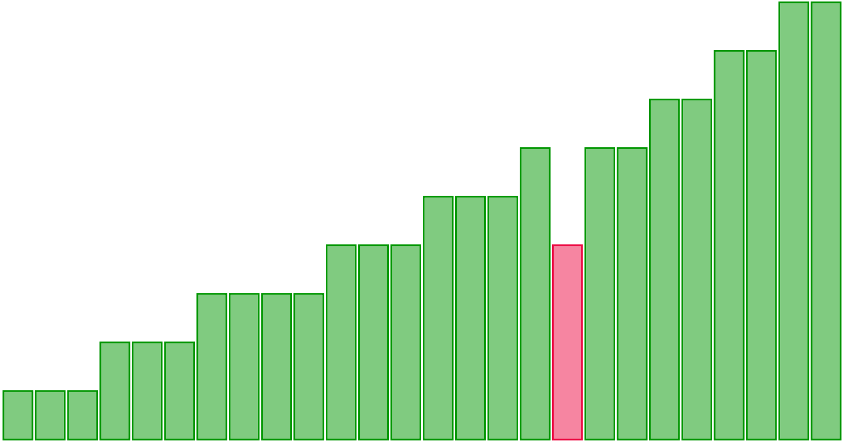

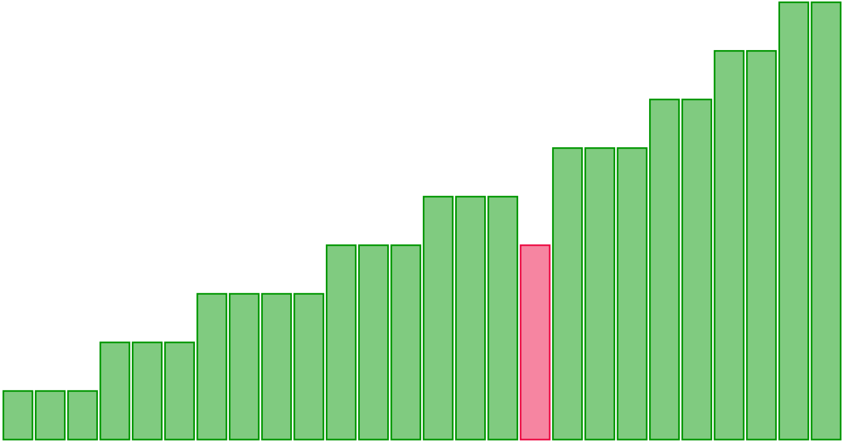

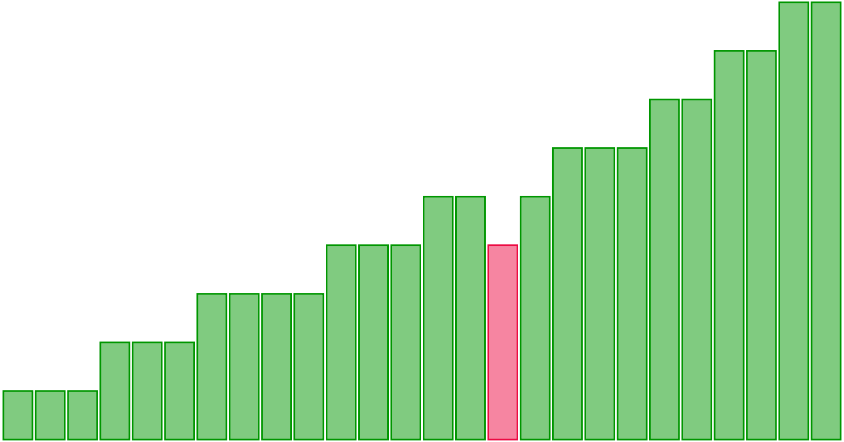

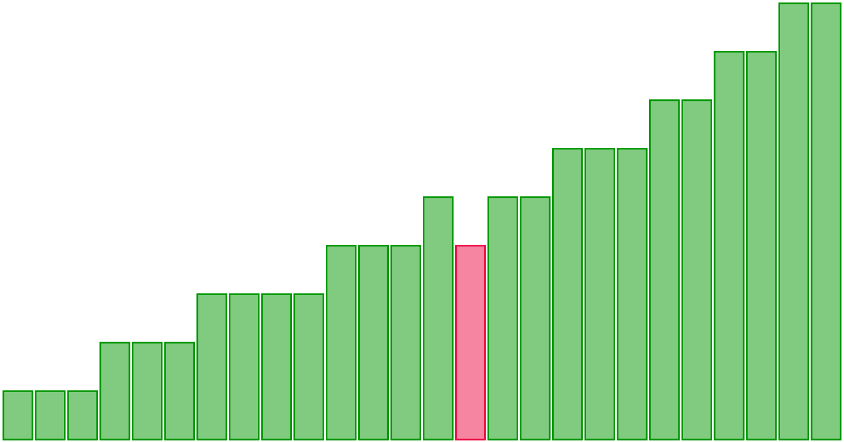

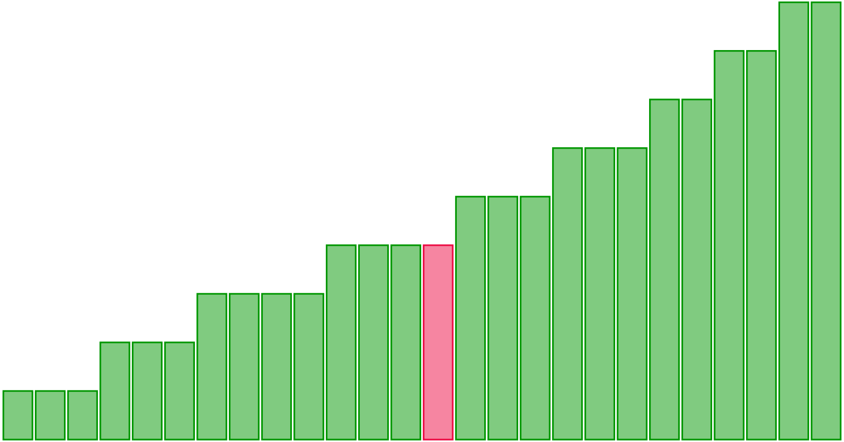

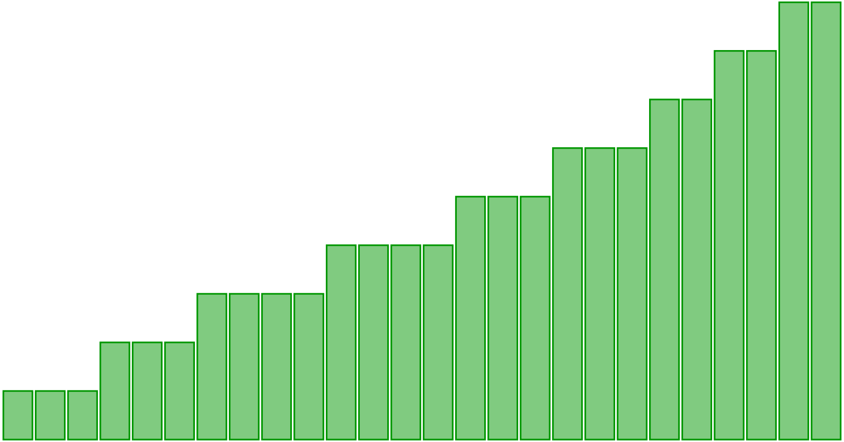

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Correctness of the insertion sort follows from this theorem with $t = |A|$.

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A more precise running time estimation. . .

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  - Average number of inversions per permutation: $\frac{N!}{2} \cdot \frac{N(N-1)}{2} \cdot \frac{1}{N!} = \frac{N(N-1)}{4} = \Theta(N^2)$