Lesson 3
Analysis of State Transitions

Introduction to Analysis of State Transitions

The cash flow calculations and models you have seen so far in the course have all assumed that the cash flows are guaranteed to occur. For example, the annuity certain example you looked at in Lesson 2 assumed that there was nothing that would stop the policy remaining in force over its entire term. However, there are many factors that could cause the actual cash flows to differ from those projected in the cash flow model. This is fundamentally what is of interest to actuaries, the uncertainty in these cash flows, how to measure this uncertainty, and how to manage this uncertainty.

So what are these factors that could cause the projected cash flows to differ? Firstly, if the premium income were invested by the insurer in assets that did not provide a certain interest rate, this would change the interest rate and the Actual Reserves held at future time periods. Perhaps the policy might have a condition that allows it to be closed down before the end of the term, and a lump sum claim made to the policyholder to cover the future claims forgone.

Apart from what was just mentioned, there are likely to be many other factors that might cause the actual cash flows to differ from those projected in an insurance policy. You might like to share some examples in the forum, where a thread has been created specifically for this topic.

It is important for actuaries to be able to model the occurrence of events that affect cash flows – this is what “Analysis of State Transitions” is all about. For the moment we’ll consider non-financial events only, such as sickness and death. Later on in the course we will consider the impact of financial events such as interest rates. Modelling these uncertain events is fundamental to what actuarial work is all about, no matter whether the actuary is working in insurance, pensions, investment, or any other line of work.

Models of “Analysis of State Transitions” are typically represented in a way similar to the following example:

```
Healthy (H)  Temporarily Ill (TI)
            \     /
    \     /       \     / 
   Permanent Ill (PI)  Dead (D)
```

Each of the boxes represents possible states for an individual, whilst lines with arrows represent possible transitions between states. For example a Healthy individual can become Temporarily Ill,
and a Temporarily Ill individual can also become Healthy. A Healthy individual or Temporarily Ill individual can become Permanently Ill, but a Permanently Ill individual cannot become Healthy or Temporarily Ill. **Absorbing** states are states that cannot be transitioned out of once they are entered (i.e. they only have arrows leading to them and not away from them). Obviously in this example the “Dead” state is absorbing; you clearly cannot exit this state once you enter it, but you can enter it from all of the other states!

These models are very important to actuaries, as the cash flows that affect individuals and the insurer will depend on the state the individual is in. For example, imagine an insurance product that requires policyholders to pay a premium to the insurer when they are healthy and take up the product, and to continue paying premiums each subsequent year that they are healthy. No claim is paid to the policyholder while they are healthy. But the product provides a regular claim to a policyholder who is temporarily ill, and a one-off claim to a policyholder who becomes permanently ill or dies. The policy ceases after one of these one-off claims. Clearly the future movements of policyholders between states will affect cash flows in (premiums) and out (claims) of the insurer, and the insurer will need to know the probabilities of movements between these states in order to ensure the Actual Reserves are sufficient to pay the claims to policyholders as they occur.

In the remainder of this week’s material we will investigate how to model transitions between the states. For simplicity, we will consider a basic two-state model only, with a single transition between the Alive and the Dead state:

![Diagram](attachment:diagram.png)

An example of a product for which this model might be useful is one that requires policyholders to pay regular premiums when they are alive and then provides a one-off claim when the policyholder dies. This is obviously a very simple product, but will provide a nice framework from which to view an example of actuarial work. In practice, actuaries will typically need to consider much more complicated models of state when working with insurance and other financial products.

**Assessment Question 3.1**

Which of the following statements best describes what an absorbing state is?

A) The state that an individual starts in  
B) A state that an individual cannot leave after they have entered it  
C) A state that has many potential paths of entry into  
D) A state that can only be entered into from one other state
The Two-State Model

Now we have our two-state model, we are ready to investigate how this model is used by actuaries in practice. We’ll start from our example of a product that requires policyholders to pay regular premiums when they are alive and then provides a one-off payment when the policyholder dies.

For this type of policy, once a policyholder dies and they receive their one-off claim from the insurer, the insurer has no further obligations in respect to that policyholder. Hence, the insurer is primarily interested in policyholders who are Alive and the probability that they might transition to the Dead state. We are now going to introduce some notation that will help us model this.

We first introduce $\mu_x$, where $\mu$ is the transition intensity of mortality per annum (i.e. from state Alive to Dead – this is also known as the force of mortality) and $x$ is the factor (or factors) that affect $\mu$. For the remainder of the course we’ll assume $x$ is age or time. We would obviously expect mortality to be affected by age, although there could be other factors such as gender, socio-economic status, etc. that affect $\mu$ but which we will ignore to keep things simple. Actuaries in practice will have much more complicated models for $\mu$ than what we will consider here.

An Example Using Car Speed

We can think of $\mu$ as somewhat equivalent to the instantaneous speed of a car. Let’s just imagine for a moment that $\mu_x$ is the speed of a car at exact time $s$. If $\mu_x$ was a constant $\mu$ (i.e. not dependent on $s$), then the distance travelled between time 0 and time $t$ would be:

$$\text{Distance} = \text{Speed} \times \text{Time} = \mu t$$

This of course assumes that speed is measured in equivalent units to time. For example, if a car travelled at 60 km/hr for 3 hours it would travel $60 \times 3 = 180$ km.

Let’s make things a little more complicated now. If $\mu_x$ is dependent on $s$ (i.e. if $\mu$ changes over time), then our formula Distance = Speed x Time no longer works as simply as it does above. We can, however, use integration techniques to calculate the distance travelled. We can see from our original example that distance is the result of the integration of speed between time 0 and time $t$, or simply the area under a plot of speed against time:
Now let us assume that $\mu_s = s$. Now the distance travelled between time 0 and time $t$ would be:

$$\text{Distance} = \int_0^t \mu_s ds = \int_0^t s \, ds = \left[ \frac{s^2}{2} \right]_0^t = \frac{t^2}{2}$$

Expressing this is a plot we can see that the area under the line $\mu_s = s$ is equal to $\frac{t^2}{2}$:
Back to the Two-State Model

What we have done with the car example is move from an instantaneous calculation $\mu_s$, to a calculation that gives us a value over a period of time $0$ to $t$. Typically actuaries are interested in something similar – they would like to know the probability that an individual who is currently alive at age $x$ will be dead at age $x+t$. This is commonly defined by actuaries as $q_x$. Equivalently, the probability that an individual who is currently alive at age $x$ will be alive at age $x+t$ is defined as $p_x$. Since these options are exhaustive (i.e. you can only be alive or dead) then $p_x = 1 - q_x$.

Intuitively, we might expect $q_x$ to be calculated as follows:

$$t q_x = \int_0^t \mu_{x+s} ds$$

This approach works in the car distance example, as there are no probabilities attached to the whether the car is driving or not. However, that is not the case here, as an individual could die at age $x+j$ between age $x$ and age $x+t$; they cannot die again between age $x+j$ and age $x+t$ and therefore $\mu_{x+s}$ is irrelevant when $s > j$ for someone who has died at age $x+j$.

Another way of demonstrating the incorrectness of this approach is through a numerical example. Let’s assume that the transition intensity of mortality $\mu$ is constant at a rate of 10% per annum.

Were the above formula correct the probability of dying in the next 15 years, $q_{15}$, would be equal to $10\% \times 15 = 1.5$, a number which is outside the possible bounds of 0 to 1 for a probability to take! Therefore we need a different approach to determining $q_x$. 
### Assessment Question 3.2

Which of the following statements best describes what a transition intensity is?

A) The state that an individual moves into  
B) The state that an individual moves out of  
C) The probability that an individual moves from one state to another over a period of time  
D) The instantaneous rate of movement from one state to another

### Assessment Question 3.3

What is the meaning of the notation $p_x$?

A) The probability that an individual aged 0 will die before age $x + t$.
B) The probability that an individual aged $x$ will die before age $x + t$.
C) The probability that an individual aged 0 will survive until age $x + t$.
D) The probability that an individual aged $x$ will survive until age $x + t$. 

Calculating Probabilities from Transition Intensities

In determining \( q_t \), it will be helpful to utilise the following conditional rule of probability for events \( A \) and \( B \):

\[
P(A | B) = \frac{P(A \cap B)}{P(B)}
\]

A Dice-Based Example

For those unfamiliar with conditional probability, we give the following simple example using dice. Imagine that two six-sided dice are rolled. \( A \) is the event where the sum of the two dice is 10 and \( B \) is the event where the number on the first dice is 5. Therefore \( A \mid B \) (which means event \( A \) given that event \( B \) has already occurred) is the event where the sum of the two dice is 10, given that the number on the first dice is 5. We want to calculate the probability of \( A \mid B \). Let’s look at this in terms of the probabilities on the right hand side of the above equation:

\[
P(A \cap B) \text{ is the joint probability of } A \text{ and } B; \text{ i.e. the probability that both } A \text{ and } B \text{ will occur. There are 36 different combinations of numbers that can be rolled on two dice (6 on the first dice multiplied by 6 on the second dice). The only combination of these that leads to both } A \text{ and } B \text{ occurring is where the first dice is 5 and the second dice is also 5. }\]

\[
\therefore P(A \cap B) = \frac{1}{36}
\]

\[
P(B) \text{ is the probability that the number on the first dice is 5. } P(B) = \frac{1}{6}
\]

Thus the required probability is:

\[
P(A \mid B) = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}
\]

This result makes sense intuitively. If we know the first dice is 5, then the only way the two dice can add up to 10 is for the second dice to be 5, which has a probability of 1/6. Note that the probability of the sum of two dice being 10 without knowing the results of one of the dice is 1/12. In this instance, knowing the result of one of the dice has given us a higher probability of the sum of the two dice being 10.
Back to the Two-State Model

In the two-state model we will let \( A \) be the event of surviving from age 0 to age \( x + t \) and \( B \) be the event of surviving from age 0 to age \( x \). Therefore \( A \setminus B \) is simply the event of surviving from age \( x \) to age \( x + t \). Event \( A \cap B \) (which means event \( A \) and event \( B \) both occurring) is just event \( A \), since surviving from age 0 to age \( x + t \) must also entail surviving from age 0 to age \( x \).

Using notation previously introduced, and the conditional probability formula, this is equivalent to stating:

\[
_{t}p_{x} = \frac{x+tP_{0}}{xP_{0}}
\]

We will use this result throughout the calculations below.

Recall that the result we want to calculate is \( _{t}q_{x} \). We determine \( _{t}q_{x} \) as follows:

\[
_{t}q_{x} = 1 - _{t}p_{x} \quad \text{which we already know}
\]

\[
_{t}q_{x} = \frac{xP_{0} - x+tP_{0}}{xP_{0}} \quad \text{by using the probability rule above}
\]

\[
\frac{_{t}q_{x}}{t} = \frac{xP_{0} - x+tP_{0}}{xP_{0} \times t}
\]

As \( t \) goes to zero, the left hand side of this equation become the instantaneous rate of mortality, i.e. the transition intensity \( \mu_{x} \). The right hand side of the equation can be rearranged so it becomes a differentiation from first principles.

\[
\lim_{t \to 0} \frac{_{t}q_{x}}{t} = \mu_{x} = -\frac{1}{xP_{0}} \lim_{t \to 0} \frac{x+tP_{0} - xP_{0}}{t}
\]

\[
\mu_{x} = -\frac{d}{dx} \frac{_{t}p_{x}}{xP_{0}}
\]

\[
= -\frac{d}{dx} \ln x_{P_{0}} \quad \text{since} \quad \frac{d}{dx} \ln f(x) = \frac{d}{dx} \frac{f(x)}{f(x)}
\]

Note that \( \ln \) is a log with base \( e \), where \( e \) is a mathematical constant approximate to 2.71828...
Given this, we can also state:

\[ \mu_{x+s} = -\frac{d}{ds} \ln x_s p_0 \]

We will now integrate both sides of this equation:

\[ -\int_0^t \mu_{x+s} ds = \ln \left[ x_{x+t} p_0 \right]_0^t = \ln x_{x+t} p_0 - \ln x_s p_0 = \ln \left( \frac{x_{x+t} p_0}{x_s p_0} \right) = \ln t p_x \]

\[ t p_x = \exp \left[ -\int_0^t \mu_{x+s} ds \right] \]

Note that \( \exp[] = e^{} \) — this result holds because \( \exp[\ln{}] = {} \)

\[ t q_x = 1 - t p_x \]

\[ t q_x = 1 - \exp \left[ -\int_0^t \mu_{x+s} ds \right] \]

The following questions will all use the above result.

**Practice Question 3.1**

An individual has a constant transition intensity of mortality of \( \mu = 0.03 \) per annum. Calculate the probability that they will die within the next 5 years.

**Assessment Question 3.4**

An individual has a constant transition intensity of mortality of \( \mu = 0.01 \) per annum. Calculate the probability that they will die within the next 20 years.

**Assessment Question 3.5**

An individual has a transition intensity of mortality for the next 10 years of \( \mu_s = 0.01s \) per annum (where \( s \) is the number of years from today). Calculate the probability that they will survive for the next 10 years.
Assessment Question 3.6 (Hard)

The Gompertz-Makeham law of mortality states that the transition intensity of mortality for humans of age \( x \) can be expressed in the form \( \mu_x = \lambda + \alpha e^{\beta x} \). It has been observed in a population that the probability of a 20 year old surviving until age 50 is equal to 0.96, and the probability of a 20 year old surviving until age 85 is equal to 0.50. Assuming that the Gompertz-Makeham law of mortality holds and that \( \beta = 0.085 \), calculate \( \lambda \) and \( \alpha \).

In order to solve this question, you will need to know the result:

\[
\int e^{f(x)} \, dx = \frac{e^{f(x)}}{f'(x)}; \quad \text{where } f(x) \text{ is a linear function of } x.
\]

Extension Question 3.1

An individual currently aged 30 experiences a transition intensity of mortality of \( \mu_x = 5 \times 10^{-4} + 2 \times 10^{-5} e^{0.085x} \), where \( x \) is equal to age. Calculate the probability that they will die between age 75 and 80.

Note that this question is asking for a deferred probability of mortality, which is commonly defined by actuaries as \( q_{n+t} \), where \( n \) is the deferral period that the individual must survive during. In this question, you are asked to calculate \( q_{45}^{30} \).

Extension Question 3.2

Show that an alternative formulation for \( q_x \) is:

\[
t_q = \int_0^t p_s \mu_{x+s} \, ds
\]

Hint: Use the result \( P(A \mid B) = \frac{P(A \cap B)}{P(B)} \) and let \( A \) be the event of dying between age \( x \) and age \( x + t \), and \( B \) be the event of surviving from age 0 to age \( x \), attempting to express \( P(A \cap B) \) in terms of \( q \) only.

Given the discussion on the relationship between distance, speed and time earlier in this Lesson, why does this result make intuitive sense?
Extension Question 3.3

Using $\mu_x = \lambda + \alpha e^{\beta x}$, calculate $q_x$ in terms of $x$, $\alpha$, $\beta$ and $\lambda$. Note that $q_x$ is equivalent to $q_x$. 

After doing this, use a spreadsheet tool to calculate $q_x$ for $x = 30, 31, 32, \ldots, 80$ and graph the results using the $\alpha$, $\beta$ and $\lambda$ from Extension Question 3.1 (i.e. $\mu_x = 5 \times 10^{-4} + 2 \times 10^{-5}e^{0.085x}$).

You might like to investigate the effect of changing the values of $\alpha$, $\beta$ and $\lambda$ on the $q_x$ values.
How Actuaries use State-Based Models

So what does this mean to an actuary in practice? If an actuary is able to determine the transition intensities $\mu$ then they can use this information to calculate probabilities of transition in their modelling. Later on in the course, this will become particularly relevant when we want to model the outcome of selling an insurance product. Actuaries typically estimate the transition intensities and probabilities by reference to relevant past data, although for the purposes of this course we will assume that transition intensities and probabilities are known and that data analysis is not required.

Although we have only looked at mathematical relationships between transition intensities and probabilities of transition in a simple two-state model, in practice most situations that will be considered by an actuary will have more than two states, which makes the mathematical modelling far more complicated.

In Lesson 4 we will consider a specific example of the simple two-state model, the life table.
Summary

- Analysis of State Transitions involve investigating the states an individual could be in that will affect the cash flows of a financial system, as well as the potential transitions between states. Absorbing states are states that cannot be transitioned out of once they are entered.

- The two-state model we will consider throughout this course looks like:

```
Alive (A) \[ \mu_x \] Dead (D)
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- $\mu_x$ is the transition intensity of mortality per annum (i.e. from state Alive to Dead – this is also known as the force of mortality), where $x$ is age or time.

- $q_x$ is the probability that an individual who is currently alive at age $x$ will be dead before age $x + t$, whilst $p_x$ is the probability that an individual who is currently alive at age $x$ will be alive at age $x + t$. They are calculated as:

\[
\begin{align*}
q_x & = 1 - p_x \\
q_x & = 1 - \exp\left( - \int_0^t \mu_{x+s} \, ds \right)
\end{align*}
\]

- Generating the above formulae required the use of the conditional probability formula:

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)}
\]